FRONTIERS OF PHYSICS

RESEARCH ARTICLE
Volume 18 / Issue 3 / 31301 / 2023

Charging and self-discharging process of a quantum battery in composite environments

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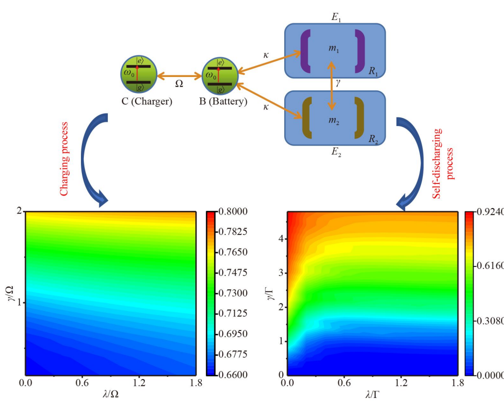
Received September 17, 2022; accepted November 25, 2022

ABSTRACT

How to improve charging processes and suppress self-discharging processes has always been one of the key issues to achieve quantum batteries with high performance. Although a quantum battery is inevitably influenced by composite environments, this situation is still little understood, particularly regarding the influence of the memory effect of the composite environments and the coupling between composite environments. In this work, we investigate the effects of the composite environments, composed of two identical parts each containing a single cavity mode decaying to a reservoir, on the charging and self-discharging processes of a quantum battery. We show that increasing the two-mode coupling can effectively enhance the charging performance (i.e., the stored energy, the charging power, ergotropy) and restrain the self-discharging process (i.e., suppressing the process of dissipating the energy). However, different from the effect of two-mode coupling, we reveal that the memory effect of the reservoir in this composite environment is unfavorable to the charging process of the quantum battery, which is in sharp contrast to previous studies where the memory effect can significantly improve the charging performance of a quantum battery. Our results may benefit to the realization of quantum batteries with high performance under the actual complex environmental noise.



Keywords quantum battery, quantum device

1 Introduction

With the development of quantum technology, the utilization of quantum resources such as coherence and entanglement may enable future technical devices to have excellent performance. A prime example is in quantum computing [1, 2] where quantum resources can be used to achieve efficient manipulation and processing of information [3–10] compared to classical computing. A natural question is whether quantum resources can be beneficial to the energy field for efficient storage of energy. To solve this problem, a quantum battery, i.e., a quantum system that receives or supplies energy [11–23], has attracted widespread attention from researchers.

Previous work has demonstrated the positive contribution of quantum resources in improving the performance of quantum batteries, such as work extraction [24–27], charging power and stored energy [28–32]. Recently,

Higher Education Press

https://doi.org/10.1007/s11467-022-1230-x

many other methods have been proposed to realize the efficient charging process of the quantum batteries [33–48]. For example, the high performance of a quantum battery can be achieved by using the two-photon charging process [44], the harmonic drive [45], and the adiabatic shortcut technology [46]. The above research focuses on how to improve the charging process of closed quantum batteries. However, in a practice scenario, the quantum battery inevitably interacts with the environment [49–53]. Therefore, it is necessary to consider how to improve the charging performance of quantum batteries under the influence of the environment.

Recently, many schemes to enhance the stored energy and charging power of open quantum batteries have been introduced [54–74]. For instance, when considering the quantum battery and charger in a non-Markovian environment, an efficient charging process can occur [70]. By manipulating the coupling strength between the quantum battery–charger system and the environment, the optimal storage process of the battery can be realized [71]. Furthermore, the best charging process of the quantum battery can be achieved by using the coherent driving field [72]. Then, to obtain the high performance of quantum batteries, in addition to considering the charging process of quantum batteries, one should also focus on how to suppress the self-discharging process [74–76] of the quantum battery to achieve a stable energy storage process of quantum batteries. Currently, researchers have utilized many technologies [77–79], such as quantum measurement on Zeno protection [77], stimulated Raman adiabatic channel technology [78] and the Floquet engineering with two bound states in the quasienergy spectrum [79] to obtain the quantum battery with long-term energy storage.

In the above research on the charging or the self-discharging process of quantum batteries, one usually considers the quantum battery in a single environment. However, in actual scenarios, the quantum system implementing quantum batteries may face complexly coupled environments [80–82]. For instance, for a nitrogen-vacancy (NV) center in bulk diamond, the shallow NV color center is affected by multiple coupled electron spin impurities [80]. In the coupled cavity array system, the dynamic behavior of a quantum system is closely related to its surrounding array of coupled cavities [81]. Moreover, in real scenarios, environments with memory effects also appear. In an all-optical experiment, the memory environment can be got by the polarization degree of freedom of photons coupled to the frequency degree of freedom representing the environment [82]. The memory effect of the environment can be also manipulated for the NV center in diamond by adjusting the spin-spin coupling between bath spins, and the surface-modified phonons coupled to the NV spin [80]. Despite this, it is currently unclear about the role of couplings between complex environmental parts and the memory effect of the complex environment on the charging and also about self-discharging processes of quantum batteries. Understanding how these environmental parameters influence the performance of an open quantum battery would provide insightful developments for manipulating a quantum battery in possible technological applications.

Therefore, this aspect deserves careful investigation, possibly starting from a paradigmatic model where it can simply emerge and be understood. Here, we choose a model which complies with the above requirements, namely, a quantum battery (a qubit) interacting at the same time with two coupled composite environments, each containing a single-mode cavity decaying to a reservoir. The composite environment we choose in this paper is a non-Markovian environment. Compared with the Markovian environment model, our environment is more conducive to improving the energy storage capacity of quantum batteries and inhibiting the self-discharge process of quantum batteries. Then using the previous circuit quantum electrodynamics technique [83] and analog all-optical set-up [84], the model can be realized experimentally. In stark contrast to previous studies that the stronger memory effect of a single reservoir environment could lead to the stronger charging performance of a quantum battery [70, 74], we show that the memory effect of the reservoir can be detrimental to the charging process of the quantum battery. Using the pseudomode theory, we explain this phenomenon from the fact that the energy of the quantum battery will flow into the environment earlier as the environmental memory effect increases. Furthermore, the influence of the coupling strength between the environmental parts on the charging and the self-discharging process of the quantum battery is also considered. We show that the charging performance and the ability to keep energy in a quantum battery for long times after charging can be enhanced by increasing the two-mode coupling.

This paper is organized as follows. In Section 2 we introduce the model of the quantum battery. Section 3 and Section 4 discusses the influence of the memory effect of the environment and the coupling between the environmental parts on the charging process and the self-discharging process of the quantum battery, respectively. The conclusions drawn from the present study are given in Section 5.

2 Physical model and approach

Our total system consists of a quantum battery B (i.e., a qubit) that interacts with two environments (E_1 and E_2), and a quantum charger C (i.e., a qubit). Each environment E_n $(n = 1, 2)$ can be modeled as a bosonic mode m_n and a zero temperature bosonic reservoir R_n. The bosonic mode m_n is coupled to the quantum battery with strength κ and decays to a zero temperature bosonic reservoir R_n, as shown in Fig. 1. It is worth mentioning that both our model and the model in Ref. [56] take the cavity field and reservoir as the environment of a quantum battery. The difference is, in our model, there is an interaction between environment E_1 and environment E_2.

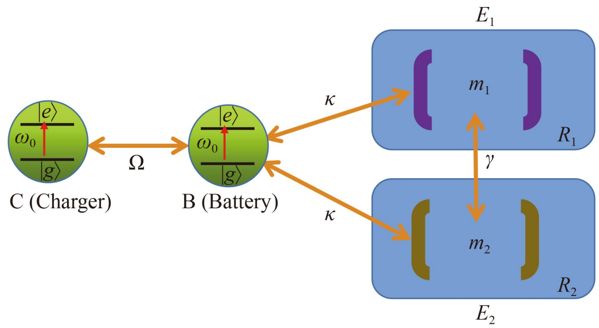


Fig. 1 Schematic diagram of a quantum battery interacting with two environments, E_n ($n = 1$ and $n = 2$), each represented by a single-mode cavity m_n decaying to a reservoir environment R_n. The coupling strength between the quantum battery and the charger is Ω, while the coupling strength between the quantum battery and the two coupled single-mode cavities is κ. The two cavity modes m_1 and m_2 are coupled with strength γ.

The interaction between the two environments relies on the coupling γ between the two boson modes, which instead plays the role of a control parameter for the performance of the quantum battery. Then the coupling strength of the quantum battery with the charger is Ω. The Hamiltonian of the total system is described as follows

$$H = H_0 + f(t)H_I, \tag{1}$$

where H_0 is the total free Hamiltonian and H_I describes the interactions between the subsystems in the overall system. The total free Hamiltonian in Eq. (1) can be written as

$$\begin{aligned} H_0 =& H_C + H_B + \sum_{n=1}^{2}(H_{m_n} + H_{R_n}) = \omega_0\sigma_C^+\sigma_C^- \\ &+ \omega_0\sigma_B^+\sigma_B^- + \sum_{n=1}^{2}\omega_0 a_n^+ a_n + \sum_{n=1}^{2}\sum_k \omega_{n,k} b_{n,k}^+ b_{n,k}, \end{aligned} \tag{2}$$

where σ_j^+ (σ_j^-) ($j = B, C$) is the Pauli raising and lowering operators for jth qubits, a_n^+ (a_n) denotes the creation (annihilation) operators for nth modes and $b_{n,k}^+$ ($b_{n,k}$) refers to the creation (annihilation) operator of the kth field mode of the reservoir R_n. Then the interaction Hamiltonian H_I in Eq. (1) can be given by

$$\begin{aligned} H_I &= H_{BC} + H_{m_1m_2} + \sum_{n=1}^{2}(H_{Bm_n} + H_{m_nR_n}) \\ &= \Omega\left(\sigma_C^+\sigma_B^- + \sigma_C^-\sigma_B^+\right) + \gamma\left(a_1a_2^+ + a_2a_1^+\right) \\ &\quad + \sum_{n=1}^{2}\kappa\left(\sigma_B^+a_n + \sigma_B^-a_n^+\right) + \sum_{n=1}^{2}\sum_k g_{n,k}\Big(a_nb_{n,k}^+ \\ &\quad + a_n^+b_{n,k}\Big), \end{aligned} \tag{3}$$

where the first term is the quantum charger–battery interaction Hamiltonian, the second term is mode-mode interaction Hamiltonian, the third term is the quantum battery-mode interaction Hamiltonian, and the last term is mode-reservoir interaction Hamiltonian. In the interaction picture, the Hamiltonian of the total system can be written as $H_{Int} = \Omega(\sigma_C^+\sigma_B^- + \sigma_C^-\sigma_B^+) + \sum_{n=1}^{2}\kappa(\sigma_B^+a_n + \sigma_B^-a_n^+) + \gamma(a_1a_2^+ + a_2a_1^+) + \sum_{n=1}^{2}\sum_k g_{n,k}(a_nb_{n,k}^+\mathrm{e}^{\mathrm{i}\Delta_{n,k}t} + a_n^+b_{n,k}\mathrm{e}^{-\mathrm{i}\Delta_{n,k}t})$, where $\Delta_{n,k} = \omega_{n,k} - \omega_0$.

Then $f(t)$ in Eq. (1) is a dimensionless function that equals 1 for $t \in [0, \tau)$ and 0 elsewhere, which is used to switch interactions on or off, and τ represents the charging/self-discharging time of the protocol. In the charging protocol of the quantum battery, we consider that for $t < 0$, the quantum battery and the charger are isolated and do not interact with the environment. At the time $t = 0$, by switching on the interaction Hamiltonian H_I, the charger C is attached to the quantum battery B and the quantum battery B begins to interact with mode m_n, and mode m_n with reservoir R_n. Since $[H_0, H_I] \neq 0$, the final energy of a quantum battery is related not only to the charger, but also to the thermodynamic work from the on/off interaction at switching times [69, 70]. Then in the time window $[0, \tau)$, a part of the energy of the quantum charger flows into the quantum battery as well as a portion of the energy of the quantum battery moves into the environment. Ultimately, at the end of the charging process, that is, at time τ when $f(t)$ returns to zero, we isolate the quantum battery system again and turn off the interaction. Given the infinite degree of freedom of the environment, we focus on the charging and self-discharging process of the quantum battery under the influence of the composite environment without discussing the amount of thermodynamic work cost [70]. For the self-discharging protocol, according to Ref. [75], by considering a quantum battery disconnected from a charger (i.e., $\Omega = 0$), the dissipation of energy from the quantum battery to the surrounding environment is studied.

To study the charging and self-discharging process of a quantum battery in composite environments, the physical quantities that characterize the performance of the quantum battery should be introduced. In the following, we will describe the performance characterization of a quantum battery charging/self-discharging process in detail.

2.1 Performance characterization of a quantum battery charging process

First, during the charging process, we are interested in characterizing how efficiently energy of the charger can be transferred into the quantum battery. To this end, we consider the stored energy of the quantum battery at the end of the charging process and the corresponding mean power (stored energy over the charging time) [29], i.e.,

$$E_B(\tau) = \mathrm{tr}\left[H_B \rho_B(\tau)\right] - \mathrm{tr}\left[H_B \rho_B(0)\right], \tag{4}$$

$$P_B(\tau) = E_B(\tau)/\tau, \tag{5}$$

where $\rho_B(\tau) = \mathrm{tr}_A(\rho_{AB}(\tau))$ is the reduced state of the quantum battery. Then, to quantify the maximal amount of energy that can be extracted from the quantum battery at the end of the charging process under the cyclic unitary operations, the ergotropy [85] is defined as

$$W_B(\tau) = \mathrm{tr}\left(\rho_B(\tau) H_B\right) - \min \mathrm{tr}\left(U \rho_B(\tau) U^{\dagger} H_B\right), \tag{6}$$

where the minimization is performed over all possible unitaries. The term $\min \mathrm{tr}\left(U \rho_B(\tau) U^{\dagger} H_B\right)$ corresponds to the expectation value $E_{\mathrm{B}}^{(p)}(\tau) = \mathrm{tr}(H_B \sigma_{\rho_B})$ of H_B computed on the passive state σ_{ρ_B} [86–90], where no amount of work can be extracted from the quantum battery in a cyclic unitary process. By introducing the passive state, Eq. (6) can be written as

$$W_B(\tau) = \mathrm{tr}\left(\rho_B(\tau) H_B\right) - \mathrm{tr}\left(\sigma_{\rho_B} H_B\right). \tag{7}$$

A detailed derivation process for the ergotropy can be found in Appendix A. Then to better evaluate the charging performance of the quantum battery, the maximum stored energy $E_{\max}$ and the maximum ergotropy $W_{\max}$ can be given by

$$E_{\max} \equiv \max_{\tau}\left[E_B(\tau)\right], W_{\max} \equiv \max_{\tau}\left[W_B(\tau)\right]. \tag{8}$$

Larger $E_{\max}$ and $W_{\max}$ are needed to stimulate the optimal the charging process of the quantum battery.

2.2 Performance characterization of a quantum battery self-discharging process

A good quantum battery should not only have good charging performance but also the strong ability to suppress the self-discharging process. To size up this ability, by considering the self-discharging condition (i.e., $\Omega = 0$), the minimum stored energy $E_{\min}$ and the minimum ergotropy $W_{\min}$ can be introduced by

$$E_{\min} \equiv \min_{\tau}\left[E_B(\tau)\right],\ W_{\min} \equiv \min_{\tau}\left[W_B(\tau)\right]. \tag{9}$$

In the self-discharging process of a quantum battery, larger $E_{\min}$ and $W_{\min}$ indicates more energy able to be maintained in the battery despite environmental effects.

Then with the above definition, we focus on improving the charging performance and suppressing the self-discharging process by controlling environmental parameters in our model (i.e., the coupling strength between the environmental parts and the memory effect of the complex environment).

3 The charging process of a quantum battery

We first investigate the role of the memory effect of the reservoir environment and the coupling strength γ between the environmental parts in the charging process of a quantum battery. To do so, we need to analyze the dynamical evolution process of the quantum battery.

We assume the initial state of the whole system is $|\Phi(0)\rangle = (c_1(0)|10\rangle_{CB} + h(0)|01\rangle_{CB}) \otimes |00\rangle_{m_1 m_2} |\overline{\mathbf{00}}\rangle_{R_1 R_2}$, where $|\overline{\mathbf{0}}\rangle_{R_n} \equiv \prod_k |0_k\rangle_{R_n}$. The total excitation in the total system is limited to one. Since the excitation number operator $N = \sum_{j=C,B} \sigma_j^+ \sigma_j^- + \sum_{i=1}^2 a_i^+ a_i + \sum_{i=1}^2 \sum_k b_{i,k}^+ b_{i,k}$ is a conserved quantum number, the total excitation number of the total system remains unchanged during evolution. Therefore, we can write a general total state vector in basis spanned by a single excitation subspace as

$$\begin{aligned}|\Phi(t)\rangle =& (c_1(t)|10\rangle_{CB} + h(t)|01\rangle_{CB}) \otimes |00\rangle_{m_1 m_2} |\overline{\mathbf{00}}\rangle_{R_1 R_2} \\ &+ c_2(t)|00\rangle_{CB}|10\rangle_{m_1 m_2}|\overline{\mathbf{00}}\rangle_{R_1 R_2} \\ &+ c_3(t)|00\rangle_{CB}|01\rangle_{m_1 m_2}|\overline{\mathbf{00}}\rangle_{R_1 R_2} \\ &+ c_{1,k}(t)|00\rangle_{CB}|00\rangle_{m_1 m_2}|\mathbf{1_k}\rangle_{R_1}|\overline{\mathbf{0}}\rangle_{R_2} \\ &+ c_{2,k}(t)|00\rangle_{CB}|00\rangle_{m_1 m_2}|\overline{\mathbf{0}}\rangle_{R_1}|\mathbf{1_k}\rangle_{R_2},\end{aligned} \tag{10}$$

where $|\mathbf{1_k}\rangle_{R_n} \equiv |0 \cdots 1_k \cdots 0\rangle_{R_n}$ means that there is one excitation in the kth mode of the reservoir R_n. According to the Schrödinger equation, we can obtain the following set of differential equations:

$$\begin{aligned}\dot{c}_1(t) &= -\mathrm{i}\Omega h(t), \\ \dot{h}(t) &= -\mathrm{i}\Omega c_1(t) - \mathrm{i}\kappa c_2(t) - \mathrm{i}\kappa c_3(t), \\ \dot{c}_2(t) &= -\mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_3(t) - \mathrm{i}g_{1,k}\mathrm{e}^{-\mathrm{i}\Delta_{1,k}t} c_{1,k}(t), \\ \dot{c}_3(t) &= -\mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_2(t) - \mathrm{i}g_{2,k}\mathrm{e}^{-\mathrm{i}\Delta_{2,k}t} c_{2,k}(t), \\ \dot{c}_{1,k}(t) &= -\mathrm{i}g_{1,k}\mathrm{e}^{\mathrm{i}\Delta_{1,k}t} c_2(t), \\ \dot{c}_{2,k}(t) &= -\mathrm{i}g_{2,k}\mathrm{e}^{\mathrm{i}\Delta_{2,k}t} c_3(t).\end{aligned} \tag{11}$$

Integrating the last two differential equations and bringing them back, we can get $\dot{c}_2(t) = -\mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_3(t) - \int_0^t \sum_k |g_{1,k}|^2 \mathrm{e}^{-\mathrm{i}\Delta_{1,k}(t-t')} c_2(t')\,\mathrm{d}t'$, $\dot{c}_3(t) = -\mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_2(t) - \int_0^t \sum_k |g_{2,k}|^2 \mathrm{e}^{-\mathrm{i}\Delta_{2,k}(t-t')} c_3(t')\,\mathrm{d}t'$. The term $\sum_k |g_{n,k}|^2 \cdot \mathrm{e}^{\mathrm{i}(\omega_0 - \omega_{n,k})(t-t')}$ is recognized as the correlation function $F(t-t')$ of the reservoir R_n. We now consider that a reservoir R_n has the form of Lorentzian spectrum density $J(\omega) = \Gamma\lambda^2 / \left\{2\pi\left[(\omega-\omega_0)^2 + \lambda^2\right]\right\}$ [51–53]. The corresponding correlation function is $F(t-t') = \Gamma\lambda \exp(-\lambda \mid t - t')$, where $1/\lambda$ represents the memory time [51–53] and Γ refers to the cavity mode-reservoir coupling strength. The above differential equations for $\dot{c}_1(t)$, $\dot{h}(t)$, $\dot{c}_2(t)$, $\dot{c}_3(t)$ can be solved numerically. By tracing other degrees of freedom of the total system, the reduced density matrix of the quantum battery B can be

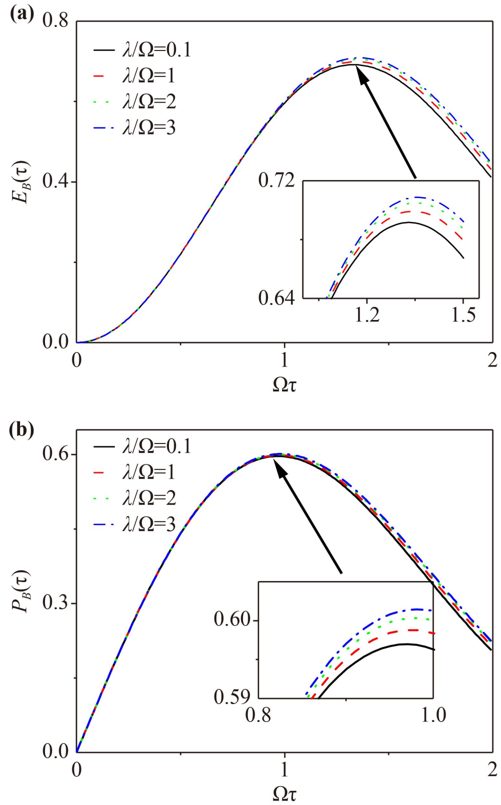


Fig. 2 **(a, b)** The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of the quantum battery as a function of the dimensionless quantity $\Omega\tau$ for different values of the parameter λ/Ω. Other parameters are chosen as (a), (b) $c_1(0)=1,\ h(0)=0,\ \gamma/\Omega=1,\ \kappa/\Omega=0.5,\ \Gamma/\Omega=1$. The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of a quantum battery are in the unit of ω_0.

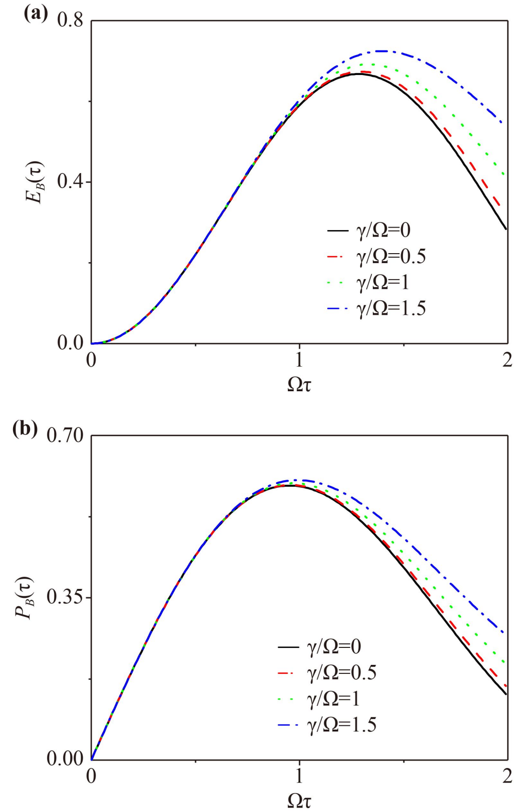


Fig. 3 **(a, b)** The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of the quantum battery as a function of the dimensionless quantity $\Omega\tau$ for different values of the coupling strength γ/Ω between the two modes. Other parameters are chosen as (a), (b) $c_1(0)=1,\ h(0)=0,\ \kappa/\Omega=0.5,$ $\lambda/\Omega=0.1,\ \Gamma/\Omega=1$. The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of a quantum battery are in the unit of ω_0.

obtained, i.e., $\rho_{ee}^B(t)=|h(t)|^2, \rho_{gg}^B(t)=1-|h(t)|^2$.

According to Eqs. (4), (5), and (6), the stored energy $E_B(\tau)$, the average charging power $P_B(\tau)$ and the extractable work $W_B(\tau)$ are given respectively by,

$$E_B(\tau)=\omega_0\,|h(\tau)|^2, \tag{12}$$

$$P_B(\tau)=\omega_0\,|h(\tau)|^2/\tau, \tag{13}$$

$$W_B(\tau)=\omega_0\left(2\,|h(\tau)|^2-1\right)\Theta\left(|h(\tau)|^2-1/2\right), \tag{14}$$

where $\Theta(x-x_0)$ is the Heaviside function, which satisfies $\Theta(x-x_0)=0$ for $x<x_0$, $\Theta(x-x_0)=1/2$ for $x=x_0$ and $\Theta(x-x_0)=1$ for $x>x_0$.

Then the influence of parameter λ/Ω, representing the memory effect of the reservoir, and the coupling strength γ/Ω between the two modes on the charging process of the quantum battery can be discussed.

Figure 2 shows the time evolution of the stored energy and the average charging power for various values of the parameter λ (smaller λ implies longer memory time of the reservoir). In contrast to previous quantum batteries charging in a simple reservoir environment [74], we are surprised to find that the decrease of the memory time (i.e., increase of λ) of the reservoir environment would be beneficial to the improvement of the energy storage and the average charging power (see the inset of Figs. 2(a) and (b) for a more evident demonstration). In fact, because the charging process is determined by both the reservoir R_n and the cavity m_n, the memory effect of the reservoir alone is not sufficient to affect the charging performance. Besides, the evolution of the stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ under different coupling strength γ/Ω are plotted in Fig. 3. As shown, both the stored energy and the average charging power can be enhanced by increasing the coupling strength between the two modes. Then as far as extractable work is concerned, according to Eqs. (12) and (14), λ/Ω and γ/Ω have similar effects on the stored energy and extractable work. Therefore, when the quantum battery is charged in a composite environment, both the weaker environmental memory effect and the larger

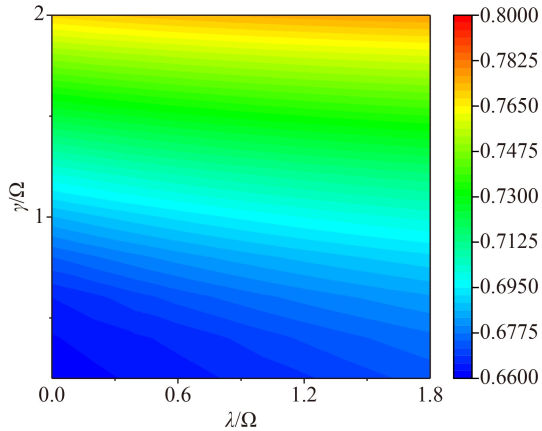


Fig. 4 Maximum stored energy $E_{\max}$ of the quantum battery as a function of the parameter λ/Ω and the coupling strength γ/Ω between the cavity mode m_1 and the cavity mode m_2. Other parameters are: $c_1(0)=1$, $h(0)=0$, $\Gamma/\Omega=1$, $\kappa/\Omega=0.5$. Maximum stored energy $E_{\max}$ of a quantum battery is in the unit of ω_0.

coupling strength between the environments should be requested to stimulate the optimal performance of the quantum battery.

To further investigate the influence of the memory effect of the reservoir environment and the coupling between the two modes on the charging process of the quantum battery, the dependence of the maximum stored energy $E_{\max}$ on λ/Ω and γ/Ω has been drawn in Fig. 4. Indeed, $E_{\max}$ decreases slightly with the decrease of λ/Ω. However, as the coupling strength γ/Ω between the two-mode increases, the maximum stored energy of the quantum battery increases significantly. This means that, when considering the initial separation state of quantum battery and charger, shorter memory time and larger coupling strength between the environmental parts are required to achieve the maximum stored energy. Then, it is worth emphasizing that the effects of λ/Ω and γ/Ω on the quantum battery charging process do not depend on the initial state of the quantum battery and the charger (i.e., separated or entangled), which is discussed in detail in Appendix B.

Now, one might still be surprised that the memory effects of the reservoir environment and the coupling between the two modes have a negative and positive effect on the charging performance of the quantum battery. To explain the longer memory time (i.e., smaller λ/Ω) of reservoir R_n and the smaller coupling strength between the two modes is not conducive to the charging process, we introduce the pseudomode theory [91–94] and the energy flow between subsystems. The pseudomode of the reservoir R_n is an auxiliary variable introduced based on the position of the pole of the reservoir's spectral distribution. The coherent interaction between the system and the reservoir can be expressed by the interaction between the system and the pseudomode, which dissipates to the Markovian reservoir. The memory effect of the reservoir is contained in the interaction between the system and the pseudomode. Then one can derive a corresponding Lindblad master equation by treating the system of interest plus the pseudomodes as an extended system. For the currently considered quantum charger C, quantum battery B, cavity fields m_n and reservoir R_n with Lorentzian spectrum (each reservoir R_n has a pseudomode), the master equation for this extended system (i.e., quantum charger C, quantum battery B, cavity fields m_n and pseudomodes l_n) satisfies

$$\dot{\rho}(t) = -\,\mathrm{i}\left[H_0^1, \rho(t)\right] - \sum_{n=1}^{2} \frac{\Gamma_n'}{2}\left[l_n^\dagger l_n \rho(t) - 2 l_n \rho(t) l_n^\dagger + \rho(t) l_n^\dagger l_n\right], \tag{15}$$

where $H_0^1 = \omega_0\sigma_C^+\sigma_C^- + \omega_0\sigma_B^+\sigma_B^- + \sum_{n=1}^2 \omega_0 a_n^+ a_n + \sum_{n=1}^2 \omega_0 l_n^+ l_n + \Omega(\sigma_C^+\sigma_B^- + \sigma_C^-\sigma_B^+) + \sum_{n=1}^2 \kappa\left(\sigma_B^+ a_n + \sigma_B^- a_n^+\right) + \gamma\left(a_1 a_2^+ + a_2 a_1^+\right) + \sum_{n=1}^2 \chi\left(a_n l_n^+ + a_n^+ l_n\right)$, with l_n^+ (l_n) being the creation (annihilation) operator of the nth pseudomode whose coupling constant with m_n is $\chi = \sqrt{\lambda\Gamma/2}$, and $\Gamma_n' = 2\lambda$ [91, 92] denotes the decay rate of the nth pseudomode. Then we define $h(t)$, $c_1(t)$, $c_n(t) (n=2,3)$, and $c_m(t)$ $(m=4,5)$ as the probability amplitudes of the quantum battery, charger, two bosonic modes, and two pseudomodes in their respective excited states, respectively. According to Eq. (15), the differential equations satisfied by these probability amplitudes can be written as $\dot{c}_1(t) = -\mathrm{i}\omega_0 c_1(t) - \mathrm{i}\Omega h(t)$, $\dot{h}(t) = -\mathrm{i}\omega_0 h(t) - \mathrm{i}\Omega c_1(t) - \mathrm{i}\kappa c_2(t) - \mathrm{i}\kappa c_3(t)$, $\dot{c}_2(t) = -\mathrm{i}\omega_0 c_2(t) - \mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_3(t) - \mathrm{i}\chi c_4(t)$, $\dot{c}_3(t) = -\mathrm{i}\omega_0 c_3(t) - \mathrm{i}\kappa h(t) - \mathrm{i}\gamma c_2(t) - \mathrm{i}\chi c_5(t)$, $\dot{c}_4(t) = (-\mathrm{i}\omega_0 - \Gamma_1'/2) c_4(t) - \mathrm{i}\chi c_2(t)$, $\dot{c}_5(t) = (-\mathrm{i}\omega_0 - \Gamma_2'/2) c_5(t) - \mathrm{i}\chi c_3(t)$. The above differential equations can be solved by using standard Laplace transformations combined with numerical simulations.

Then to witness the energy flow between the quantum battery and the subsystems, the witness $M(\tau)$ can be defined as

$$M(\tau) \equiv \frac{\mathrm{d}\sum_{n=1}^5 |c_n(\tau)|^2}{\mathrm{d}\tau} + \Gamma_1' |c_4(\tau)|^2 + \Gamma_2' |c_5(\tau)|^2 . \tag{16}$$

If the population of all subsystems other than the battery drops and this decline cannot compensate for its dissipation to the Markovian reservoir, $M(\tau) < 0$ would occur. This means that the energy is flowing from the subsystems to the quantum battery. Conversely, when $M(\tau) > 0$, the energy of the quantum battery is dissipated to the subsystems, resulting in a poor charging process. To clarify the influence of the memory effect of the reservoir environment and the coupling strength between the two modes on the charging performance, we plot the change of the witness $M(\tau)$ as a function of $\Omega\tau$ for different λ/Ω and γ/Ω in Figs. 5(a) and (b). In each case, the $M(\tau)$ will change sign (i.e. from negative to positive) which means the transition of the direction of

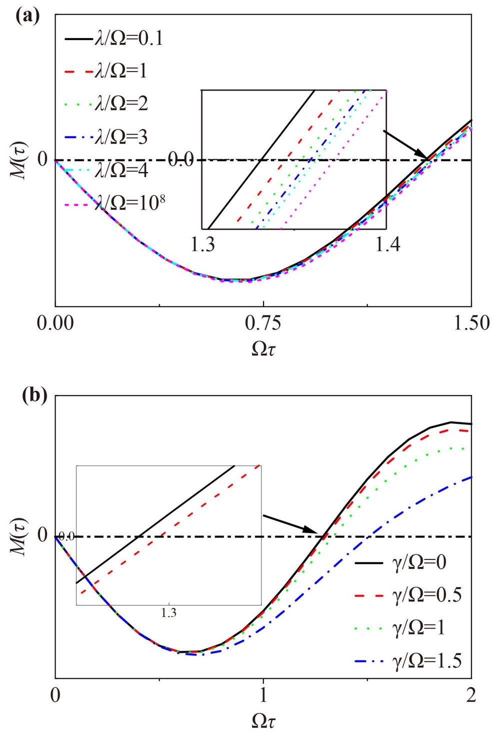


Fig. 5 **(a)** The witness $M(\tau)$ as a function of the dimensionless quantity $\Omega\tau$ for different values of λ/Ω. **(b)** The witness $M(\tau)$ as a function of the dimensionless quantity $\Omega\tau$ for different values of γ/Ω. Other parameters are: (a) $c_1(0)=1$, $h(0)=0$, $\gamma/\Omega=\Gamma/\Omega=1$, $\kappa/\Omega=0.5$; (b) $c_1(0)=1$, $h(0)=0$, $\lambda/\Omega=0.1$, $\Gamma/\Omega=1$, $\kappa/\Omega=0.5$.

energy flow (i.e., from towards the battery to outwards the battery). As shown, the small λ/Ω or the small γ/Ω will result in the earlier transition moment which indices a shorter time of energy flow from the subsystem towards the battery. This shorter time of energy flow towards the battery is supposed to be the main reason why the memory effect or the small coupling strength between the two modes is unfavorable for the charging process.

4 The self-discharging process of a quantum battery

Besides charging performance, the ability to store energy for a long time after charging should also be considered to achieve excellent performance of a quantum battery. In reality, due to the influence of the environment, the self-discharging process, during which the energy of the quantum battery is dissipated into the environment, is detrimental but inevitable. Therefore, it is essential to consider how to suppress the self-discharging process of a quantum battery and preserve the energy for a long time. In this section, we study how to utilize the memory effect of the reservoir environment and adjust the coupling strength between the composite environments to suppress the self-discharging process of the quantum battery.

The system in this self-discharging process can be modeled by taking the limit $\Omega\to 0$ in Fig. 1 and setting the quantum battery at state $|e\rangle_B$, the cavity modes and the reservoir in vacuum state (i.e., $|00\rangle_{m_1m_2}|\overline{\mathbf{00}}\rangle_{R_1R_2}$). The evolution state of this system at any moment is

$$\begin{aligned}|\psi(t)\rangle = {} & u_1(t)|e\rangle_B|00\rangle_{m_1m_2}|\overline{\mathbf{00}}\rangle_{R_1R_2}\\ & + u_2(t)|g\rangle_B|10\rangle_{m_1m_2}|\overline{\mathbf{00}}\rangle_{R_1R_2}\\ & + u_3(t)|g\rangle_B|01\rangle_{m_1m_2}|\overline{\mathbf{00}}\rangle_{R_1R_2}\\ & + u_{1,k}(t)|g\rangle_B|00\rangle_{m_1m_2}|\mathbf{1_k}\rangle_{R_1}|\overline{\mathbf{0}}\rangle_{R_2}\\ & + u_{2,k}(t)|g\rangle_B|00\rangle_{m_1m_2}|\overline{\mathbf{0}}\rangle_{R_1}|\mathbf{1_k}\rangle_{R_2},\end{aligned} \tag{17}$$

where $|\overline{\mathbf{0}}\rangle_{R_n}$ and $|\mathbf{1_k}\rangle_{R_n}$ are defined in the same way as Eq. (10). Then, by taking the same steps as done in solving the charging process of the quantum battery, the reduced density matrix of the quantum battery can be obtained, i.e., $\rho_{\mathrm{B}}(t)=|u_1(t)|^2|e\rangle\langle e|_{\mathrm{B}}+[1-|u_1(t)|^2]\mid g\rangle\langle g|_{\mathrm{B}}$. Combining Eqs. (12) and (14), the stored energy and the ergotropy can be written as

$$E_B(\tau)=\omega_0\left|u_1(\tau)\right|^2, \tag{18}$$

$$W_B(\tau)=\omega_0\left(2\left|u_1(\tau)\right|^2-1\right)\Theta\left(\left|u_1(\tau)\right|^2-1/2\right). \tag{19}$$

In the following, we can use the minimum stored energy $E_{\min}$ and the minimum extractable work $W_{\min}$ (i.e., Eq. (9)) to study the condition needed for a quantum battery to store energy for a long time in composite environments.

The effects of the memory effect of the reservoir environment and the coupling strength between the two coupled single-mode cavities on the minimum stored energy $E_{\min}$ and the minimum extractable work $W_{\min}$ are discussed. In Fig. 6, we show that the minimum stored energy $E_{\min}$ can be retained at a higher value as the memory time of the reservoir environment and the coupling strength between the two modes increase. This means that a longer reservoir environment memory time and greater coupling strength γ/Γ between the environments will effectively inhibit the self-discharging process of the quantum battery, thus realizing the purpose preserving energy for a long time. As for the ergotropy of the quantum battery, the variation of the minimum extractable work $W_{\min}$ with λ/Γ and γ/Γ is shown in Fig. 7. Similar to the case of the minimum stored energy, the larger the memory time of the reservoir environment and the coupling γ/Γ of the two modes are requested to enhance the minimum extractable work of the quantum battery. Therefore, different from obtaining the optimal quantum battery charging process, the longer reservoir memory time and larger two-mode coupling are needed to suppress this self-discharging process.

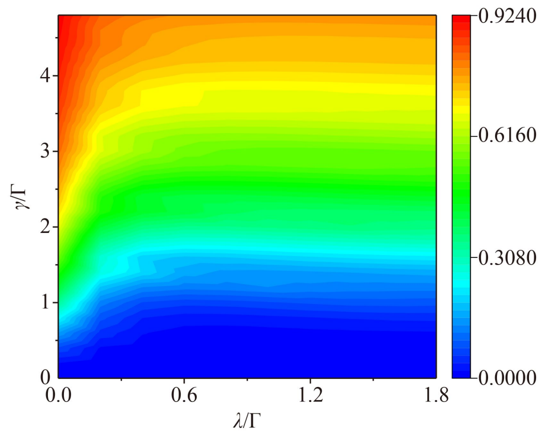


Fig. 6 Minimum stored energy $E_{\min}$ of the quantum battery as a function of the parameter λ/Γ and the coupling strength γ/Γ between the cavity mode m_1 and the cavity mode m_2. Other parameters are: $\kappa/\Gamma = 0.5$. Minimum stored energy $E_{\min}$ of a quantum battery is in unit of ω_0.

To clearly explain why long memory time and large two-mode coupling strength are beneficial to restrain the self-discharging process of a quantum battery, we plot the witness $M(\tau)$ as a function of the dimensionless quantity $\Gamma\tau$ for different values of λ/Γ or γ/Γ by considering the self-discharging condition (i.e., $\Omega = 0$) in Eq. (16). As shown in Figs. 8(a) and (b), by fixing the value of λ/Γ or γ/Γ, the witness $M(\tau)$ will change from a positive value to a negative value with the increase of $\Gamma\tau$, which means an information backflow from the cavity modes to the quantum battery at a certain transition time. And it is worth noting that the small λ/Γ or the large γ/Γ will result in the earlier transition moment, which implies the earlier time of energy flow from cavity modes towards the battery. The earlier energy flow from the cavity modes to the battery is considered to be the main reason why the memory effect of the reservoir or the coupling strength between two modes is conducive to resisting the self-discharging process.

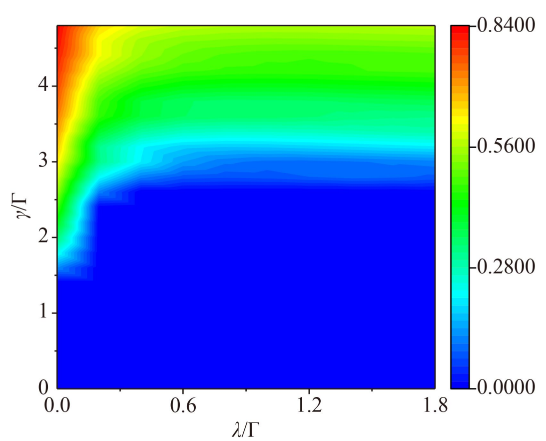


Fig. 7 Minimum ergotropy $W_{\min}$ of the quantum battery as a function of the parameter λ/Γ and the coupling strength γ/Γ between the cavity mode m_1 and the cavity mode m_2. Other parameters are: $\kappa/\Gamma = 0.5$. Minimum ergotropy $W_{\min}$ of a quantum battery is in unit of ω_0.

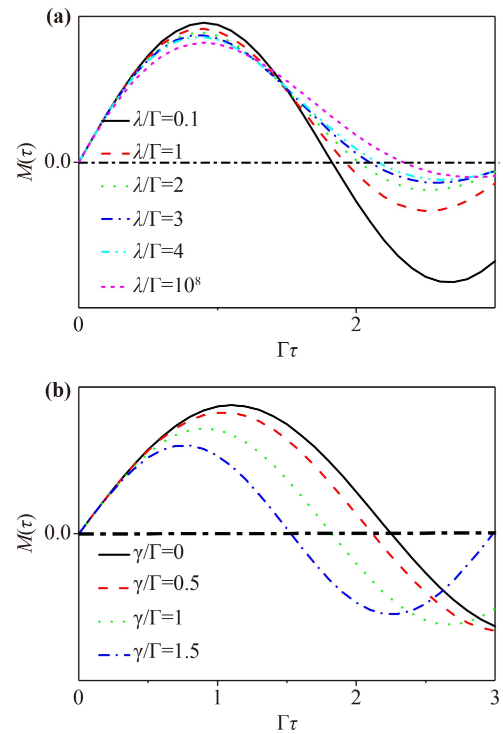


Fig. 8 **(a)** The witness $M(\tau)$ as a function of the dimensionless quantity $\Gamma\tau$ for different values of λ/Γ in the self-discharging process. **(b)** The witness $M(\tau)$ as a function of the dimensionless quantity $\Gamma\tau$ for different values of γ/Γ in the self-discharging process. Other parameters are: (a) $\gamma/\Gamma = 1$, $\kappa/\Gamma = 0.5$; (b) $\lambda/\Gamma = 0.1$, $\kappa/\Gamma = 0.5$.

5 Conclusion

In this paper, we have discussed the influences of the memory effect of the reservoir and the coupling between two parts of multiple environments on the charging and the self-discharging process of a quantum battery. In particular, we have considered the charging and the self-discharging process of the quantum battery in two composite environments, each containing a single-mode cavity decaying to a reservoir. For the charging process of the quantum battery, we have shown that increasing two-mode coupling can improve the charging performance (i.e., the stored energy, the average power, and the ergotropy) of the quantum battery. Different from two-mode coupling, we have demonstrated that the memory effect of the reservoir environment is not conducive to the charging process of the quantum battery, which is in sharp contrast to the previous studies where the memory effect of the reservoir environment can greatly improve the charging performance of quantum battery [74]. Using pseudomodes theory, we have revealed that this unconventional behavior is due to the earlier the quantum battery dissipates energy into the environment as memory time increases. Moreover, we have also studied the self-discharging process of the quantum battery in composite environments. We show that increasing the memory effect of the reservoir environment and the coupling strength of the two-mode coupling can effectively suppress the self-discharging process of the quantum battery, thus realizing the long-term storage of energy of the quantum battery in the composite environment.

Our findings evidence that when the environment is composite the underlying physical mechanisms may be counterintuitive. We note that the coupling between the two modes has the same effect on the charging process and resisting the self-discharging process of the quantum battery, which is different from the memory effect of the reservoir. Thus, environmental coupling is revealed as an effective tool to improve the performance of the open quantum battery. It is worth noting that our system has the advantage of making the effects of the memory effect of the reservoir and the coupling between the environmental parts on the performance of a quantum battery emerges in a clear way and, at the same time, of being simple enough to find feasibility within the current experimental technologies, for instance, in circuit QED [83] or in simulating all-optical setups [84]. Our results may help for realizing the optimal charging and long-term stable energy storage of the quantum battery in composite environments.

Acknowledgements This work was supported by the National Natural Science Foundation of China under grant Nos. 12204348, 12074027, 11434015, 61227902, 61835013, 11611530676, and KZ201610005011, the National Key R&D Program of China under grant No. 2016YFA0301500, and SPRPCAS under grant Nos. XDB01020300 and XDB21030300.

Appendix A: The ergotropy functional

Let ρ_B be the density matrix of a system characterized by the Hamiltonian H_B, denoted by their spectral decomposition

$$\rho_B^{(p)} \equiv \sum r_n \left|r_n\right\rangle\!\left\langle r_n\right|, H_B \equiv \sum e_n \left|e_n\right\rangle\!\left\langle e_n\right|, \tag{A1}$$

where $\{|r_n\rangle\}_n$ and $\{|e_n\rangle\}_n$ represent the eigenvectors of ρ and H, respectively, and $r_0 \geqslant r_1 \geqslant \cdots \geqslant r_n$ and $e_0 \leqslant e_1 \leqslant \cdots \leqslant e_n$ are the associated eigenvalues, which have been properly ordered. The passive counterpart of ρ_B is defined as the following density matrix:

$$\sigma_{\rho_B} \equiv \sum r_n \left|e_n\right\rangle\!\left\langle e_n\right|. \tag{A2}$$

Accordingly, the ergotropy of the state ρ_B can be conveniently expressed as

$$W_B(\tau) = \mathrm{tr}\left(\rho_B(\tau)H_B\right) - \mathrm{tr}\left(\sigma_{\rho_B}H_B\right). \tag{A3}$$

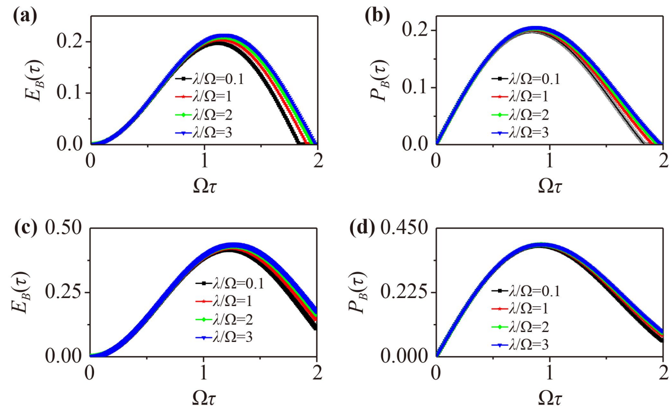


Fig. B1 **(a–d)** The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of the quantum battery as a function of the dimensionless quantity $\Omega\tau$ for different values of the parameter λ/Ω. Other parameters are chosen as (a), (b) $c_1(0)=\sqrt{3}/2,\ h(0)=1/2,\ \gamma/\Omega=1,\ \kappa/\Omega=0.5,\ \Gamma/\Omega=1$; (c), (d) $c_1(0)=\sqrt{14}/4,\ h(0)=\sqrt{2}/4,\ \kappa/\Omega=0.5,\ \gamma/\Omega=1,\ \Gamma/\Omega=1$. The stored energy $E_B(\tau)$ and the average charging power $P_B(\tau)$ of a quantum battery are in unit of ω_0.

Appendix B: Charging performance of a quantum battery when the system is in initially entangled states

In this section, we consider the effects of the memory effect of the reservoir environment and the coupling between the environments on the charging performance of the quantum battery when the system composed of the quantum battery and the charger is in different initial states. We assume that the system is initially in state $|\Phi(0)\rangle = c_1(0)|10\rangle_{CB} + h(0)|01\rangle_{CB}$, where $|c_1(0)|^2 + |h(0)|^2 = 1$. Since we consider the charging process of the quantum charger to the quantum battery, the initial internal energy of the quantum charger is greater than the initial internal energy of the quantum battery (i.e., $|c_1(0)|^2 > |h(0)|^2$). For convenience, we first choose two different initial entangled states ($c_1(0)=\sqrt{3}/2$ or $c_1(0)=\sqrt{14}/4$) in Fig. B1 to study the effect of the memory effect of the reservoir environment on the charging performance of the quantum battery. According to the charging process of the quantum battery (that is, the third part of this paper) and Eqs. (12) and (13), the stored energy $E_B(\tau)=\omega_0|h(\tau)|^2$ and the average charging power $P_B(\tau)=\omega_0|h(\tau)|^2/\tau$ of the quantum battery can be obtained. In the following, for the different initial states, we can analyze the influence of the memory effect of the reservoir environment on the charging performance of the quantum battery. As shown in Figs. B1(a)–(d), we show that both the stored energy and the average power increase with the increase of the parameter λ/Ω for different initial entangled states. This means that, for different initial entangled states, the increase in the memory time (i.e., $1/\lambda$) of the reservoir environment is

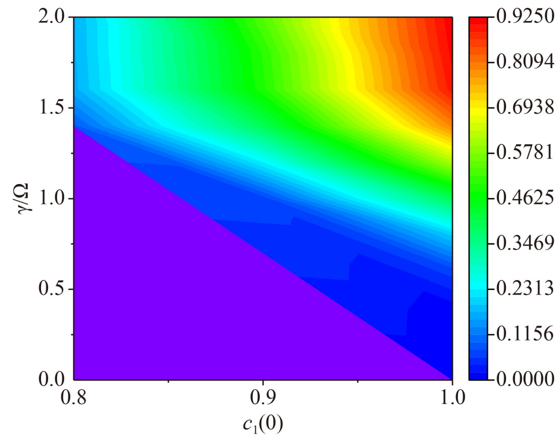


Fig. B2 Maximum stored energy $E_{\max}$ of the quantum battery as a function of γ/Ω and $c_1(0)$. Other parameters are: $\lambda/\Omega = 0.1$, $\Gamma/\Omega = 1$, $\kappa/\Omega = 0.5$. Maximum stored energy $E_{\max}$ of the quantum battery is in the unit of ω_0.

not conducive to the charging process of the quantum battery.

Furthermore, for the different initial entangled states, the influence of the coupling of the environmental part on the charging process of the quantum battery is also discussed. The variation of maximum stored energy with the coupling γ/Ω of the two-mode and the weight $c_1(0)$ of the state $|10\rangle_{CB}$ is described in Fig. B2. The purple area in Fig. B2 indicates that the quantum battery has no energy storage capacity (i.e., $E_{\max} \leq 0$). However, the maximum energy storage of the quantum battery can be stimulated and then increases with the increase of γ/Ω. This means that the optimal charging process of a quantum battery under different initial entangled states of the system can be achieved by utilizing environmental coupling. The above results are consistent with those in our paper. Therefore, the results in this paper should also be applicable to the system in other initial entangled states.

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