

topological modes on the interface of two semi-ladders. These highly adjustable electrical circuits and flexible phases open the door to smart, tunable and adaptive topological metamaterials.

2 The model and results

The considered nonlinear topological model is a one-dimensional circuit under periodic boundary condition (PBC). As schematically shown by Fig. 1(a), the diatomic unit cell consists of two identical LC resonators, whose inductance and capacitance are $L = 1 \mu\text{H}$, and $C = 22 \text{ pF}$, respectively. The other ends of the resonators are grounded such that the functionalities are the analog of mechanical oscillators in elastic networks [57–60]. The resonators are connected by nonlinear voltage-dependent capacitors $C_1(V)$ and linear capacitors $C_2 = 25 \text{ pF}$ that serve as the inter-cell nonlinear couplings and intra-cell linear couplings, respectively. Here, the nonlinear capacitors $C_1(V)$ are made up of two varactor diodes that yield mirror symmetry [53], and the voltage dependence is obtained by detecting the resonant frequency of an LC resonator that is constructed from an inductor of $1 \mu\text{H}$ and the nonlinear capacitor $C_1(V)$. By changing the bias voltage, the corresponding resonant frequencies are measured by network analyzer (KEYSIGHT 5061B). Derived from the resonant frequencies, the voltage dependence of the capacitor $C_1(V)$ is plotted by blue dots in Fig. 1(c), with the maximum and minimum values $C_{1\text{max}} = 37.7 \text{ pF}$ and $C_{1\text{min}} = 5 \text{ pF}$, respectively. We further fit these experimentally measured data using Gaussian function, as shown by the red curve in Fig. 1(c), for numerical computations of the nonlinear topological phases and transitions. For a cascaded circuit of N unit cells, the Lagrangian reads

$$\begin{aligned} \mathbf{L} \left(\left\{ V_n^{(1)}, V_n^{(2)} \right\} \right) \\ = \sum_{n=1}^N \left[\mathbf{L}_n - U_1 \left(V_{n-1}^{(2)} - V_n^{(1)} \right) - U_2 \left(V_n^{(1)} - V_n^{(2)} \right) \right], \end{aligned} \quad (1)$$

where \mathbf{L}_n is the total Lagrangian of the two LC resonators of the n th unit cell, $U_{j=1,2}$ denote the potential energies of the nonlinear and linear capacitors, and $V_n^{(j=1,2)}$ are the two voltage fields of the unit cell, as marked in Fig. 1(a). While the potential energy of linear capacitors is harmonic, the energy of nonlinear capacitors $U_1(V) = \int_0^V (V-x)C_1(x)dx$ is strongly anharmonic for large biased voltage V , which forbids the availability of linear analysis. Due to the intrinsic structural symmetry of the nonlinear capacitor and the ladder circuit, the Lagrangian stays invariant under reflection transformation

$$\mathbf{L} \left(\left\{ V_n^{(1)}, V_n^{(2)} \right\} \right) = \mathbf{L} \left(\left\{ V_{-n}^{(2)}, V_{-n}^{(1)} \right\} \right). \quad (2)$$

As we show below, reflection symmetry of the lattice Lagrangian fundamentally quantizes the nonlinear Berry phase, whose non-trivial integer value guarantees the emergence of nonlinear topological interface modes.

The nonlinear dynamics of the considered circuit follow from the Lagrangian equations of motion, which are expressed by four-field generalized nonlinear Schrödinger equations [61]. Spatially repetitive structures enjoy the nonlinear extension of Bloch theorem [56, 61], whose spatial-temporal periodic voltage oscillations take the format of plane-wave nonlinear normal modes [62] $V_k = \left(V_k^{(1)}(\omega t - nk), V_k^{(2)}(\omega t - nk + \phi_k) \right)$. Here, ω and k are the frequency and wave number respectively, $V_k^{(j=1,2)}(\theta)$ are 2π -periodic wave components, and ϕ_k characterizes the relative phase between these two wave components. The frequencies of plane-wave nonlinear normal modes $\omega = \omega(k, A)$ are controlled both by wavenumber k and mode amplitudes A , which naturally deviate from their linear counterparts as nonlinearity grows.

Nonlinear normal modes yield reflection symmetry in Eq. (2), from which Berry phase of nonlinear normal modes are guaranteed to pick quantized values (see Ref. [56] and Electronic Supplementary Materials [61] for details),

$$\begin{aligned} \gamma(A) = \oint_{\text{BZ}} dk \frac{\sum_l l \left| v_{l,k}^{(2)} \right|^2 \partial_k \phi_k + i \sum_j v_{l,k}^{(j)*} \partial_k v_k^{(j)}}{\sum_{l',j'} l' \left| v_{l',k}^{(j')} \right|^2} = n\pi, \\ n = 0 \text{ or } 1. \end{aligned} \quad (3)$$

Here, A stands for the amplitude of the nonlinear voltage modes, and $v_{l,k}^{(j)}(\theta) = (2\pi)^{-1} \int_0^{2\pi} e^{il\theta} V_k^{(j)} d\theta$ is the l -th Fourier component of $V_k^{(j)}$. This quantized geometric phase serves as the topological index of the nonlinear circuit dynamics, where $\gamma = \pi$ and $\gamma = 0$ indicate topologically nontrivial and trivial phases, respectively. Upon the increase of mode amplitudes A , $\gamma(A)$ cannot change continuously from π to 0 due to its topological nature. Nevertheless, it experiences abrupt jumps between distinct integer multiples of π as the nonlinear bandgap closes and reopens at the topological transition amplitude A_c . This nonlinear topological transition can be intuitively understood by referring to the transitions of linear SSH circuit in Fig. 1(b), whose intercell and intracell couplings are C_3 and C_2 , respectively. When $C_2 < C_3$, the intracell coupling is weaker ($C_2 > C_3$, the intracell coupling is stronger), the linear topological number is in the non-trivial (trivial) phase. Likewise, the nonlinear Berry phase is in the non-trivial (trivial) phase when $C_2 < C_1(V)$ for weaker intracell coupling [$C_2 > C_1(V)$ for stronger intracell coupling], as we discuss below.

As shown in Fig. 1(d), we numerically compute the nonlinear band gap in the circuit system with the unit

cells addressed in Fig. 1(a). The nonlinear band gap experiences topological phase transition as voltage amplitudes rise. In the linear regime, the initial bandgap opens, and the topological number $\gamma(A=0) = \pi$ indicates that the circuit system is in the non-trivial phase. As voltage amplitudes rise, topological invariance states that quantized nonlinear Berry phase should stay unchanged as $\gamma(A < A_c) = \pi$, provided that the nonlinear band gap remains open, where $A_c = 2.97$ V is the topological phase transition amplitude. The nonlinear gap closes at this critical amplitude A_c , as pictorially depicted by the vanishing gap in Fig. 1(d), where nonlinear Berry phase becomes ill-defined. We define the degree of nonlinearity $\frac{C_1(0)-C_1(A)}{C+C_1(0)+C_2}$ by comparing the nonlinear part of $C_1(A)$ and the linear part of all capacitors $C, C_1(0)$ and C_2 . At the transition amplitude A_c , the degree of nonlinearity reads 0.332, which demonstrates the strongly nonlinear regime of the underlying circuit dynamics [56, 62]. The bandgap reopens above A_c [Fig. 1(d)], whose topological index is well-defined again to pick the integer value $\gamma(A > A_c) = 0$ in the trivial phase.

Figure 2 addresses both the theoretical and experimental transition amplitudes of the topological index in the strongly nonlinear circuit dynamics, as we treat the linear capacitor C_2 as the varying parameter in the horizontal axis. As the amplitude of the voltage fields grows, integer-valued topological Berry phase jumps from $\gamma = \pi$ to 0, as indicated by the nonlinear topological phase transition of the unit cell structure in Fig. 1(a). The theoretical scenario of the transition amplitude is based on the matching condition of the frequencies $\omega(k = \pi, \phi_\pi = 0, A_c) = \omega(k = \pi, \phi_\pi = \pi, A_c)$ of the nonlinear normal modes at the time-reversal-invariant-momentum $k = \pi$ with even ($\phi_\pi = 0$) and odd ($\phi_\pi = \pi$) parities. Given the nonlinear capacitor $C_1(V)$ of Fig. 1(a), we theoretically compute a series of topological transition

amplitudes A_c by varying the linear capacitor C_2 , and plot the relationship between transition amplitudes and the linear capacitor using the blue curve in Fig. 2. For example, the transition amplitude for $C_2 = 23$ pF is 3.65 V, whose topological transition is captured by the inset of Fig. 2. For $C_2 > C_{1\max} = 37.7$ pF, the topological phase stays trivial for all voltage amplitudes, and thus the system cannot experience nonlinear topological transition. As C_2 drops below $C_{1\max} = 37.7$ pF, increasing amplitudes are needed to achieve the topological phase transition. Meanwhile, we experimentally probe these transition amplitudes by identifying the emergence of nonlinear topological interface modes. The experimentally measured transition amplitudes for $C_2 = 15$ pF, 18 pF, 20 pF, 22 pF, 25 pF, 27 pF, 32 pF are denoted by square marks in Fig. 2, which qualitatively agree with the aforementioned simulation results. Furthermore, we investigate how the nonlinear topological transition amplitude is affected by fluctuations in the nonlinear capacitors C_1 with a range of $\pm 10\%$. In Fig. 2, the lower bound of the blue area indicates that the theoretical curve for the topological transition voltage exhibits better agreement with experimental measurements for a value of $0.9C_1$. Deviations between theory and experiment may also arise from fluctuations in the linear coupling strength of C_2 , on-site resonators L and C , and resistance, which is set to zero in theory but non-zero in experiments.

Based on these numerical demonstrations of topological phases and transitions in the nonlinear circuit model, we experimentally conduct the corresponding nonlinear topological physics in real space. According to the nonlinear extension of bulk–boundary correspondence [63], topological physics can be manifested by the emergence and absence of nonlinear topological modes on the interface of two semi-lattices. To observe the evolution

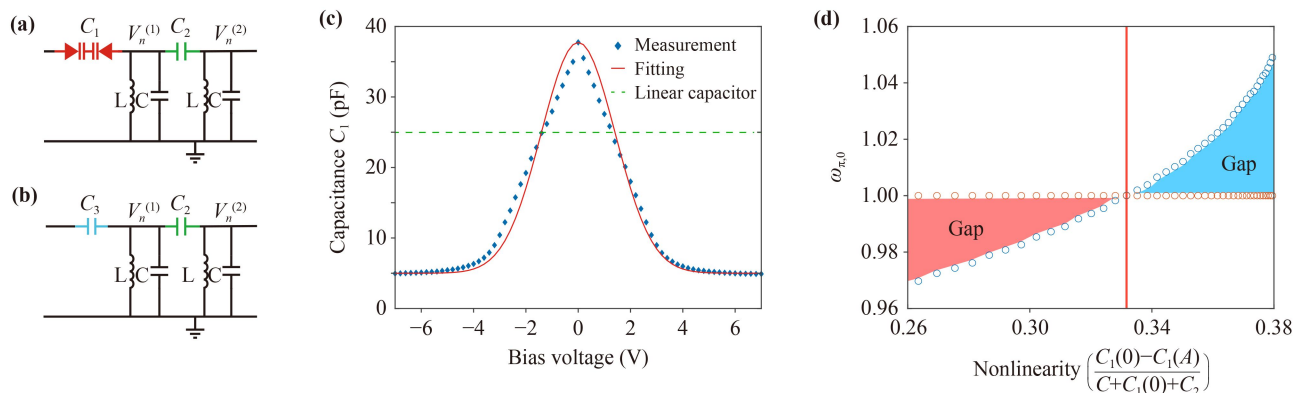


Fig. 1 (a) The unit cell of the underlying nonlinear topoelectrical circuit. (b) The unit cell of the linear circuit that helps to construct an interface between two semi-lattices in Fig. 3. (c) The voltage-dependence of the nonlinear capacitors, where the dots represent the data measured by the network analyzer, and the fitting Gaussian curve is adopted for numerical computations. (d) The numerical calculation of the nonlinear band gap and topological phase transitions of the circuit unit cell in (a) for growing nonlinearity as voltage amplitude increases. At the transition point marked by the vertical red line, the degree of nonlinearity (horizontal axis) reads 0.332, which grants the considered topoelectrical circuit the strongly nonlinear regime.

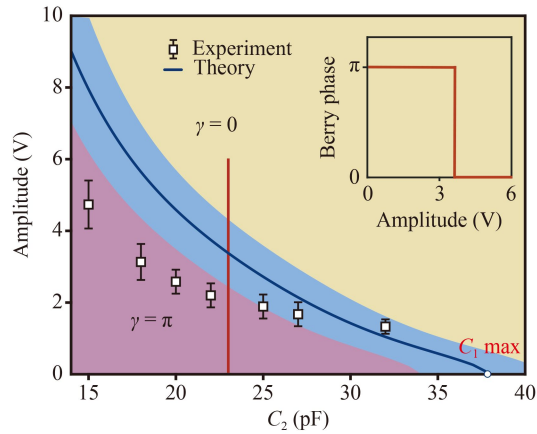


Fig. 2 Topological phase diagram of the nonlinear electrical circuit in Fig. 1(a). In the horizontal axis, we vary the parameter of the linear capacitor C_2 from 14.0 pF to 37.0 pF. The vertical axis represents the amplitude of responding voltage fields. The blue curve denotes the numerical result of the topological transition voltage amplitudes for varying linear capacitor C_2 . The blue area depicts the numerical result of the topological transition voltage under the influence of fluctuations in the nonlinear capacitors C_1 with a range of $\pm 10\%$. The experimentally measured transition voltages are depicted by the square marks with error bars. The inset illustrates the transition of the topological index when $C_2 = 23$ pF.

of topological interface modes, we build two prototypes in Fig. 3 and Fig. 4, and experimentally investigate the spatial profile of the impedance along the circuit board in response to external excitation power.

The electrical circuits are built on the Printed Circuit Board, with the tolerance of chip capacitors and chip inductors $\pm 5\%$ and $\pm 10\%$, respectively. We measure the impedance response of circuit system by generating a chirp voltage signal from a function generator (KEYSIGHT 33600A), and subsequently enlarge the signal by a power amplifier (Minicircuit ZHL-6A-S+). The impedance is measured by the frequency response of the voltage and current on the top end of the LC resonators by an oscilloscope (KEYSIGHT DSOX4054A) controlled by a computer. We probe the voltage responses in all unit cells and measure their local impedance by raising the excitation power from 0.038 V to 7.84 V, to experimentally measure the responding interface modes.

The first prototype in Fig. 3(a) considers two semi-infinite ladder circuits, whose unit cells are enclosed by the green and red dashed boxes respectively, to construct a mutual interface between them. We encircle the unit cells of the left-sided and right-sided semi-lattices using the green and red dashed boxes, respectively. These gauge choices of the unit cells yield open boundary conditions on both sides of the experimental circuit board in Fig. 3(b). Other unit cell choices of the left semi-lattice may cause problems, because left the

open boundary can slice the unit cell at $N = -8$ into half, making it un-defined.

In Fig. 3(b), we experimentally build the circuit board based on the design principle of Fig. 3(a), where both the left and right sides of the interface contain 4 unit cells. On the right side, the unit cells of the semi-lattice are composed of purely linear electrical elements with $C_2 = 25$ pF and $C_3 = 37$ pF as the intracell and intercell couplings. The topological number is fixed at $\gamma_{\text{right}} = \pi$, and the linear band gap is marked by the blue dashed box on the right semi-lattice of Figs. 3(c–e). On the left side, the intracell and intercell couplings are C_1 (V) and C_2 , respectively. In the weakly nonlinear regime, we approximate $C_1(V = 0.038 \text{ V}) = C_3$, and hence, the topological number of the left semi-lattice is $\gamma_{\text{left}} = 0$ is in line with the linear SSH model. This index is different from the right semi-lattice, because the unit cell choices are different on the two sides of the interface to yield open boundary conditions. As a result, $C_1(V = 0.038 \text{ V}) = C_3$, C_2 and C_3 , together appear alternatively in real space to constitute a lattice without an interface, and no topological interface modes are expected in the weakly nonlinear regime. As pictorially manifested by the dark band gap that ranges from 15.36 MHz to 17.88 MHz in Fig. 3(c), topological voltage modes cannot arise on the interface, which is in line with purely linear SSH models. In Fig. 3(d), the interface is on the verge of nonlinear topological phase transition for the external triggering power at 3.09 V, which approaches the transition amplitude $A_c = 2.97$ V, and the left nonlinear band gap closes. As the amplitude further rises to 7.84 V in Fig. 3(e), the intracell coupling of the left semi-lattice, $C_1(V \approx 7.84 \text{ V}) \approx 5$ pF, becomes weaker. The nonlinear band gap reopens above the topological transition amplitude, as depicted by the blue dashed box in the left semi-lattice of Fig. 3(e), leading to the topological numbers $(\gamma_{\text{left}}, \gamma_{\text{right}}) = (\pi, \pi)$ in the large-amplitude regime. Since the band gaps of the left and right semi-lattices mismatch, nonlinear interface modes only arise on the right semi-lattice (left semi-lattice) within the frequency range between 15.36 MHz and 17.88 MHz (between 17.88 MHz and 19.70 MHz) as the same frequency is in the conducting band on the other side of the interface, which enables bulk mode excitations. These experimental results can be verified using LTspice simulations in Figs. 3(f, g, h), where the nonlinear capacitors $C_1(V)$ are numerically replaced by purely linear ones of 37 pF in Fig. 3(c), 25 pF in Fig. 3(d), and 15 pF in Fig. 3(e), respectively.

In the second prototype, namely Fig. 4(a), the interface is composed by two semi-ladder circuits that are mirror-reflection of one another. Following the unit cell convention in Fig. 3, the unit cells in Fig. 4 are encircled by the left and right green dashed boxes. This gauge choice is not only compatible with open boundary conditions, but also yields mirror symmetry regarding the interface. Given that the external power is 0.065 V in Fig. 4(c), stronger

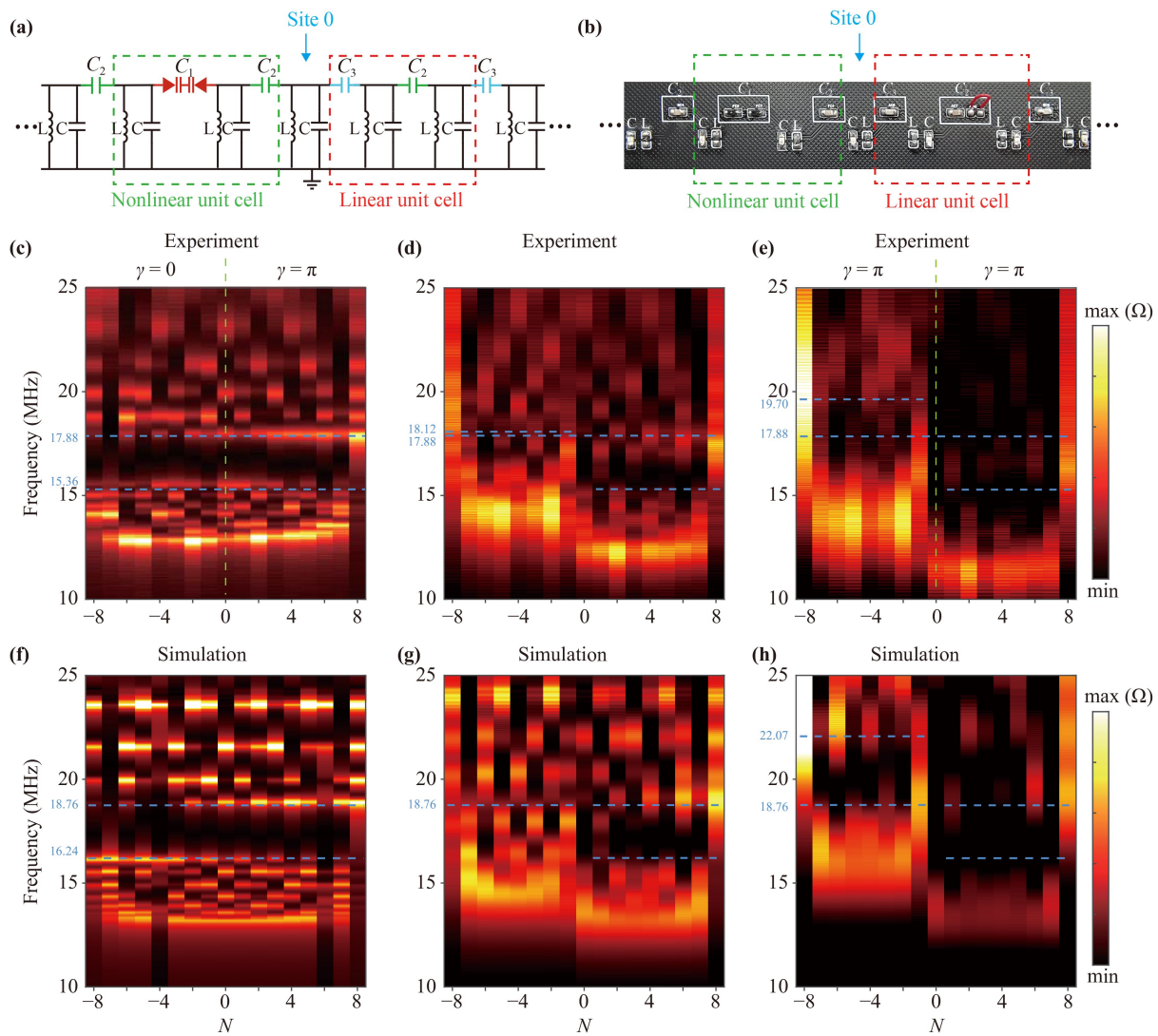


Fig. 3 (a) Schematic illustration of the first prototype of the nonlinear topoelectrical circuit, whose interface connects the left and right half-lattices. Encircled by the red dashed box, the right-sided unit cells are composed of linear elements, as shown in Fig. 1(b). The left-sided semi-lattice, as enclosed by the green dashed box, is composed of unit cells whose intra-cell and inter-cell couplings are switched comparing with Fig. 1(a). (b) Photograph of the experimentally constructed circuit board in the design in (a), whose right and left sides contain 4 unit cells each. (c) The absence of topological interface mode in the small amplitude regime. (d) Topological interface mode on the verge of nonlinear topological phase transition. The mode amplitude reads 3.09 V, which approaches the transition point $A_c = 2.97$ V. (e) Nonlinear topological interface mode becomes clearer for amplitude at 7.84 V. (f–h) Impedance diagram of variable capacitor replaced by linear capacitor 37 pF in (f), 25 pF (g), and 15 pF in (h).

capacitors, $C_1 (V = 0.065 \text{ V}) \approx C_3$, are connected to the interface, whose topological phases of the numbers $(\gamma_{\text{left}}, \gamma_{\text{right}}) = (\pi, \pi)$ are analogous to linear SSH models. We observe the nonlinear topological interface mode, which is also in line with the linear topological interface modes of linear SSH circuits. The topological mode becomes blur in Fig. 4(d) when the external power 1.91 V approaches the critical point $A_c = 2.97$ V, as indicated by the closure of the nonlinear band gap. As the exciting power further grows to 6.17 V in Fig. 4(e), the interface is in the non-topological regime, which manifests a nonlinear localized mode. This mode is not topological

because the frequency can shift into the nonlinear band by tuning the coupling parameters, as indicated by the interface studies of nonlinear topological mechanics [55, 57]. These nonlinear topological physics can be verified by performing numerical simulations in Figs. 4(f–h), where the nonlinear capacitors, $C_1 (V)$, are now replaced by linear capacitors (37 pF, 25 pF, and 15 pF) in the calculations of LTspice. It is worth emphasizing that all these nonlinear topological phases, transitions, and interface modes are flexibly controlled by the external input power without entangling/disentangling the hardware of the platform.

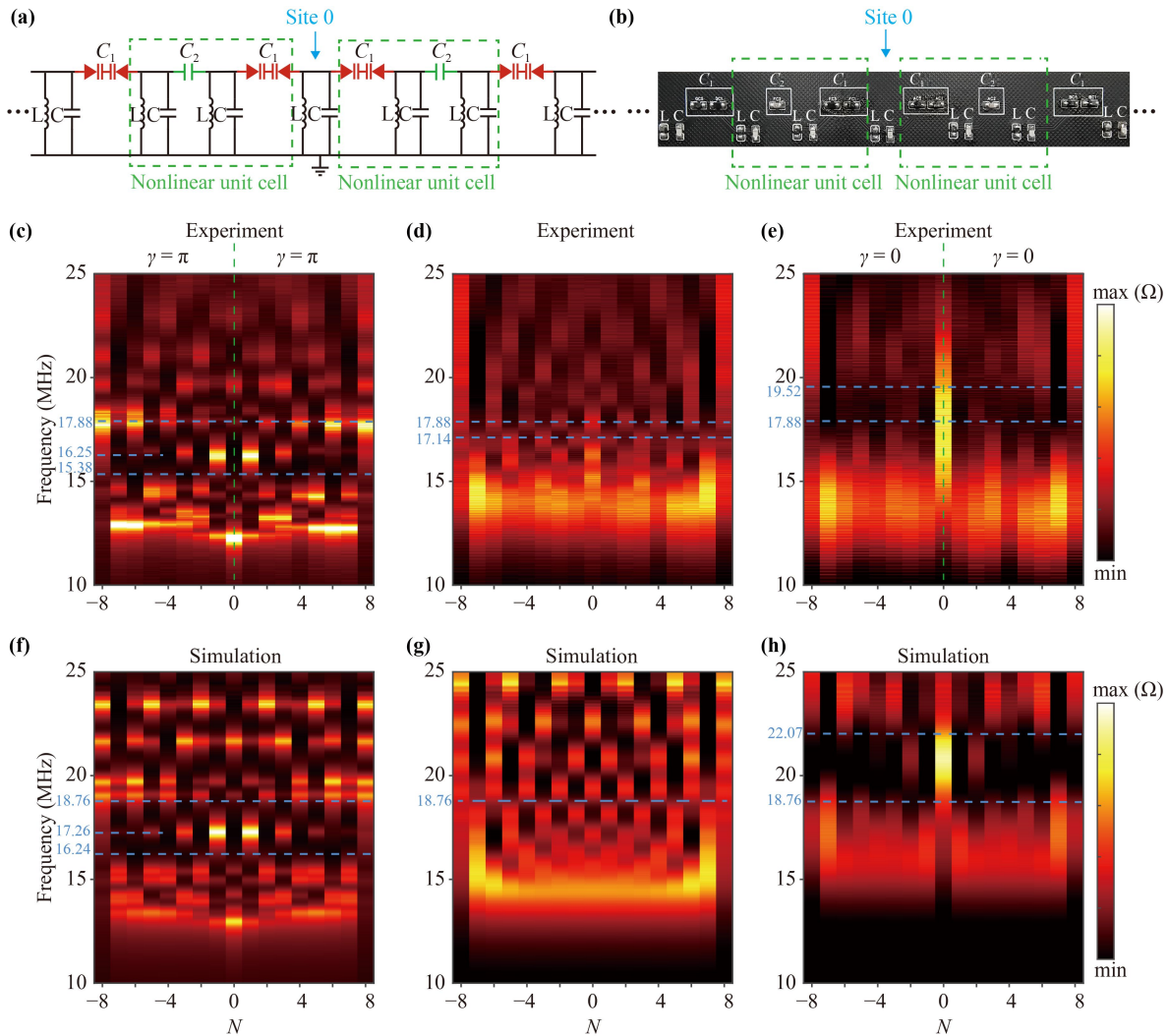


Fig. 4 (a) Schematic illustration of the second prototype of the nonlinear topoelectrical circuits. As encircled by the green dashed boxes, the unit cells of the two semi-lattices are designed from Fig. 1(a). (b) Experimental setup of (a). (c) The interface hosts a topological mode with the excitation power of 0.065 V. (d) Nonlinear topological interface mode on the verge of disappearance, as the external power 1.91 V approaches the topological transition amplitude $A_c = 2.97$ V. (e) For the power that further rises to 6.17 V, trivial localized modes take place on the interface. (f–h) Impedance diagram of variable capacitor replaced by linear capacitor 37 pF in (f), 25 pF (g), and 15 pF in (h).

3 Conclusion

In summary, we construct and experimentally demonstrate nonlinear topological modes on the interface of two cascaded semi-ladder electrical circuits. Nonlinear Berry phase is quantized by reflection symmetry of the underlying circuit structure, and guarantees nonlinear topological interface modes in the non-trivial regime. Amplitude-induced topological transitions are naturally manifested from the conversion between topologically non-trivial and trivial interface modes, whose topological transition voltage amplitudes are in good agreement between experimental measurements and simulations from nonlinear topological band theory. Our prototype establishes flexible metamaterials with amplitude-controlled

rich topological phases and transitions and are readily extendable to higher dimensional platforms.

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Author contributions All authors contributed extensively to the work presented in this paper. Jijie Tang, and Feng Li carried out the experiments. Fangyuan Ma, and Di Zhou provided the theory and calculations. Di Zhou, and Feng Li wrote the paper and Supplemental Material.

Conflict of interest The authors declare that they have no conflict

of interest.

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