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Sharing quantum nonlocality in star network scenarios

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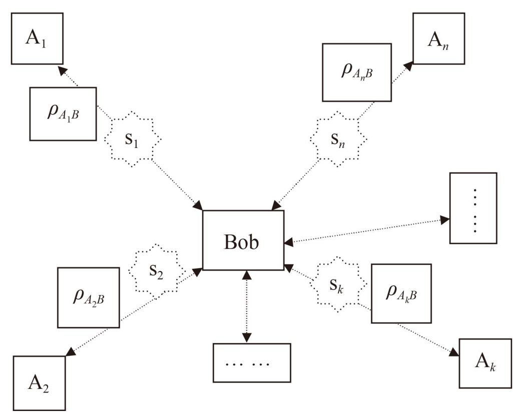
ABSTRACT

The Bell nonlocality is closely related to the foundations of quantum physics and has significant applications to security questions in quantum key distributions. In recent years, the sharing ability of the Bell nonlocality has been extensively studied. The nonlocality of quantum network states is more complex. We first discuss the sharing ability of the simplest bilocality under unilateral or bilateral POVM measurements, and show that the nonlocality sharing ability of network quantum states under unilateral measurements is similar to the Bell nonlocality sharing ability, but different under bilateral measurements. For the star network scenarios, we present for the first time comprehensive results on the nonlocality sharing properties of quantum network states, for which the quantum nonlocality of the network quantum states has a stronger sharing ability than the Bell nonlocality.



Keywords Bell nonlocality, quantum network, nonlocality sharing, POVM measurements

1 Introduction

Raised by Einstein *et al.* [1] in discussions on the incompleteness of quantum mechanics in 1935, quantum Bell nonlocality [2] has been extensively studied and played significant roles in a variety of quantum tasks [3]. Recent studies have shown that quantum nonlocality is also a unique quantum resource for some device-independent quantum tasks such as the key distribution [4], random expansion [5], random amplification [6] and in the related experiments [7–9].

In 2015 the authors in Ref. [10] studied the fundamental limits on the shareability of the Bell nonlocality with many independent observers [11, 12], by asking whether a single pair of entangled qubits could generate a long sequence of nonlocal correlations under sequential measurements on one of the qubits. This phenomenon is known as the sharing ability of quantum nonlocality. The problem is that, given an initial bipartite nonlocally correlated quantum state shared by Alice and Bob, if the nonlocal correlation can be kept under (local) unilateral or bilateral measurements by Alice or/and Bob. A state is said to be of stronger nonlocal sharing ability if the nonlocal correlation can be kept under more local measurements. Since then a series of theoretical [13–24] and experimental [25–27] results on Bell's nonlocality sharing have been obtained. In Ref. [14] with equal sharpness two-outcome measurements the authors show

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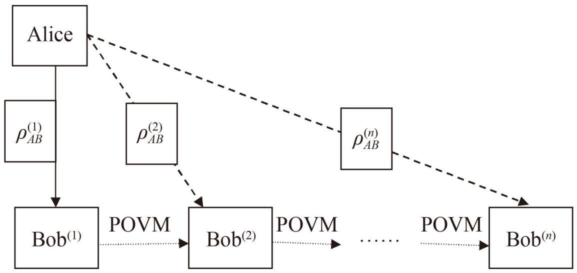

Fig. 1 Shareability of Bell nonlocality under unilateral measurements. Without losing generality, in this schematic diagram, we choose Bob to make a series of measurements, after $n-1$ measurements, we finally observe whether there is quantum nonlocality of $\rho_{AB}^{(n)}$ (the quantum state composed of Bob_n and Alice).

that at most two Bobs can share the Bell nonlocality of a maximally entangled state with a single Alice. It has been shown that at most two Bobs can share the nonlocality with a single Alice by using the local realist inequalities with three and four dichotomic measurements per observer [15]. More recently, by an elegant measurement strategy, the authors in Ref. [21] show that, contrary to the previous expectations [10, 13], there is no limit on the number of independent Bobs to have an expected violation of the CHSH inequality with one Alice. A class of initial two-qubit states, including all pure two-qubit entangled states, has been presented which are capable of achieving CHSH inequality violations under unlimited local measurements. This fact has recently been illustrated for the case of higher dimensional bipartite pure states [22]. However, most of the studies mentioned above are focused on the case of one-sided (unilateral) sequential measurements (See Fig. 1). Recently, Cheng et al. [28, 29] explored the Bell nonlocality sharing in bilateral sequential measurements (see Fig. 2), in which a pair of entangled states is distributed to multiple Alices and Bobs. It is shown that when the observers A_1 and B_1 each select their Positive-Operator-Valued-Measure (POVM) with equal probabilities, the nonlocality sharing between Alice_1–Bob_1 and Alice_2–Bob_2 is impossible.

With the development of quantum technology, quantum systems of medium scale have attracted much attention [30–34], such as quantum computation with noisy intermediate-scale quantum processors [35, 36]. The

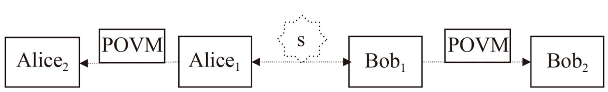

Fig. 2 Shareability of Bell nonlocality under bilateral measurement. Alice_1 and Bob_1 share an initial quantum state with Bell nonlocality via a quantum resource distributor S. They both perform measurements on their own side and send the particles to Alice_2 and Bob_2, respectively. One identifies the nonlocality sharing between Alice_2 and Bob_2 and so on.

researches on such quantum systems have spawned another kind of nonlocality that may be stronger than the Bell nonlocality – the quantum network nonlocality [37–48]. In Ref. [44] it was proved that any connected network consisting of entangled pure states can exhibit genuine many-body quantum Bell nonlocality. In Ref. [40] the authors presented an inequality to certify the nonlocality of a star-shaped quantum network. In Ref. [46] the problem of nonlocal correlation in tree tensor networks has been studied in detail. It was shown in Ref. [48] that in a large class of networks with no inputs, suitably chosen quantum color matching strategies can lead to non-local correlations that cannot be produced in classical ways.

Recently, the nonlocality sharing problem has been mainly studied for the Bell nonlocality. It would be also of significance to investigate the nonlocality sharing for quantum networks. In Ref. [49] the authors investigated network nonlocality sharing in the extended bilocal scenario via bilateral weak measurements. Interestingly, when the both states ρ_{AB} and ρ_{BC} are two-qubit maximally entangled pure state, $|\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, by bilateral weak measurements the network nonlocality sharing can be revealed from the multiple violation of the Branciard–Rosset–Gisin–Pironio (BRGP) inequalities [39] of any Alice_2–Bob–$\mathrm{Charlie}_2$, which has no counterpart in the case of Bell nonlocality sharing scenario.

In this paper, first we study the nonlocality sharing ability of bilocality quantum networks under unilateral and bilateral average measurements. Moreover, based on bilocality methodologies, we investigate comprehensively the problem of nonlocality sharing ability of star quantum networks under unilateral and bilateral average measurements. Our results incorporate the results of Ref. [49] and greatly generalize the range of quantum network states.

2 One basic fact — nonlocal of bilocality scenario

We first recall the simplest quantum network — bilocality scenario (see Fig. 3) which is given by three observers Alice, Bob and Charlie, two sources S_1 and S_2, each source sends a bipaitite quantum state [37, 50]. Consider that Alice receives measurement setting (or input) x, while Bob gets input y, and Charlie z. Upon receiving

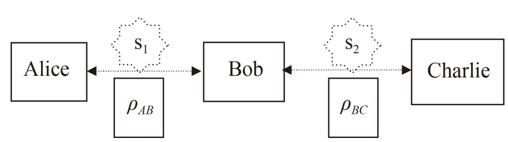

Fig. 3 The bilocality scenario. Resource S_1 distributes entangled state ρ_{AB} to Alice and Bob; resource S_2 distributes entangled state ρ_{BC} to Bob and Charlie.

their inputs, each party provides a measurement result (an output), denoted by a for Alice, b for Bob and c for Charlie. In this context, the observed statistics is said to be 2-local when

$$p(a,b,c|x,y,z) = \int \mathrm{d}\lambda \mathrm{d}\mu q_1(\lambda)q_2(\mu)p(a|x,\lambda)p(b|y,\lambda,\mu)p(c|z,\mu), \quad (1)$$

where λ and μ are hidden variables related to sources S_1 and S_2, independent shared random variables distributed according to the densities $q_1(\lambda)$ and $q_2(\mu)$, respectively.

The set of 2-local correlations is non-convex. In order to efficiently characterize the 2-local set, non-linear Bell inequalities are required. In Refs. [37, 39], first kind non-linear inequalities that allow one to efficiently capture 2-local correlations were derived. They are better than linear inequalities. Consider that each party measures two possible dichotomic observables $(a,b,c,x,y,z=0,1)$. It follows that any bilocal hidden variable (BLHV) model described by Eq. (1) must fulfill the bilocality inequality:

$$S_{biloc} \equiv \sqrt{|I|} + \sqrt{|J|} \le 2, \quad (2)$$

where $I \equiv \langle (A_0+A_1)B_0(C_0+C_1)\rangle$, $J \equiv \langle (A_0-A_1)B_1(C_0-C_1)\rangle$, $\langle A_xB_yC_z\rangle = \sum_{a,b,c=0,1}(-1)^{a+b+c}p(a,b,c|x,y,z)$, $\langle O\rangle = Tr(O\rho)$ denotes the mean value of the observable O with respect to the measured state ρ. Here, I and J are the expected values given by the union observables A_0 and A_1 (B_0 and B_1, C_0 and C_1) associated with Alice (Bob, Charlie) which are all Hermitian operators with spectrum in $[-1,1]$. The violation of this inequality implies the network nonlocality of the state.

In Refs. [50, 51], the authors considered the following two-qubit state shared by Alice and Bob,

$$\rho_{AB} = \frac{1}{4}\left(I_4 + \boldsymbol{r\sigma}\otimes I_2 + \boldsymbol{s}I_2\otimes\boldsymbol{\sigma} + \sum_{i,j} t_{ij}^{AB}\sigma_i\otimes\sigma_j\right),$$

where $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are the standard Pauli matrices. Here the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ represent the Bloch vectors of Alice's and Bob's reduced states, respectively. While $t_{i,j}^{AB}$, $i,j \in x,y,z$, are the entries of the correlation matrix t^{AB} [52], I_m stands for the $m\times m$ identity matrix. The state ρ_{BC} shared by Bob and Charlie can be expressed in a similar way. Then the maximal value of S_{biloc} is shown to be

$$S_{biloc}^{max} = 2\sqrt{\delta_1\eta_1 + \delta_2\eta_2}, \quad (3)$$

where δ_1 and δ_2 ($\delta_1 \ge \delta_2$) are the two largest eigenvalues of the matrix $R^{AB} = \sqrt{t^{AB\dagger}t^{AB}}$. Similarly, η_1 and η_2 ($\eta_1 \ge \eta_2$) are the two largest eigenvalues of the matrix $R^{BC} = \sqrt{t^{BC\dagger}t^{BC}}$. Moreover, according to the Horodecki criterion [53], the maximal CHSH value for ρ_{AB} is given by $S_{AB}^{max} = 2\sqrt{\delta_1^2+\delta_2^2} = 2\|\boldsymbol{\delta}\|$, where $\boldsymbol{\delta} = (\delta_1,\delta_2)^{\mathrm{T}}$ with T denoting the transpose. Similarly, for ρ_{BC} one has $S_{BC}^{max} = 2\sqrt{\eta_1^2+\eta_2^2} = 2\|\boldsymbol{\eta}\|$. Then it follows from Eq. (3) that

$$S_{biloc}^{max} = 2\sqrt{\boldsymbol{\delta}\cdot\boldsymbol{\eta}} \le 2\sqrt{\|\boldsymbol{\delta}\|\|\boldsymbol{\eta}\|} \le \sqrt{S_{AB}^{max}S_{BC}^{max}}. \quad (4)$$

In the following, we will use (2), (3) and (4) to judge whether a quantum state in bilocality scenario is still nonlocal after unilateral or bilateral measurements.

3 Nonlocality sharing under unilateral measurement in bilocality scenario

We first introduce the nonlocality sharing under unilateral measurement. To begin with, Alice (Alice$_1$) shares an arbitrary entangled bipartite state ρ_{AB} ($\rho_{AB}^{(1)}$) with Bob. Alice proceeds by choosing a uniformly random input, performing the corresponding measurement and recording the outcome. Denote the binary input and output of Alice$_k$ (Bob) by $x^{(k)}$ (Y) and A_x (B), respectively. Suppose Alice$_1$ performs the measurement according to $x^{(1)} = x$ with the outcome $A_{(1)} = a$. With equal probabilities over the inputs and outputs of Alice$_1$, the post-measurement unnormalized state shared between Alice$_2$ and Bob is given by

$$\rho_{AB}^{(2)} = \frac{1}{2}\Sigma_{a,x}\left(\sqrt{A_{a|x}^{(1)}}\otimes I_2\right)\rho_{AB}^{(1)}\left(\sqrt{A_{a|x}^{(1)}}\otimes I_2\right),$$

where $A_{a|x}$ is the positive operator-valued measure (POVM) with respect to the outcome a of Alice$_{(1)}$'s measurement for input x, I_2 is the 2×2 identity matrix. Repeating this process, one gets the state $\rho_{AB}^{(k)}$ shared between Alice$_{(k)}$ and Bob for $k=1,2,\cdots,n$. This process is called unilateral measurement, see the schematic diagram shown in Fig. 4. Our main goal is to judge whether the quantum network state composed of $\rho_{AB}^{(n)}$

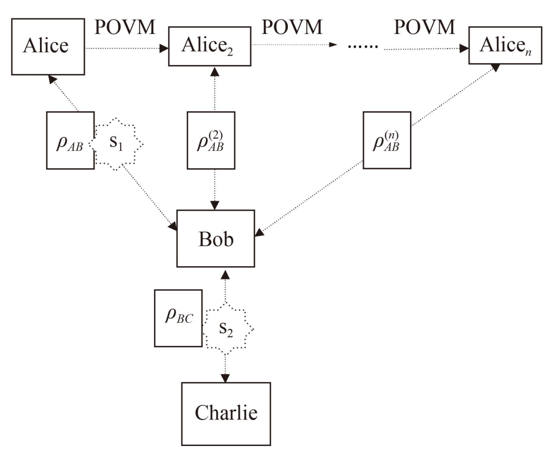

Fig. 4 Nonlocality sharing under unilateral measurement in bilocality scenario. Here, we choose Alice's side for a series of measurements. In this network structure and selective measurements, the initial state is $\rho_{AB}\otimes\rho_{BC}$ and the final state is $\rho_{AB}^{(n)}\otimes\rho_{BC}$.

and ρ_{BC} has network nonlocality.

We employ the POVMs with measurement operators $\{E, I-E\}$, where $E=\frac{1}{2}(I_2+\gamma\sigma_{\vec{r}})$, $\vec{r}\in R^3$ with $\|\boldsymbol{r}\|=1$, $\sigma_{\vec{r}}=r_1\sigma_x+r_2\sigma_y+r_3\sigma_z$, $\gamma\in[0,1]$ is the sharpness of the measurement. For each $k=1,2,\cdots,n$, Alice$_k$'s POVMs are given by

$$A_{0|0}=\frac{1}{2}[I_2+\cos\theta\sigma_z], \tag{5}$$

$$A_{0|1}^k=\frac{1}{2}[I_2+\gamma_k\sin\theta\sigma_x] \tag{6}$$

for some $\theta\in(0,\frac{\pi}{4}]$, $k=1,2,\cdots,n$. Bob's POVMs are given by

$$B_{0|0}=\frac{1}{2}[I_2+(\cos\theta\sigma_z+\sin\theta\sigma_x)], \tag{7}$$

$$B_{0|1}=\frac{1}{2}[I_2+(\cos\theta\sigma_z-\sin\theta\sigma_x)]. \tag{8}$$

After Alice's side makes a finite number of sequential POVMs measurements, we get the state $\rho_{AB}^{(n)}$ and the following conclusion.

Theorem 1. *If ρ_{AB} is an arbitrary entangled two-qubit pure state and ρ_{BC} is the maximally entangled state $|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, then for each $n\in\mathbb{N}$, the quantum network state composed of $\rho_{AB}^{(n)}$ and ρ_{BC} has network nonlocality, where $\rho_{AB}^{(n)}$ stands for the state between Alice$_n$ (after $n-1$ consecutive measurement by Alice) and Bob.*

Proof. In order to prove that the quantum network state composed of $\rho_{AB}^{(n)}$ and ρ_{BC} has network nonlocality, it is only necessary to prove that they violate the bilocality inequality Eq. (2). With respect to the Alice$_k$'s POVMs, let us define the expectation operators $A_x^k=A_{0|x}^k-A_{1|x}^k$ and $B_y=B_{0|y}-B_{1|y}$ for $x,y=0,1$. Following the idea of the proof in Ref. [21], it is easy to infer that there is a strong recursive relationship between $\rho_{AB}^{(n)}$ and ρ_{AB}. The CHSH value associated with state shared between Alice$^{(n)}$ and Bob can be similarly written as

$$I_{CHSH}^n = 2^{2-n}\left(\gamma_n\sqrt{\delta_2}\sin\theta+\sqrt{\delta_1}\cos\theta\prod_{j=1}^{n-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right).$$

For any pure state, its corresponding correlation matrix in Bloch representation has the maximum eigenvalue $\delta_1=1$. According to the conclusion of Ref. [21], the Bell nonlocality can be shared under unilateral measurements. This means that there are γ_k and θ, which makes $\rho_{AB}^{(n)}$ violate the CHSH inequality. Hence, according to the Horodecki criterion [53], $S_{AB}^{(n)max}=2\sqrt{(\delta_1^{(n)})^2+(\delta_2^{(n)})^2}>2$.

On the other hand, the corresponding η_1 and η_2 for the maximally entangled pure state ρ_{BC} are both 1 [52]. Therefore, Eq. (3) can be written as $S_{biloc}^{(n)max}=2\sqrt{\delta_1^{(n)}+\delta_2^{(n)}}$. Since $|\delta_1^{(n)}|\le 1$ and $|\delta_2^{(n)}|\le 1$, we have $\sqrt{\delta_1^{(n)}+\delta_2^{(n)}}\ge\sqrt{(\delta_1^{(n)})^2+(\delta_2^{(n)})^2}$. That is, $S_{biloc}^{(n)max}>2$, which completes the proof. □

We note that for an entangled two-qubit mixed state ρ_{AB} with $\delta_1=1$ and $\delta_2>0$, according to the conclusion of Ref. [21], the Bell nonlocality of ρ_{AB} also can be shared under unilateral measurements. Therefore, similar to the proof of Theorem 1, we get

Theorem 2. *If ρ_{AB} is an entangled two-qubit mixed state with $\delta_1=1$ and $\delta_2>0$, and ρ_{BC} is the maximally entangled state $|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, then for each $n\in\mathbb{N}$, the quantum network state composed of $\rho_{AB}^{(n)}$ and ρ_{BC} has network nonlocality, where $\rho_{AB}^{(n)}$ stands for the state between Alice$_n$ (after $n-1$ consecutive measurement by Alice) and Bob.*

4 Nonlocal sharing under bilateral measurement of bilocality scenario

Furthermore, we also allow Charlie to be able to measure his party. To begin with, Bob and Charlie (Charlie$_1$) share an arbitrary entangled bipartite state ρ_{BC} ($\rho_{BC}^{(1)}$). Charlie proceeds by choosing a uniformly random input, performing the corresponding measurement and recording the outcome. Denote the binary input and output of Charlie$_l$ (Bob) by $z^{(l)}$ (Y) and C_z (B), respectively. Suppose Charlie$_1$ performs the measurement according to $z^{(1)}=z$ with the outcome $C_{(1)}=c$. Averaged over the inputs and outputs of Charlie$_1$, the post-measurement unnormalized state shared between Bob and Charlie$_2$ is given by

$$\rho_{BC}^{(2)}=\frac{1}{2}\Sigma_{c,z}\left(\sqrt{I_2\otimes C_{c|z}^{(1)}}\right)\rho_{BC}^{(1)}\left(I_2\otimes\sqrt{C_{c|z}^{(1)}}\right),$$

where $C_{c|z}$ is of the form of $A_{a|x}$. Repeating this process, one gets the state $\rho_{BC}^{(l)}$ shared between Bob and Charlie$^{(l)}$.

Let

$$C_{0|0}=\frac{1}{2}(I_2+\cos\theta\sigma_z), \tag{9}$$

$$C_{0|1}^l=\frac{1}{2}(I_2+\gamma_l\sin\theta\sigma_x) \tag{10}$$

for some $\theta\in(0,\frac{\pi}{4}]$, $l=1,2,\cdots,m$. Defining the expectation operators $C_z=C_{0|z}-C_{1|z}$ and $B_y=B_{0|y}-B_{1|y}$ for $z,y=0,1$, we have the CHSH value of the state shared between Bob and Charlie$^{(l)}$,

$$I_{CHSH}^{l} = 2^{2-l}\left[\gamma_l\sqrt{\eta_2}\sin\theta+\sqrt{\eta_1}\cos\theta\prod_{j=1}^{l-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right].$$

Concerning the question whether the quantum network state composed $\rho_{AB}^{(n)}$ and $\rho_{BC}^{(m)}$ (see Fig. 5) has quantum network nonlocality, we have the following conclusion.

Theorem 3. *If $\rho_{AB}=\rho_{BC}$ is an entangled two-qubit pure state, then for each $n\in\mathbb{N}$, the quantum network state composed of $\rho_{AB}^{(n)}$ and $\rho_{BC}^{(n)}$ has network nonlocality, where $\rho_{AB}^{(n)}$ stands for the state between Alice$_n$ (after $n-1$ consecutive measurement by Alice) and Bob, $\rho_{BC}^{(n)}$ stands for the state between Charlie$_n$ (after $n-1$ consecutive measurement by Charlie) and Bob.*

Proof. To prove the theorem one needs to prove that the bilocality inequality Eq. (2) is violated. Since the Bell nonlocality can be shared for the quantum state ρ_{AB} under unilateral measurements, $\rho_{AB}^{(n)}$ would violate the CHSH inequality. According to the Horodecki criterion [53], $S_{AB}^{(n)max}=2\sqrt{(\delta_1^{(n)})^2+(\delta_2^{(n)})^2}>2$.

On the other hand, similar to ρ_{AB}, for the state ρ_{BC} we also have $S_{BC}^{(m)max}=2\sqrt{(\delta_1^{(m)})^2+(\delta_2^{(m)})^2}>2$ for any $m\in\mathbb{N}$. Therefore, Eq. (3) now can be written as $S_{biloc}^{(n,m)max}=2\sqrt{\delta_1^{(n)}\delta_1^{(m)}+\delta_2^{(n)}\delta_1^{(m)}}$.

When $n=m$, $S_{biloc}^{(n,n)max}=2\sqrt{(\delta_1^{(n)})^2+(\delta_2^{(n)})^2}>2$. That is, $S_{biloc}^{(n,n)max}>2$, which completes the proof. □

Remark 1: When $n=m$, we can easily get that the quantum network nonlocality is sharable. This phenomenon does not exist in Bell's nonlocality sharing [28]. If $n\neq m$, the inequality $S_{biloc}^{(n,m)max}>2$ implies that the quantity S_{biloc}^{max} in (3) is greater than 2. It should be noted that generally the equality cannot be attained in the inequality (4). Therefore, it is difficult to judge whether Alice$_n$–Bob–Charlie$_m$ can share network nonlocality from the violation of the inequality (2).

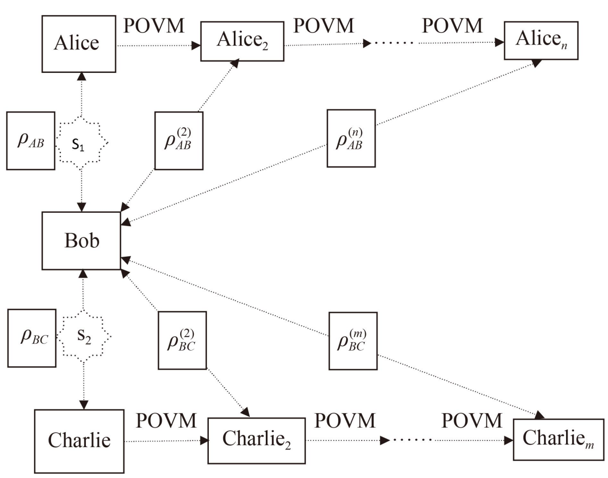

Fig. 5 Nonlocal sharing under bilateral measurement of bilocality Scenario. Here, we choose Alice and Charlie for a series of measurements. In this case of network topology and selective measurements, the initial state is $\rho_{AB}\otimes\rho_{BC}$ and the final state is $\rho_{AB}^{(n)}\otimes\rho_{BC}^{(m)}$.

Remark 2: In Ref. [49] the sources S_1 and S_2 send pairs of particles in the maximally entangled state, $\rho_{AB}=\rho_{BC}=|\psi\rangle\langle\psi|$. Then the network nonlocality sharing between Alice$_1$–Bob–Charlie$_1$ and Alice$_2$–Bob–Charlie$_2$ can be observed. Here, by incorporating this conclusion into Theorem 3, we see that the state does not need to be maximally entangled, and the network nonlocality sharing between Alice$_n$–Bob–Charlie$_n$ can be obtained for any $n\in\mathbb{N}$.

Similarly, we can easily come to the following conclusion.

Theorem 4. *If $\rho_{AB}=\rho_{BC}$ is any entangled two-qubit mixed state with $\delta_1=1$ and $\delta_2>0$, then for each $n\in\mathbb{N}$ the quantum network state composed of $\rho_{AB}^{(n)}$ and $\rho_{BC}^{(n)}$ has network nonlocality.*

5 Nonlocality sharing in the star network scenario

The n-partite star network [40] is the natural extension of the bilocality scenario, which is composed of n sources sharing a quantum state between one of the n nodes $A_1,A_2,\cdots,A_n$ and a central node Bob (see Fig. 6). The bilocality scenario corresponds to the particular case of $n=2$.

The classical description of the correlations in this scenario is characterized by the probability decomposition,

$$p(\{a_i\}_{i=1}^n,b|\{x_i\}_{i=1}^n,y) = \int\left(\prod_{i=1}^n \mathrm{d}\lambda_i p(\lambda_i)p(a_i|x_i,\lambda_i)\right)p\left(b|y,\{\lambda_i\}_{i=1}^n\right). \quad (11)$$

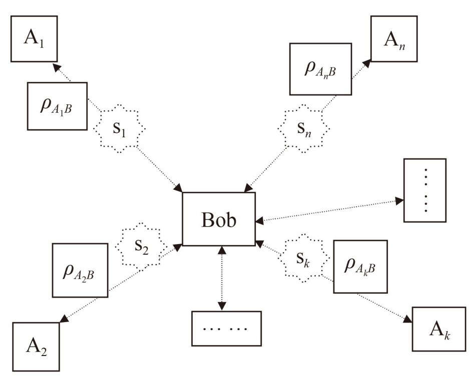

Fig. 6 The star network scenario. Here, we choose Bob as the intermediate node, and the source S_i distributes the quantum state ρ_{A_iB} to A_i and Bob, $i=1,2,\cdots,n$.

As shown in Ref. [19] the following n-locality inequality holds,

$$\mathcal{N}_{star} = |I|^{1/n} + |J|^{1/n} \le 1, \tag{12}$$

where

$$I = \frac{1}{2^n} \sum_{x_1,\cdots,x_n} \langle A^1_{x_1} A^2_{x_2} \cdots A^n_{x_n} B_0 \rangle,$$

$$J = \frac{1}{2^n} \sum_{x_1,\cdots,x_n} (-1)^{\sum_i x_i} \langle A^1_{x_1} A^2_{x_2} \cdots A^n_{x_n} B_1 \rangle,$$

$$\langle A^1_{x_1} A^2_{x_2} \cdots A^n_{x_n} B_y \rangle = \sum_{a_1,\cdots,a_n,b} (-1)^{b+\sum_i a_i} p(\{a_i\}^n_{i=1}, b|\{x_i\}^n_{i=1}, y).$$

According to Ref. [51], with respect to the generic quantum state $\rho_{A_1B} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$, the maximal value of $\mathcal{N}_{star}$ is given by

$$\mathcal{N}_{star} = \sqrt{\left(\prod_{i=1}^{n} t_1^{A_i}\right)^{1/n} + \left(\prod_{i=1}^{n} t_2^{A_i}\right)^{1/n}}, \tag{13}$$

where $t_1^{A_i}$ and $t_2^{A_i}$ are the two largest (positive) eigenvalues of the matrix $t^{A_iB\dagger} t^{A_iB}$ with $t_1^{A_i} \ge t_2^{A_i}$.

6 Nonlocality sharing under unilateral measurements in star network quantum states

The schematic diagram of this situation is shown in Fig. 7, where Alice makes sequential POVM measurements as described in Section A in the bilocality scenario, and one gets the sequential quantum states $\rho_{AB}^{(k)}$, $k = 1, 2, \cdots, m$. We need to verify the network nonlocality of the quantum network state $\rho \equiv \rho_{AB}^{(m)} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$.

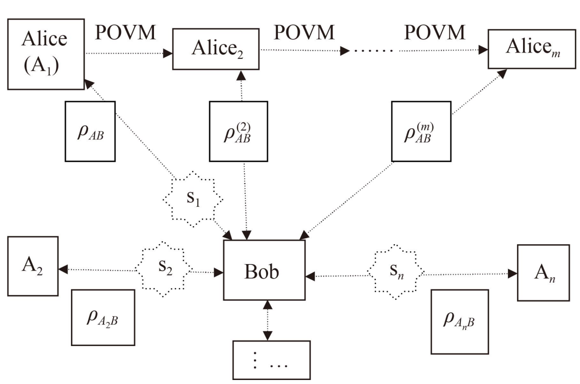

Fig. 7 Nonlocal sharing under unilateral measurements of star network quantum states. Here, we choose Alice's side for a series of measurements. In this network structure and selective measurements, the initial state is $\rho_{AB} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$ and the final state is $\rho_{AB}^{(m)} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$.

Theorem 5. *If ρ_{AB} is an entangled pure state or an entangled mixed state with $\delta_1 = 1$ and $\delta_2 > 0$, and $\rho_{A_2B} = \cdots = \rho_{A_nB} = |\psi\rangle\langle\psi|$, then for each $n, m \in \mathbb{N}$ the quantum network state $\rho \equiv \rho_{AB}^{(m)} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$ has network nonlocality.*

Proof. If ρ_{AB} is an entangled pure state or an entangled mixed state with $\delta_1 = 1$ and $\delta_2 > 0$, by the bi-locality case we have that $\rho_{AB}^{(m)}$ violates the CHSH inequality. Then according to the Horodecki criterion, we have $S_{AB}^{(m)max} = 2\sqrt{(\delta_1^{(m)})^2 + (\delta_2^{(m)})^2} > 2$. On the other hand, $\rho_{A_2B}, \cdots, \rho_{A_nB}$ are all maximally entangled pure states. Therefore, for the quantum network state $\rho = \rho_{AB}^{(m)} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$, Eq. (13) can be expressed as $\mathcal{N}_{star} = \sqrt{(\delta_1^{(m)})^2 + (\delta_2^{(m)})^2} = \frac{S_{AB}^{(m)max}}{2} > 1$. Moreover, the inequality (12) is violated, and the quantum network state ρ has network nonlocality. □

7 Nonlocality sharing under multilateral measurements in star network quantum states

A schematic diagram of this situation is shown in Fig. 8. Alice and Charlie make sequential POVM measurements as described in Section B in bilocality scenario. We get the sequential quantum states $\rho_{A_1B}^{(k)}$ and $\rho_{A_2B}^{(l)}$, $k = 1, 2, \cdots, m$, $l = 1, 2, \cdots, t$ and so on. Our problem is to identify the network nonlocality of the quantum network state

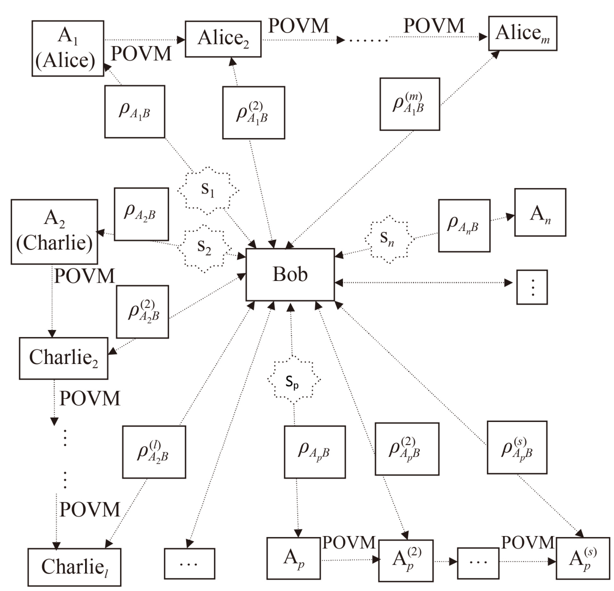

Fig. 8 Nonlocality sharing under multilateral measurements of star network quantum states. Here, we choose $A_1(Alice), A_2(Charlie), \cdots, A_p$ for a series of measurements. In this network structure and selective measurements, the initial state is $\rho_{A_1B} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_nB}$ and the final state is $\rho_{A_1B}^{(m)} \otimes \rho_{A_2B}^{(l)} \otimes \cdots \otimes \rho_{A_pB}^{(s)} \otimes \cdots \otimes \rho_{A_nB}$.

$$\rho \equiv \rho_{A_1B}^{(m)} \otimes \rho_{A_2B}^{(l)} \otimes \cdots \otimes \rho_{A_pB}^{(s)} \otimes \rho_{A_{p+1}B} \otimes \cdots \otimes \rho_{A_nB}.$$

From the analysis on the bilocality situation above and the inequality (12) and (13) of the nonlocality for the star quantum network, it is not difficult to obtain the following conclusion.

Theorem 6. *If $\rho_{A_1B} = \rho_{A_2B} = \cdots = \rho_{A_pB}$ is an entangled pure state or an entangled mixed state with $\delta_1 = 1$ and $\delta_2 > 0$, for $p = 2, 3, \cdots, n$, and $\rho_{A_{p+1}B} = \cdots = \rho_{A_nB} = |\psi\rangle\langle\psi|$, then for each $m, l, \cdots, s \in \mathbb{N}$ the quantum network state $\rho \equiv \rho_{A_1B}^{(m)} \otimes \rho_{A_2B}^{(l)} \otimes \cdots \otimes \rho_{A_pB}^{(s)} \otimes \rho_{A_{p+1}B} \otimes \cdots \otimes \rho_{A_nB}$ has network nonlocality.*

8 Conclusions and discussion

Quantum nonlocality is a distinctive feature of quantum mechanics. The nonlocal characteristics of quantum networks are more complicated due to their non-convexity, nonlinearity, etc. The research on the nonlocality sharing ability of quantum networks has important theoretical significance for developments of such as quantum repeaters. Since nonlocality is an important quantum resource, starting from a nonlocally correlated quantum state, it would be quite desirable to be able to maintain the nonlocality under sequential measurements. In the bilocality case, we have shown that under unilateral measurements, the nonlocality can be shared under any times of sequential measurements. With the same measurements and the same times of measurements on both sides of Alices and Charlies, it is possible for arbitrarily many Alices and Charlies to share the locality with a single Bob by using a pure or mixed entangled state. We have also investigated the shareability of star network nonlocality. It has been shown that from our measurement schemes the nonlocality of the star network quantum states can be shared under unilateral or multilateral measurements.

Our results may also highlight researches on sharing general multipartite quantum nonlocalities [54] and other quantum correlations such as quantum steerability [55], entanglement [56, 57] and coherence [58, 59]. Our approach may also suggest the related applications in randomness generation [60], quantum teleportation [61], random access codes [62], quantum key distribution [63–65], quantum digital signatures [66] and quantum communication [67]. It would be also interesting to explore the nonlocality sharing ability of high-dimensional [22, 23] or multipartite quantum network states in other network scenarios. Moreover, we have constructed the dichotomic POVM measurement operators in terms of the Pauli operators. As the Pauli operators are easily implemented in experiments, the POVM operators we constructed may have potential advantages in some specific experimental implementations [25–27].

Note added While completing this manuscript, we became aware of Refs. [68–70] studied the same topic and got similar results. However, we all used different methods. Compared with our strict theoretical proofs, Ref. [68] used symmetric and anti-symmetric methods, Ref. [69] mainly used weak measurement theory, Ref. [70] focused on experimental simulation.

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References

1. A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete, *Phys. Rev.* 47(10), 777 (1935)
2. H. Cao and Z. Guo, Characterizing Bell nonlocality and EPR steering, *Sci. China Phys. Mech. Astron.* 62(3), 30311 (2019)
3. N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* 86(2), 419 (2014)
4. A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, *Phys. Rev. Lett.* 98(23), 230501 (2007)
5. S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, *Nature* 464(7291), 1021 (2010)
6. R. Colbeck and R. Renner, Free randomness can be amplified, *Nat. Phys.* 8(6), 450 (2012)
7. M. H. Li, X. Zhang, W. Z. Liu, S. R. Zhao, B. Bai, Y. Liu, Q. Zhao, Y. Peng, J. Zhang, Y. Zhang, W. J. Munro, X. Ma, Q. Zhang, J. Fan, and J. W. Pan, Experimental realization of device-independent quantum randomness expansion, *Phys. Rev. Lett.* 126(5), 050503 (2021)
8. W. Z. Liu, M. H. Li, S. Ragy, S. R. Zhao, B. Bai, Y. Liu, P. J. Brown, J. Zhang, R. Colbeck, J. Fan, Q. Zhang, and J. W. Pan, Device-independent randomness expansion against quantum side information, *Nat. Phys.* 17(4), 448 (2021)
9. L. K. Shalm, Y. Zhang, J. C. Bienfang, C. Schlager, M. J. Stevens, M. D. Mazurek, C. Abellán, W. Amaya, M. W. Mitchell, M. A. Alhejji, H. Fu, J. Ornstein, R. P. Mirin, S. W. Nam, and E. Knill, Device-independent randomness expansion with entangled photons, *Nat. Phys.* 17(4), 452 (2021)
10. R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Multiple observers can share the nonlocality of half of

an entangled pair by using optimal weak measurements, *Phys. Rev. Lett.* 114(25), 250401 (2015)
11. J. S. Bell, On the Einstein–Podolsky–Rosen paradox, *Physics* 1(3), 195 (1964)
12. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, *Phys. Rev. Lett.* 23(15), 880 (1969)
13. S. Mal, A. Majumdar, and D. Home, Sharing of nonlocality of a single member of an entangled pair of qubits is not possible by more than two unbiased observers on the other wing, *Mathematics* 4(3), 48 (2016)
14. A. Shenoy H., S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Unbounded sequence of observers exhibiting Einstein–Podolsky–Rosen steering, *Phys. Rev. A* 99, 022317 (2019)
15. D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Facets of bipartite nonlocality sharing by multiple observers via sequential measurements, *Phys. Rev. A* 99(2), 022305 (2019)
16. S. Datta and A. S. Majumdar, Sharing of nonlocal advantage of quantum coherence by sequential observers, *Phys. Rev. A* 98(4), 042311 (2018)
17. C. Ren, T. Feng, D. Yao, H. Shi, J. Chen, and X. Zhou, Passive and active nonlocality sharing for a two-qubit system via weak measurements, *Phys. Rev. A* 100(5), 052121 (2019)
18. A. Kumari and A. K. Pan, Sharing nonlocality and nontrivial preparation contextuality using the same family of Bell expressions, *Phys. Rev. A* 100(6), 062130 (2019)
19. S. Saha, D. Das, S. Sasmal, D. Sarkar, K. Mukherjee, A. Roy, and S. S. Bhattacharya, Sharing of tripartite nonlocality by multiple observers measuring sequentially at one side, *Quantum Inform. Process.* 18(2), 42 (2019)
20. K. Mohan, A. Tavakoli and N. Brunner, Sequential random access codes and self-testing of quantum instruments, *New J. Phys.* 21, 083034 (2019)
21. P. J. Brown, and R. Colbeck, Arbitrarily many independent observers can share the nonlocality of a single maximally entangled qubit pair, *Phys. Rev. Lett.* 125(9), 090401 (2020)
22. T. Zhang and S. M. Fei, Sharing quantum nonlocality and genuine nonlocality with independent observables, *Phys. Rev. A* 103(3), 032216 (2021)
23. T. Zhang, Q. Luo, and X. Huang, Quantum Bell nonlocality cannot be shared under a special kind of bilateral measurements for high-dimensional quantum states, *Quantum Inform. Process.* 21(10), 350 (2022)
24. S. Mukherjee and A. K. Pan, Semi-device-independent certification of multiple unsharpness parameters through sequential measurements, *Phys. Rev. A* 104(6), 062214 (2021)
25. M. J. Hu, Z. Y. Zhou, X. M. Hu, C. F. Li, G. C. Guo, and Y. S. Zhang, Observation of non-locality sharing among three observers with one entangled pair via optimal weak measurement, *npj Quantum Inf.* 4(1), 63 (2018)
26. M. Schiavon, L. Calderaro, M. Pittaluga, G. Vallone, and P. Villoresi, Three-observer Bell inequality violation on a two-qubit entangled state, *Quantum Sci. Technol.* 2(1), 015010 (2017)
27. T. Feng, C. Ren, Y. Tian, M. Luo, H. Shi, J. Chen, and X. Zhou, Observation of nonlocality sharing via not-so-weak measurements, *Phys. Rev. A* 102(3), 032220 (2020)
28. S. Cheng, L. Liu, T. J. Baker, and M. J. W. Hall, Limitations on sharing Bell nonlocality between sequential pairs of observers, *Phys. Rev. A* 104(6), L060201 (2021)
29. S. Cheng, L. Liu, T. J. Baker, and M. J. W. Hall, Recycling qubits for the generation of Bell nonlocality between independent sequential observers, *Phys. Rev. A* 105(2), 022411 (2022)
30. F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, et al., Quantum supremacy using a programmable superconducting processor, *Nature* 574(7779), 505 (2019)
31. H. S. Zhong, H. Wang, Y. H. Deng, M. C. Chen, L. C. Peng, Y. H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, P. Hu, X. Y. Yang, W. J. Zhang, H. Li, Y. Li, X. Jiang, L. Gan, G. Yang, L. You, Z. Wang, L. Li, N. L. Liu, C. Y. Lu, and J. W. Pan, Quantum computational advantage using photons, *Science* 370(6523), 1460 (2020)
32. H. S. Zhong, Y. H. Deng, J. Qin, H. Wang, M. C. Chen, L. C. Peng, Y. H. Luo, D. Wu, S. Q. Gong, H. Su, Y. Hu, P. Hu, X. Y. Yang, W. J. Zhang, H. Li, Y. Li, X. Jiang, L. Gan, G. Yang, L. You, Z. Wang, L. Li, N. L. Liu, J. J. Renema, C. Y. Lu, and J. W. Pan, Phase-programmable Gaussian Boson sampling using stimulated squeezed light, *Phys. Rev. Lett.* 127(18), 180502 (2021)
33. M. Gong, S. Wang, C. Zha, M. C. Chen, H. L. Huang, et al., Quantum walks on a programmable two-dimensional 62-qubit superconducting processor, *Science* 372(6545), 948 (2021)
34. Y. Wu, W. S. Bao, S. Cao, F. Chen, M. C. Chen, et al., Strong quantum computational advantage using a superconducting quantum processor, *Phys. Rev. Lett.* 127(18), 180501 (2021)
35. J. M. Liang, S. Q. Shen, M. Li, and S. M. Fei, Quantum algorithms for the generalized eigenvalue problem, *Quantum Inform. Process.* 21(1), 23 (2022)
36. J. M. Liang, S. J. Wei, and S. M. Fei, Quantum gradient descent algorithms for nonequilibrium steady states and linear algebraic systems, *Sci. China Phys. Mech. Astron.* 65(5), 250313 (2022)
37. C. Branciard, N. Gisin, and S. Pironio, Characterizing the nonlocal correlations created via entanglement swapping, *Phys. Rev. Lett.* 104(17), 170401 (2010)
38. D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acín, Quantum networks reveal quantum nonlocality, *Nat. Commun.* 2(1), 184 (2011)
39. C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, *Phys. Rev. A* 85(3), 032119 (2012)
40. A. Tavakoli, P. Skrzypczyk, D. Cavalcanti, and A. Acin, Nonlocal correlations in the star-network configuration, *Phys. Rev. A* 90(6), 062109 (2014)
41. M. X. Luo, Computationally efficient nonlinear Bell inequalities for quantum networks, *Phys. Rev. Lett.* 120(14), 140402 (2018)
42. M. O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine quantum nonlocality in the triangle network, *Phys. Rev. Lett.* 123(14), 140401 (2019)

43. A. Tavakoli, A. Pozas-Kerstjens, M. X. Luo, and M. O. Renou, Bell nonlocality in networks, *Rep. Prog. Phys.* 85(5), 056001 (2022)
44. P. Contreras-Tejada, C. Palazuelos, and J. I. de Vicente, Genuine multipartite nonlocality is intrinsic to quantum networks, *Phys. Rev. Lett.* 126(4), 040501 (2021)
45. L. Y. Hsu and C. H. Chen, Exploring Bell nonlocality of quantum networks with stabilizing and logical operators, *Phys. Rev. Res.* 3(2), 023139 (2021)
46. L. Yang, X. Qi, and J. Hou, Nonlocal correlations in the tree-tensor-network configuration, *Phys. Rev. A* 104(4), 042405 (2021)
47. A. Pozas-Kerstjens, N. Gisin, and A. Tavakoli, Full network nonlocality, *Phys. Rev. Lett.* 128(1), 010403 (2022)
48. M. O. Renou and S. Beigi, Nonlocality for generic networks, *Phys. Rev. Lett.* 128(6), 060401 (2022)
49. W. Hou, X. Liu, and C. Ren, Network nonlocality sharing via weak measurements in the extended bilocal scenario, *Phys. Rev. A* 105(4), 042436 (2022)
50. N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, *Phys. Rev. A* 96, 020304(R) (2017)
51. F. Andreoli, G. Carvacho, L. Santodonato, R. Chaves, and F. Sciarrino, Maximal qubit violation of n-locality inequalities in a star-shaped quantum network, *New J. Phys.* 19(11), 113020 (2017)
52. J. I. de Vicente, Separability criteria based on the Bloch representation of density matrices, *Quantum Inf. Comput.* 7(7), 624 (2007)
53. R. Horodecki, P. Horodecki, and M. Horodecki, Violating Bell inequality by mixed states: Necessary and sufficient condition, *Phys. Lett. A* 200(5), 340 (1995)
54. C. Ren, X. Liu, W. Hou, T. Feng, and X. Zhou, Nonlocality sharing for a three-qubit system via multilateral sequential measurements, *Phys. Rev. A* 105(5), 052221 (2022)
55. S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Steering a single system sequentially by multiple observers, *Phys. Rev. A* 98(1), 012305 (2018)
56. A. Bera, S. Mal, A. Sen(De), and U. Sen, Witnessing bipartite entanglement sequentially by multiple observers, *Phys. Rev. A* 98(6), 062304 (2018)
57. A. G. Maity, D. Das, A. Ghosal, A. Roy, and A. S. Majumdar, Detection of genuine tripartite entanglement by multiple sequential observers, *Phys. Rev. A* 101(4), 042340 (2020)
58. S. Datta and A. S. Majumdar, Sharing of nonlocal advantage of quantum coherence by sequential observers, *Phys. Rev. A* 98(4), 042311 (2018)
59. M. L. Hu, J. R. Wang, and H. Fan, Limits on sequential sharing of nonlocal advantage of quantum coherence, *Sci. China Phys. Mech. Astron.* 65(6), 260312 (2022)
60. F. J. Curchod, M. Johansson, R. Augusiak, M. J. Hoban, P. Wittek, and A. Acin, Unbounded randomness certification using sequences of measurements, *Phys. Rev. A* 95(2), 020102 (2017)
61. S. Roy, A. Bera, S. Mal, A. Sen(De), and U. Sen, Recycling the resource: Sequential usage of shared state in quantum teleportation with weak measurements, *Phys. Lett. A* 392, 127143 (2021)
62. K. Mohan, A. Tavakoli, and N. Brunner, Sequential random access codes and self-testing of quantum measurement instruments, *New J. Phys.* 21(8), 083034 (2019)
63. Y. F. Yan, L. Zhou, W. Zhong, and Y. B. Sheng, Measurement-device-independent quantum key distribution of multiple degrees of freedom of a single photon, *Front. Phys.* 16(1), 11501 (2021)
64. Y. M. Xie, Y. S. Lu, C. X. Weng, X. Y. Cao, Z. Y. Jia, Y. Bao, Y. Wang, Y. Fu, H. L. Yin, and Z. B. Chen, Breaking the rate-loss bound of quantum key distribution with asynchronous two-photon interference, *PRX Quantum* 3(2), 020315 (2022)
65. J. Gu, X. Y. Cao, Y. Fu, Z. W. He, Z. J. Yin, H. L. Yin, and Z. B. Chen, Experimental measurement-device-independent type quantum key distribution with flawed and correlated sources, *Sci. Bull. (Beijing)* 67(21), 2167 (2022)
66. H. L. Yin, Y. Fu, C. L. Li, C. X. Weng, B. H. Li, J. Gu, Y. S. Lu, S. Huang, and Z. B. Chen, Experimental quantum secure network with digital signatures and encryption, *Natl. Sci. Rev.* nwac228 (2022)
67. Z. D. Ye, D. Pan, Z. Sun, C. G. Du, L. G. Yin, and G. L. Long, Generic security analysis framework for quantum secure direct communication, *Front. Phys.* 16(2), 21503 (2021)
68. S. S. Mahato and A. K. Pan, Pan, Sharing nonlocality in a quantum network by unbounded sequential observers, *Phys. Rev. A* 106, 042218 (2022)
69. J. H. Wang, Y. J. Wang, L. J. Wang, and Q. Chen, Network nonlocality sharing via weak measurements in the generalized star network configuration, *Phys. Rev. A* 106, 052412 (2022)
70. Y. L. Mao, Z.D. Li, A. Steffinlongo, B. Guo, B. Liu, S. Xu, N. Gisin, A. Tavakoli, and J. Fan, Recycling nonlocality in a quantum network, arXiv: 2202.04840 (2022)