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Inequality relations for the hierarchy of quantum correlations in two-qubit systems

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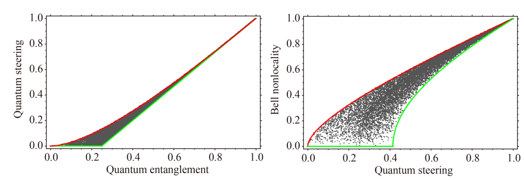
ABSTRACT

Entanglement, quantum steering and Bell nonlocality can be used to describe the distinct quantum correlations of quantum systems. Because of their different characteristics and application fields, how to divide them quantitatively and accurately becomes particularly important. Based on the sufficient and necessary criterion for quantum steering of an arbitrary two-qubit T-state, we derive the inequality relations between quantum steering and entanglement as well as between quantum steering and Bell nonlocality for the T-state. Additionally, we have verified those relations experimentally.



Keywords entanglement, quantum steering, Bell nonlocality, inequality relation

1 Introduction

The famous Einstein–Podolsky–Rosen (EPR) paradox was first introduced in Ref. [1], which indicated that there are some conflicts between quantum mechanics and local realism. In order to answer the EPR paradox, Schrödinger introduced the notion of entanglement [2]. Since then, entanglement has received extensive attentions [3]. From the perspective of mathematics, an entangled state refers to it cannot be represented by a probabilistic mixture of product states. With the development of the entanglement theory, some researches about entanglement measures have been proposed, such as entanglement of formation [4], relative entropy of entanglement [5], concurrence [4], and negativity [6]. As its miraculous feature, entanglement has been used to accomplish quantum computers [7], quantum teleportation [8], cryptographic communications [9], and quantum error correction [10].

With the deepening of research, quantum steering [11, 12] and Bell nonlocality [13, 14] were revealed, which are the stronger quantum correlations than entanglement. The relevant investigations have ushered in an epoch of relentless exploration for quantum correlations and other issues [15–42]. Quantum steering and Bell nonlocality are the crucial resources for accomplishing quantum information tasks in practice. For quantum steering, the most striking feature is the asymmetry [15]. This characteristic plays a vital role in the one-sided device-independent quantum key distribution task [16–18]. Besides, by virtue of Bell nonlocality, quantum information tasks, such as the device-independent quantum key distribution [19], communication complexity [20], and genuine random number generation [21], can be realized.

According to the hierarchy of quantum correlations (mainly including entanglement, quantum steering and Bell nonlocality), the steerable states form a strict

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subset of entangled states, and also form a strict superset of Bell nonlocal states [12]. Quintino *et al.* [22] proved that entanglement, quantum steering and Bell nonlocality are genuinely different based on general measurements. On one hand, the sufficient criteria of quantum steering based on entanglement have been constructed [23–25]. For instance, the constraint relation between steerability and concurrence indicates that quantum steering can be detected via calculating the concurrence and purity [23]; the mapping criterion between quantum steering and entanglement shows that quantum steering of a two-qubit state can be detected by detecting entanglement of the newly constructed state [24, 25]. On the other hand, the detection of Bell nonlocality can be achieved by calculating entanglement [26–30]. Verstraete *et al.* [26] was the first to investigate the regions of possible extremal Bell inequality violations for a given value of the concurrence. Later, the constraint relations between Bell nonlocality and entanglement (mainly including entanglement of formation, relative entropy of entanglement, concurrence and negativity) were studied in Refs. [27–30].

At present, the sufficient and necessary criterion (SNC) of quantum steering has been given in the form of critical radius [38, 39]. This SNC criterion is analytic and numerically computable in the case of two-qubit T-states. And it can detect all steerable states from two-qubit T-states, which cannot be achieved by other sufficient criteria. However, the research on the relation between the SNC of quantum steering and other quantum correlations (entanglement and Bell nonlocality) is still lacking. Based on this SNC, we make an investigation on the relations between quantum steering and entanglement as well as between quantum steering and Bell nonlocality for two-qubit T-states. The main conclusions are listed as follows: the upper and lower boundaries of quantum steering can be exactly denoted as the monotone functions of entanglement; the upper and lower boundaries of Bell nonlocality are also the monotone functions of quantum steering. These two conclusions quantify the hierarchical relation among these quantum correlations. In other words, the lower boundaries of quantum steering and Bell nonlocality can be used to detect quantum steering and Bell nonlocality respectively; the upper boundaries of quantum steering and Bell nonlocality reveal that the steerable states form a strict subset of entangled states, and also form a strict superset of Bell nonlocal states. Meanwhile, we verify these relations by the experiments. The results from experiment fit the theoretical prediction pretty well. Therefore, our study quantitatively describes the internal relation among these quantum correlations and is helpful to find all steerable states from two-qubit T-states.

2 Theoretical preparation

In general, a two-qubit state ϱ can be described by a density matrix, which is a positive (semi-definite) unit-trace operator over $H_{\rm A}\otimes H_{\rm B}$, where $H_{\rm A}$ and $H_{\rm B}$ are 2-dimensional (2D) Hilbert spaces. If we use the Pauli matrices $\{\sigma_i\}_{i=0}^3$ (with $\sigma_0=\mathbb{1}$) as the basis of the Hilbert space $H_{\rm A}$ ($H_{\rm B}$), the state ϱ can be written as

$$\varrho=\frac{1}{4}\sum_{i,j=0}^{3}\Theta_{ij}\sigma_i\otimes\sigma_j, \tag{1}$$

where $\Theta_{ij}={\rm Tr}\left[\varrho\left(\sigma_i\otimes\sigma_j\right)\right]$ denote the elements of the matrix Θ. The matrix Θ is usually rewritten as the Bloch tensor form

$$\Theta=\begin{pmatrix}1 & \boldsymbol{b}^{\rm T}\\ \boldsymbol{a} & T\end{pmatrix}, \tag{2}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are Alice's and Bob's Bloch vectors, and T is the correlation matrix of the state ϱ. In order to conveniently represent the correlation matrix T of a two-qubit state ϱ, we use the notation $T(\varrho)$. Specially, when both $\boldsymbol{a}$ and $\boldsymbol{b}$ are zero vectors ($\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{0}$), the state is called as a two-qubit T-state. And we use ρ to denote a general two-qubit T-state in this paper.

2.1 Measure of entanglement

Concurrence is usually used as a measure for entanglement of two-qubit states. For an arbitrary pure state $|\psi\rangle$, its concurrence can be defined as [4]

$$E\left(|\psi\rangle\right)=|\langle\psi|\tilde{\psi}\rangle|, \tag{3}$$

where $|\tilde{\psi}\rangle=(\sigma_2\otimes\sigma_2)|\psi\rangle^*$ represents the spin-flipped state of $|\psi\rangle$. Here σ_2 is the Pauli-y matrice, and $|\psi\rangle^*$ is the complex conjugate state of $|\psi\rangle$. For an arbitrary two-qubit state ϱ, its spin-flipped state $\tilde{\varrho}$ can be given by $\tilde{\varrho}=(\sigma_2\otimes\sigma_2)\varrho^*(\sigma_2\otimes\sigma_2)$. Considering its decomposition $\varrho=\sum_n p_n|\psi_n\rangle\langle\psi_n|$, the concurrence is defined by the convex-roof as follows [10, 40]:

$$E\left(\varrho\right)=\min_{\{p_n,|\psi_n\rangle\}}\sum_n p_nE\left(|\psi_n\rangle\right). \tag{4}$$

The minimization is taken over all possible decompositions ϱ into pure states. An analytic solution of concurrence can be calculated by [4]

$$E\left(\varrho\right)=\max\left\{0,\ 2\lambda_{\max}\left(\sqrt{\varrho\tilde{\varrho}}\right)-{\rm Tr}\left(\sqrt{\varrho\tilde{\varrho}}\right)\right\}, \tag{5}$$

where $\lambda_{\max}(X)$ is the maximum eigenvalue of the matrix X. Specially, for an arbitrary two-qubit T-state ρ, we have $\rho=\tilde{\rho}$. Therefore, its concurrence can be given by

$$E\left(\rho\right)=\max\left\{0,\ \ 2\lambda_{\max}\left(\rho\right)-1\right\}. \tag{6}$$

2.2 Measure of quantum steering

In general, the maximum violation degree of the steering inequality is used to quantify the steerablity of a quantum state. However, the violations of the steering inequalities with finite measurement can only be considered as sufficient conditions for quantum steering. Nguyen *et al.* [38, 39] presented the SNC of quantum steering for two-qubit T-states. For an arbitrary two-qubit T-state ρ, the critical radius $R(\rho)$ can be reduced to

$$R(\rho) = 2\pi N_T(\rho)\left|\det[T(\rho)]\right|, \tag{7}$$

where $N_T^{-1}(\rho) = \iint dS(\boldsymbol{v})\left\|T^{-1}(\rho)\boldsymbol{v}\right\|^{-4}$, and $\|X\| = \sqrt{\mathrm{Tr}(XX^\dagger)}$ is the norm of the matrix X. Here the integration runs over the surface of the unit sphere. Considering the unit vector $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \begin{pmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\end{pmatrix}^{\mathrm{T}}$, we have $\mathrm{d}S(\boldsymbol{v}) = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. This SNC indicates that a two-quibt T-state ρ can be steerable if and only if the state ρ meets the condition $R(\rho) < 1$. It has been proved in Ref. [41] that this condition $R(\rho) < 1$ is equivalent to the condition $\iint \mathrm{d}S(\boldsymbol{v})\|T(\rho)\boldsymbol{v}\| > 2\pi$.

In our recent work [42], we have also put forward the SNC of quantum steering for two-qubit T-states ρ by calculating the maximum violation $F(\rho)$ of EPR steering inequality with infinite projection measurements and constructing a local hidden state model. And the maximum violation $F(\rho)$ can be denoted as

$$F(\rho) = \frac{1}{4\pi}\iint \mathrm{d}S(\boldsymbol{v})\|T(\rho)\boldsymbol{v}\|. \tag{8}$$

This SNC indicates that a two-quibt T-state ρ can be steerable if and only if the state ρ meets the relation $F(\rho) > 1/2$. Therefore, the maximum violation degree of EPR steering inequality with infinite projection measurements can be used to quantify the steering of a two-qubit T-state, i.e., EPR steering $\mathcal{F}(\rho)$ can be characterized as

$$\mathcal{F}(\rho) = \frac{\max\{F(\rho) - 1/2, 0\}}{F_{\max} - 1/2}, \tag{9}$$

where $F_{\max} = \max_\rho F(\rho)$. Both entanglement and Bell nonlocality are known to satisfy convexity and local unitary transformation invariance. Likewise, quantum steering has these two properties [42].

Property 1 If a two-qubit T-state ρ can be written as an ensemble $\rho = \sum_i p_i\rho_i$ which is made up of T-states ρ_i with the probability p_i, then the EPR steering $\mathcal{F}(\rho)$ conforms to the following relation:

$$\mathcal{F}(\rho) \leqslant \sum_i p_i\mathcal{F}(\rho_i). \tag{10}$$

Property 2 Given a two-qubit T-state ρ, we consider a family of states ρ' which are formed by the local unitary transformation applied to the original state ρ, i.e., $\rho' = (U_{\mathrm{A}}\otimes U_{\mathrm{B}})\rho(U_{\mathrm{A}}\otimes U_{\mathrm{B}})^\dagger$, where U_{A} and U_{B} are the unitary transformations on Alice's and Bob's sides, respectively. For these states, we have

$$\mathcal{F}(\rho') = \mathcal{F}(\rho). \tag{11}$$

2.3 Measure of Bell nonlocality

Violation of Bell inequality in quantum mechanics tells us that quantum correlations are quite different from classical correlations. In the case of two-qubit states, Clauser–Horne–Shimony–Holt (CHSH) inequality is a well-known Bell inequality and has the important property that an arbitrary two-qubit pure state violates the CHSH inequality if and only if it is entangled [43]. Considering the Hilbert space $\mathcal{H}_{\mathrm{AB}} = \mathcal{H}_{\mathrm{A}}\otimes\mathcal{H}_{\mathrm{B}}$, the Bell-operator associated with the CHSH inequality can be given by

$$B = M_{\mathrm{A}_1}\otimes(M_{\mathrm{B}_1} + M_{\mathrm{B}_2}) + M_{\mathrm{A}_2}\otimes(M_{\mathrm{B}_1} - M_{\mathrm{B}_2}), \tag{12}$$

where M_{A_1}, M_{A_2} and M_{B_1}, M_{B_2} are the projection measurements on sides A and B, respectively. For a general two-qubit state ϱ, the CHSH inequality can be expressed as

$$|\mathrm{Tr}(\varrho B)| \leqslant 2. \tag{13}$$

In terms of the Horodecki's theorem [43], the maximum expected value of the Bell-operator has the following form

$$\max_{\{M_{\mathrm{A}_1},M_{\mathrm{A}_2},M_{\mathrm{B}_1},M_{\mathrm{B}_2}\}}|\mathrm{Tr}(\varrho B)| = 2\sqrt{G(\varrho)}, \tag{14}$$

where $G(\varrho) = \mathrm{Tr}\left[T^{\mathrm{T}}(\varrho)T(\varrho)\right] - \lambda_{\min}\left[T^{\mathrm{T}}(\varrho)T(\varrho)\right]$, and $\lambda_{\min}\left[T^{\mathrm{T}}(\varrho)T(\varrho)\right]$ is the minimum eigenvalue of the symmetric matrix $T^{\mathrm{T}}(\varrho)T(\varrho)$. Obviously, the CHSH inequality holds when the state ϱ meets the relation $G(\varrho) \leqslant 1$. In other words, the state ϱ is Bell nonlocal if and only if $G(\varrho) > 1$. Thus, Bell nonlocality $\mathcal{G}(\varrho)$ can be characterized as

$$\mathcal{G}(\varrho) = \sqrt{\frac{\max\{0,\ G(\varrho) - 1\}}{G_{\max} - 1}}, \tag{15}$$

where $G_{\max} = \max_\varrho G(\varrho)$.

3 Experimental preparation

In this section, we briefly introduce our experimental setup which is used to provide some quantum states to investigate our relations below. A linear optical setup is selected as it has a mature technology to prepare and control quantum states. One can encode state $|0\rangle$ (or $|1\rangle$) into the horizontal (vertical) polarization of the photon.

Figure 1 is a diagrammatic sketch of our experimental setup. It can be divided into three main parts: (a) source stage, (b) unbalanced Mach–Zehnder device(UMZ) stage, and (c) tomography stage. In our experiment, (a) source stage is made of an optical maser (OM), three half-wave plates (HWPs), a polarizing beam splitter (PBS), a beam splitter (BS), four mirrors, two type-I β-barium borate crystals (BBOs) and a reflecting prism (RP). First, a beam of polarized light (130 mW, 405 nm), which is emitted by an OM, will be modulated by a HWP which is designed to change the portion between $|H\rangle$ and $|V\rangle$. Then the PBS can divide the pumped beam into two directions, the transmitted light along with its original direction is state $|H\rangle$ while the reflected light is state $|V\rangle$. The sequential two HWPs will convert state $|H\rangle$ to $|D\rangle$ and state $|V\rangle$ to $|A\rangle$, where $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. The states $|D\rangle$ and $|A\rangle$ are gathered by a BS. So we reach the states $\rho(p) = p|D\rangle\langle D| + (1-p)|A\rangle\langle A|$. Such states pass through two type-I β-BBO crystals $(6.0\ \mathrm{mm} \times 6.0\ \mathrm{mm} \times 0.5\ \mathrm{mm})$ with its optic axis cut at 29.2° will take place a process of spontaneous parametric down-conversion [44], then become to $\rho(p) = p|\psi^+\rangle\langle\psi^+| + (1-p)|\psi^-\rangle\langle\psi^-|$ where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|HH\rangle \pm |VV\rangle)$, the parameter p is controlled by the first HWP. By the source stage, a set of two-qubit entangled states are generated. (b) Unbalanced Mach–Zehnder device (UMZ) stage is made of two BSs, two attenuators (ATTs), two mirrors and a HWP. It is designed to prepare our desired states. The generated photons passing through the BBOs split into two directions: A and B. Photon in A side keeps its status while photon in B side will be divided into two parts: b1 and b2. There is an ATT in every part aim into adjust the portion between those two parts. Importantly, the UMZ module [45] are used in B side to generate states $|\varphi^\pm\rangle$. Therefore, we obtain the states

$$\rho(p,q) = p\left(q\left|\psi^+\rangle\langle\psi^+\right| + (1-q)\left|\varphi^+\rangle\langle\varphi^+\right|\right) + (1-p)\left(q\left|\psi^-\rangle\langle\psi^-\right| + (1-q)\left|\varphi^-\rangle\langle\varphi^-\right|\right), \tag{16}$$

where $|\varphi^\pm\rangle = \frac{1}{\sqrt{2}}(|HV\rangle \pm |VH\rangle)$ are Bell states, the parameter q is controlled by the two ATTs. (c) Tomography stage is made of two quarter-wave plates (QWPs), two HWPs, two interference filter (IFs), two PBSs, four single photon detectors (SPDs) and a coincidence. The combination of HWPs, QWPs and PBSs can realize a set of measurements so that we can complete a tomography process. In this work, the states we prepare are a set of typical two-qubit T-states, the rest of the experiment is continued with those typical states.

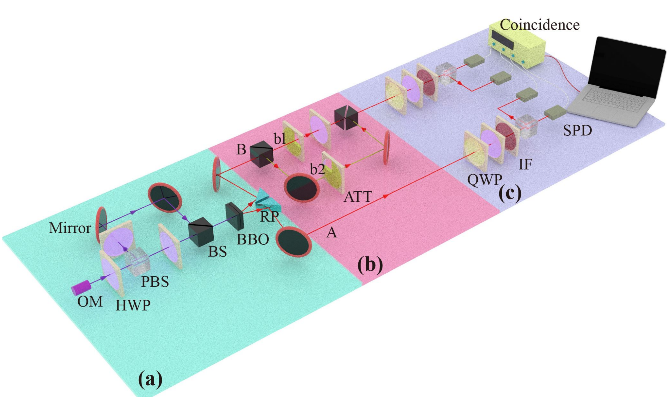

Fig. 1 A diagrammatic sketch of experimental setup: **(a)** source stage, **(b)** unbalanced Mach–Zehnder device stage and **(c)** tomography stage. OM is short for optical maser, HWP for half-wave plate, PBS for polarizing beam splitter, BS for beam splitter, BBO for type-I β-barium borate crystal, RP for reflecting prism, ATT for attenuator, QWP for quarter wave plater, IF for 3-nm interference filter and SPD for single photon detector.

4 Relations among quantum correlations

The hierarchy of quantum correlations indicates that the steerable states form a strict subset of entangled states, and also form a strict superset of Bell-nonlocal states. However, the hierarchy does not quantify their differences in these quantum correlations. Therefore, it is an important task to investigate the relation between entanglement and quantum steering, as well as the relation between quantum steering and Bell nonlocality. Before we do that, we consider two special types of two-qubit T-states as examples, with the aim at clarifying that the relation between these three quantum correlations depends on the structure of the quantum states.

Example I : Werner state. This state can be described as

$$\mathrm{W}(\alpha) = \alpha\left|\psi^+\rangle\langle\psi^+\right| + (1-\alpha)\frac{\mathbb{1}}{2}\otimes\frac{\mathbb{1}}{2}, \tag{17}$$

where the parameter α represents the probability of $|\psi^+\rangle$. For the Werner state $\mathrm{W}(\alpha)$, its EPR steering can be represented by the concurrence, i.e., $\mathcal{F}[\mathrm{W}(\alpha)] = \frac{1}{3}\max\{4E[\mathrm{W}(\alpha)]-1, 0\}$, and its Bell nonlocality can be represented by the EPR steering, i.e., $\mathcal{G}[\mathrm{W}(\alpha)] = \sqrt{\frac{1}{2}\max\{0, \mathcal{F}^2[\mathrm{W}(\alpha)] + 2\mathcal{F}[\mathrm{W}(\alpha)] - 1\}}$. Obviously, the Werner state is steerable if and only if its concurrence exceeds $1/4$; the Werner state is Bell nonlocal if and only if its EPR steering exceeds $\sqrt{2}-1$.

Example II : Maximally nonlocal mixed state. This state can be generally written as

$$\mathrm{MNM}(e) = \frac{1+e}{2}\left|\psi^+\rangle\langle\psi^+\right| + \frac{1-e}{2}\left|\psi^-\rangle\langle\psi^-\right|, \tag{18}$$

where $0 \leqslant e \leqslant 1$. Clearly the concurrence is equal to the Bell nonlocality for the state $\mathrm{MNM}(e)$, i.e., $E[\mathrm{MNM}(e)] = \mathcal{G}[\mathrm{MNM}(e)] = e$. And the EPR steering can be denoted as $\mathcal{F}[\mathrm{MNM}(e)] = \frac{e^2}{\sqrt{1-e^2}}\ln\frac{1+\sqrt{1-e^2}}{e}$. Therefore, this state has EPR steering if and only if it is entanged; this state possess Bell nonlocality iff it is steerable. This shows that the three quantum correlations are equivalent for the state $\mathrm{MNM}(e)$.

The results of examples I and II clarify that the relation among these three quantum correlations depends on the structure of the quantum states. In order to get a more general relation, we will investigate the relation between concurrence and EPR steering, and the relation between EPR steering and Bell nonlocality for any two-qubit T-states in two subsections.

4.1 Inequality relation between concurrence and EPR steering

An arbitrary two-qubit T-state ρ has a decomposition in which each pure state has the same entanglememt [30], i.e., $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$. Here, $E(|\psi_i\rangle)=E(\rho)$ for each pure state $|\psi_i\rangle$. According to Ref. [40], the spin-flipped state $\tilde{\rho}$ can be denoted as $\tilde{\rho}=\sum_i p_i|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$. For the two-qubit T-state ρ, we have $\rho=\tilde{\rho}$. It indicates that the T-state has a special decomposition, i.e.,

$$\rho=\frac{1}{2}(\rho+\tilde{\rho})=\sum_i p_i\psi_i, \tag{19}$$

where $\psi_i=\frac{1}{2}\left(|\psi_i\rangle\langle\psi_i|+|\tilde{\psi_i}\rangle\langle\tilde{\psi_i}|\right)$ denotes a 2-rank T-state. According to Eqs. (3) and (6), we obtain an invariability that concurrence of the T-state ψ_i is equal to concurrence of the pure state $|\psi_i\rangle$, i.e., $E(\psi_i)=E(|\psi_i\rangle)$. Obviously, for each state ψ_i, we have $E(\psi_i)=E(\rho)$. And the eigenvalue of the matrix $\sqrt{T^{\mathrm{T}}(\psi_i)T(\psi_i)}$ can be given by $\Lambda(\psi_i)=diag\{1,\ E(\psi_i),\ E(\psi_i)\}=diag\{1,\ E(\rho),\ E(\rho)\}$. Therefore, these states $\{\psi_i\}_i$ have the same EPR steering. And the EPR steering can be reduced to $\mathcal{F}(\psi_i)=\frac{E^2(\rho)}{\sqrt{1-E^2(\rho)}}\ln\frac{1+\sqrt{1-E^2(\rho)}}{E(\rho)}$ for each state ψ_i. Based on the convexity of EPR steering as shown in Eq. (10), the EPR steering $\mathcal{F}(\rho)$ for the T-state ρ conforms to the relation $\mathcal{F}(\rho)\leqslant\mathcal{F}(\psi_i)$. Therefore, EPR steering is upper bounded by

$$\mathcal{F}(\rho)\leqslant\frac{E^2(\rho)}{\sqrt{1-E^2(\rho)}}\ln\frac{1+\sqrt{1-E^2(\rho)}}{E(\rho)}. \tag{20}$$

Eq. (20) reveals that the EPR steering possesses the upper boundary, which means that it may have a lower boundary. In order to seek its lower boundary, we consider three equally entangled Bell diagonal states, i.e.,

$$\begin{aligned}\rho_1&=\frac{1}{4}\left(\mathbb{1}_4+c_1\sigma_1\otimes\sigma_1-c_2\sigma_2\otimes\sigma_2+c_3\sigma_3\otimes\sigma_3\right),\\ \rho_2&=\frac{1}{4}\left(\mathbb{1}_4+c_2\sigma_1\otimes\sigma_1-c_3\sigma_2\otimes\sigma_2+c_1\sigma_3\otimes\sigma_3\right),\\ \rho_3&=\frac{1}{4}\left(\mathbb{1}_4+c_3\sigma_1\otimes\sigma_1-c_1\sigma_2\otimes\sigma_2+c_2\sigma_3\otimes\sigma_3\right),\end{aligned} \tag{21}$$

where $c_1\geqslant c_2\geqslant c_3\geqslant 0$ and $c_1+c_2+c_3>1$. Obviously, these states have the same EPR steering, i.e., $\mathcal{F}(\rho_1)=\mathcal{F}(\rho_2)=\mathcal{F}(\rho_3)$. In fact, the Werner state $\mathrm{W}(\alpha)$ can be produced by equal probability mixing of these three states ρ_1, ρ_2 and ρ_3, i.e., $\mathrm{W}(\alpha)=(\rho_1+\rho_2+\rho_3)/3$, where the probability $\alpha=(c_1+c_2+c_3)/3$. It is not difficult to find that the entanglement values of these states are the same, i.e., $E(\rho_1)=E(\rho_2)=E(\rho_3)=E[\mathrm{W}(\alpha)]=(3\alpha-1)/2$. Based on the convexity of EPR steering as shown in Eq. (10), the EPR steering conforms to the relation $\mathcal{F}(\rho_1)\geqslant\mathcal{F}[\mathrm{W}(\alpha)]=\max\{2\alpha-1,0\}$. Obviously, for the three Bell diagonal states $\{\rho_i\}_{i=1}^3$, the relation between the concurrence and EPR steering can be denoted as $\mathcal{F}(\rho_i)\geqslant\frac{1}{3}\max\{4E(\rho_i)-1,0\}$. Since both entanglement and EPR steering are not affected by local unitary operations, Therefore, for any two-qubit T-state ρ, EPR steering is lower bounded by

$$\mathcal{F}(\rho)\geqslant\frac{1}{3}\max\{4E(\rho)-1,0\}. \tag{22}$$

The upper boundary, corresponding to Eq. (20), reveals that the steerable states form a strict subset of entangled states; and the lower boundary of EPR steering, corresponding to Eq. (22), can be used to detect quantum steering. The upper and lower boundaries of EPR steering make up of the inequality relation between concurrence and EPR steering. If we only know the amount of concurrence of a given state, it is difficult to quantity the EPR steering. By using the inequality relation between the concurrence and EPR steering, we can obtain the upper and lower boundaries of the EPR steering and the class of states that can achieve the upper and lower boundaries. In the following, we verify the inequality relation by the experiments and numerical methods.

From experimental perspective, we prepare 30 quantum states to investigate Eqs. (22) and (20). We select parameters $\{p|\,0,0.05,0.1,0.15,0.2\}$ and $\{q|\,0,0.1,0.2,0.3,0.4,0.5\}$, combine with Eq. (16), we can obtain 30 different two-qubit T-states. To illustrate the accuracy of our experiment, the fidelities of the prepared states are calculated by $\bar{F}=\mathrm{Tr}\sqrt{\sqrt{\rho(p,q)}\rho\sqrt{\rho(p,q)}}$, where ρ is the desired state and $\rho(p,q)$ is the prepared state from the experiment, and we show them in Table 1.

The error bar is given by the Poisson distribution of the optical maser. In our experiment, we achieve a high average fidelity $\bar{F}=0.9982\pm0.0008$. This high fidelity guarantees the reliability of our experiment. By our experimental datum, we calculate their concurrence and steering and plot their relation in Fig. 2 with green dots. The blue line is the lower bound which indicated from Eq. (22) and the red line is the upper bound which indicated from Eq. (20). These green dots verify the relation between entanglement and EPR steering like we predicted in Eqs. (22) and (20). As we showed, all the experimental states are complying with Eqs. (22) and (20).

Table 1 The fidelities of experimental prepared states.

	$q=0$	$q=0.1$	$q=0.2$
$p=0$	0.9976 ± 0.0005	0.9979 ± 0.0018	0.9984 ± 0.0014
$p=0.05$	0.9970 ± 0.0001	0.9970 ± 0.0010	0.9993 ± 0.0005
$p=0.1$	0.9971 ± 0.0002	0.9969 ± 0.0021	0.9996 ± 0.0002
$p=0.15$	0.9970 ± 0.0001	0.9994 ± 0.0003	0.9990 ± 0.0003
$p=0.2$	0.9966 ± 0.0002	0.9986 ± 0.0007	0.9994 ± 0.0003
	$q=0.3$	$q=0.4$	$q=0.5$
$p=0$	0.9986 ± 0.0013	0.9992 ± 0.0007	0.9988 ± 0.0008
$p=0.05$	0.9976 ± 0.0009	0.9968 ± 0.0026	0.9988 ± 0.0008
$p=0.1$	0.9987 ± 0.0004	0.9953 ± 0.0017	0.9986 ± 0.0006
$p=0.15$	0.9987 ± 0.0004	0.9987 ± 0.0006	0.9990 ± 0.0004
$p=0.2$	0.9980 ± 0.0005	0.9990 ± 0.0003	0.9993 ± 0.0002

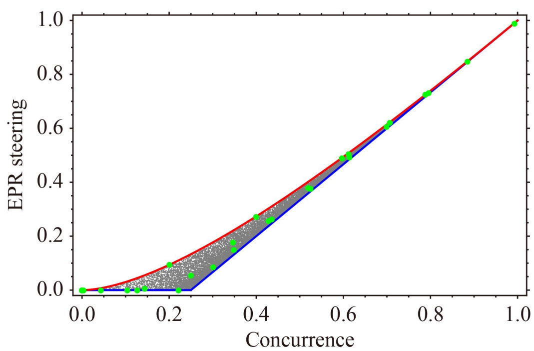

Fig. 2 EPR steering $\mathcal{F}(\rho)$ versus concurrence $E(\rho)$ for two-qubit T-states ρ. The upper bound (red solid line) can be achieved by the 2-rank T-states. And the lower bound (blue solid line) can be achieved by the Werner states. EPR steering $\mathcal{F}(\rho)$ along the Y axis, and concurrence $E(\rho)$ along the X axis for 5×10^4 randomly generated two-qubit T-states, where we show them with gray dots, by using a specific Mathematica package. The green dots are obtained by the experimental process with a set of Bell-diagonal states.

In order to verify that more states also satisfy this inequality relation, we investigate lots of randomly generated two-qubit T-states, which are plotted in Fig. 2 with gray dots. The result (as shown in Fig. 2) shows that the lower and upper boundaries of EPR steering can be achieved by the Werner state $\mathrm{W}(\alpha)$ and the 2-rank T-state $\psi=\frac{1}{2}\left(|\psi\rangle\langle\psi|+|\tilde{\psi}\rangle\langle\tilde{\psi}|\right)$, respectively. This shows that the upper and lower boundaries of the EPR steering are correct.

4.2 Inequality relation between EPR steering and Bell nonlocality

As before, we start investigating the relation between EPR steering and Bell nonlocality for any two-qubit T-states, and write the upper and lower boundaries of Bell nonlocality as certain functions of EPR steering.

To obtain the upper boundary of Bell nonlocality, we firstly consider a 2-rank T-state $\delta=\sum_{i=1}^{3}p_i\rho_i$, where these states ρ_1, ρ_2 and ρ_3, as shown in Eq. (21), are equally entangled Bell diagonal states, and the corresponding probabilities can be given by

$$p_1=\frac{(c_1-c_2)^2+(c_1-c_3)^2+(1-c_1)(2c_1-c_2-c_3)}{(c_1-c_2)^2+(c_2-c_3)^2+(c_1-c_3)^2},$$
$$p_2=\frac{(c_1-c_2)^2+(c_2-c_3)^2+(1-c_2)(2c_2-c_1-c_3)}{(c_1-c_2)^2+(c_2-c_3)^2+(c_1-c_3)^2},$$
$$p_3=\frac{(c_1-c_3)^2+(c_2-c_3)^2+(1-c_3)(2c_3-c_1-c_3)}{(c_1-c_2)^2+(c_2-c_3)^2+(c_1-c_3)^2}. \tag{23}$$

Obviously, these states ρ_1, ρ_2 and ρ_3 have the same entanglement, i.e., $E(\rho_1)=E(\rho_2)=E(\rho_3)=\frac{1}{2}(c_1+c_2+c_3-1)$. Thus, the EPR steering can be denoted as $\mathcal{F}(\delta)=\frac{E^2(\rho_i)}{\sqrt{1-E^2(\rho_i)}}\ln\frac{1+\sqrt{1-E^2(\rho_i)}}{E(\rho_i)}$. It is not difficult to find that the EPR steering of these states are the same, i.e., $\mathcal{F}(\rho_1)=\mathcal{F}(\rho_2)=\mathcal{F}(\rho_3)$. In addition, these states ρ_1, ρ_2 and ρ_3 have the same Bell nonlocality. And the Bell nonlocality can be reduced to $\mathcal{G}(\rho_1)=\mathcal{G}(\rho_2)=\mathcal{G}(\rho_3)=\sqrt{\max\{0,c_1^2+c_2^2-1\}}$. Based on the convexity of EPR steering as shown in Eq. (10), the 2-rank T-state δ conforms to the relation $\mathcal{F}(\rho_i)\geqslant\mathcal{F}(\delta)$. For all two-qubit states ϱ, entanglement as the upper bound for Bell nonlocality [30], i.e., $\mathcal{G}(\varrho)\leqslant E(\varrho)$. Thus, the relation between EPR steering and Bell nonlocality can be denoted as $\mathcal{F}(\rho_i)\geqslant\frac{\mathcal{G}^2(\rho_i)}{\sqrt{1-\mathcal{G}^2(\rho_i)}}\ln\frac{1+\sqrt{1-\mathcal{G}^2(\rho_i)}}{\mathcal{G}(\rho_i)}$ for these Bell diagonal states ρ_i. Since both Bell nonlocality and EPR steering are not affected by local unitary operations, Bell nonlocality of any T-state ρ is upper bounded by

$$\mathcal{G}(\rho)\leqslant f^{(-1)}[\mathcal{F}(\rho)], \tag{24}$$

where $f^{(-1)}(x)$ denotes the inverse function of $f(x)=\frac{x^2}{\sqrt{1-x^2}}\ln\frac{1+\sqrt{1-x^2}}{x}$.

To obtain the lower boundary of Bell nonlocality, we consider the Bell diagonal state ρ_2 as shown in Eq. (21). Obviously, the state ρ_2 has a decomposition which is composed of three equally entangled Werner states, i.e., $\rho_2=\sum_{i=1}^{3}p_i\mathrm{W}_i$. Here, these Werner states can be expressed as

$$\mathrm{W}_1 = \frac{1}{4}\left[\mathbb{1}_4 + \alpha\left(-\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right)\right],$$
$$\mathrm{W}_2 = \frac{1}{4}\left[\mathbb{1}_4 + \alpha\left(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right)\right],$$
$$\mathrm{W}_3 = \frac{1}{4}\left[\mathbb{1}_4 + \alpha\left(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3\right)\right], \tag{25}$$

where $\alpha = c_1 + c_2 - c_3$. And the corresponding probabilities can be given by $p_1 = \frac{1}{2\alpha}(c_1 - c_3)$, $p_2 = \frac{1}{2\alpha}(c_1 + c_2)$ and $p_3 = \frac{1}{2\alpha}(c_2 - c_3)$, respectively. Obviously, these states have the same EPR steering, i.e., $\mathcal{F}(\mathrm{W}_1) = \mathcal{F}(\mathrm{W}_2) = \mathcal{F}(\mathrm{W}_3) = \max\{0, 2\alpha - 1\}$. Based on the convexity of EPR steering as shown in Eq. (10), the Bell diagonal state ρ_2 conforms to the relation $\mathcal{F}(\rho_2) \leqslant \mathcal{F}(\mathrm{W}_1) = \max\{0, 2(c_1 + c_2 - c_3) - 1\}$. Besides, the Bell nonlocality can be reduced to $\mathcal{G}(\rho_2) = \sqrt{\max\{0, c_1^2 + c_2^2 - 1\}}$. Therefore, for Bell diagonal state ρ_2, the relation between EPR steering and Bell nonlocality can be denoted as $\mathcal{F}(\rho_2) \leqslant \sqrt{2[1 + \mathcal{G}^2(\rho_2)]} - 1$, equivalently $\mathcal{G}(\rho_2) \geqslant \sqrt{\frac{1}{2}\max\{0, \mathcal{F}^2(\rho_2) + 2\mathcal{F}(\rho_2) - 1\}}$. Since both Bell nonlocality and EPR steering are not affected by local unitary operations, Bell nonlocality of any T-state ρ is lower bounded by

$$\mathcal{G}(\rho) \geqslant \sqrt{\frac{1}{2}\max\{0, \mathcal{F}^2(\rho) + 2\mathcal{F}(\rho) - 1\}}. \tag{26}$$

The upper boundary, corresponding to Eq. (24), reveals that the steerable states form a strict superset of Bell nonlocal states; and the lower boundary of Bell nonlocality, corresponding to Eq. (26), can be used to detect Bell nonlocality.

From experimental perspective, we also prepare 30 quantum states to investigate Eqs. (26) and (24). We select parameters $\{p | 0, 0.05, 0.1, 0.15, 0.2\}$ and $\{q | 0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ like we did before, combine with Eq. (16), we can obtain 30 different two-qubit T-states. Then, we calculate their steering and Bell nonlocality and plot their relation in Fig. 3 with green dots. The blue line is the lower bound which indicated from Eq. (26) and the red line is the upper bound which indicated from Eq. (24). These green dots verify the relation between EPR steering and Bell nonlocality like we predicted in Eqs. (26) and (24). As we showed, all the experimental states are complying with Eq. (26) and Eq. (24).

In order to show the more general situation of T-states comprehensively, we investigate lots of randomly generated two-qubit T-states, with gray dots in Fig. 3, to obtain the relation between EPR steering and Bell nonlocality. The result (as shown in Fig. 3) shows that the lower and upper boundaries of Bell nonlocality can be also achieved by the Werner state and the 2-rank T-state, respectively. This shows that the upper and lower boundaries of the Bell nonlocality are correct.

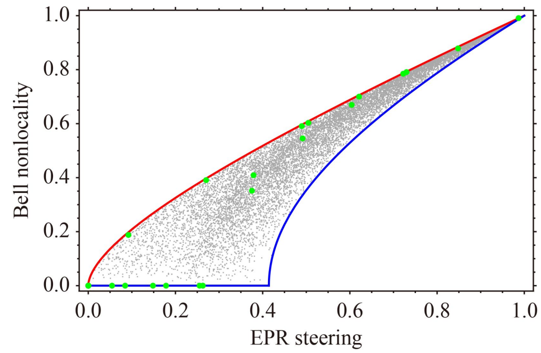

Fig. 3 Bell nonlocality $\mathcal{G}(\rho)$ versus EPR steering $\mathcal{F}(\rho)$ for two-qubit T-states ρ. The upper bound (red solid line) can be achieved by the 2-rank T-states. The lower bound (blue solid line) can be achieved by the Werner states. Bell nonlocality $\mathcal{G}(\rho)$ along the Y axis, and EPR steering $\mathcal{F}(\rho)$ along the X axis for 5×10^4 randomly generated two-qubit T-states, where we show them with gray dots, by using a specific Mathematica package. The green dots are obtained by the experimental process with a set of Bell-diagonal states.

5 Conclusion

In this paper, we have derived the inequality relations between entanglement and quantum steering as well as between quantum steering and Bell nonlocality for two-qubit T-states. We use some T-states to test these relations with a set of Bell-diagonal states experimentally. The inequality relations show that the upper and lower boundaries of quantum steering can be exactly expressed as the monotone functions of entanglement, and that the upper and lower boundaries of Bell nonlocality can be also represented by the monotone functions of quantum steering. One of the more interesting findings is that the upper and lower boundaries can be achieved by the 2-rank T-states and the Werner states, respectively. These results indicate that these quantum correlations are equivalent if and only if the rank of a two-qubit T-state is not greater than 2. Specifically, the lower boundaries, corresponding to Eqs. (22) and (26), can be used to detect quantum steering and Bell nonlocality, respectively. And the upper boundaries, corresponding to Eqs. (20) and (24), reveal that the steerable states form a strict subset of entangled states, and also form a strict superset of Bell nonlocal states. Our researches may be helpful to make more accurate use of these quantum correlations in future quantum information tasks.

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References

1. A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered

complete, *Phys. Rev.* 47(10), 777 (1935)

2. E. Schrödinger, Discussion of relations between separated systems, *Math. Proc. Camb. Philos. Soc.* 31(4), 555 (1935)
3. R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* 81(2), 865 (2009)
4. W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, *Phys. Rev. Lett.* 80(10), 2245 (1998)
5. M. Piani, Relative entropy of entanglement and restricted measurements, *Phys. Rev. Lett.* 103(16), 160504 (2009)
6. A. Miranowicz and A. Grudka, Ordering two-qubit states with concurrence and negativity, *Phys. Rev. A* 70(3), 032326 (2004)
7. A. Ekert and R. Jozsa, Quantum computation and Shor' s factoring algorithm, *Rev. Mod. Phys.* 68(3), 733 (1996)
8. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.* 70(13), 1895 (1993)
9. N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, *Rev. Mod. Phys.* 74(1), 145 (2002)
10. C. H. Bennett, D. P. DiVincenzo, J. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, *Phys. Rev. A* 54(5), 3824 (1996)
11. H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, entanglement, nonlocality, and the Einstein–Podolsky–Rosen paradox, *Phys. Rev. Lett.* 98(14), 140402 (2007)
12. R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, *Rev. Mod. Phys.* 92(1), 015001 (2020)
13. J. S. Bell, On the Einstein–Podolsky–Rosen paradox, *Physics* 1(3), 195 (1964)
14. N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* 86(2), 419 (2014)
15. W. X. Zhong, G. L. Cheng, and X. M. Hu, One-way Einstein–Podolsky–Rosen steering via atomic coherence, *Opt. Express* 25(10), 11584 (2017)
16. C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, *Phys. Rev. A* 85(1), 010301(R) (2012)
17. B. Opanchuk, L. Arnaud, and M. D. Reid, Detecting faked continuous-variable entanglement using one-sided device independent entanglement witnesses, *Phys. Rev. A* 89(6), 062101 (2014)
18. N. Walk, S. Hosseini, J. Geng, O. Thearle, J. Y. Haw, S. Armstrong, S. M. Assad, J. Janousek, T. C. Ralph, T. Symul, H. M. Wiseman, and P. K. Lam, Experimental demonstration of Gaussian protocols for one-sided device-independent quantum key distribution, *Optica* 3(6), 634 (2016)
19. C. M. Zhang, M. Li, H. W. Li, Z. Q. Yin, D. Wang, J. Z. Huang, Y. G. Han, M. L. Xu, W. Chen, S. Wang, P. Treeviriyanupab, G. C. Guo, and Z. F. Han, Decoy-state measurement-device independent quantum key distribution based on the Clauser–Horne–Shimony–Holt inequality, *Phys. Rev. A* 90(3), 034302 (2014)
20. Č. Brukner, M. Żukowski, J. W. Pan, and A. Zeilinger, Bell's inequalities and quantum communication complexity, *Phys. Rev. Lett.* 92(12), 127901 (2004)
21. S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, *Nature* 464(7291), 1021 (2010)
22. M. T. Quintino, T. Vertesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Inequivalence of entanglement, steering, and Bell nonlocality for general measurements, *Phys. Rev. A* 92(3), 032107 (2015)
23. X. G. Fan, H. Yang, F. Ming, D. Wang, and L. Ye, Constraint relation between steerability and concurrence for two-qubit states, *Ann. Phys.* 533(8), 2100098 (2021)
24. C. Chen, C. L. Ren, X. J. Ye, and J. L. Chen, Mapping criteria between nonlocality and steerability in qudit–qubit systems and between steerability and entanglement in qubit-qudit systems, *Phys. Rev. A* 98(5), 052114 (2018)
25. D. Das, S. Sasmal, and S. Roy, Detecting Einstein–Podolsky–Rosen steering through entanglement detection, *Phys. Rev. A* 99(5), 052109 (2019)
26. F. Verstraete and M. M. Wolf, Entanglement versus bell violations and their behavior under local filtering operations, *Phys. Rev. Lett.* 89(17), 170401 (2002)
27. K. Bartkiewicz, B. Horst, K. Lemr, and A. Miranowicz, Entanglement estimation from Bell inequality violation, *Phys. Rev. A* 88(5), 052105 (2013)
28. B. Horst, K. Bartkiewicz, and A. Miranowicz, Two-qubit mixed states more entangled than pure states: Comparison of the relative entropy of entanglement for a given nonlocality, *Phys. Rev. A* 87(4), 042108 (2013)
29. K. Bartkiewicz, K. Lemr, A. Černoch, and A. Miranowicz, Bell nonlocality and fully entangled fraction measured in an entanglement-swapping device without quantum state tomography, *Phys. Rev. A* 95(3), 030102(R) (2017)
30. Z. F. Su, H. S. Tan, and X. Y. Li, Entanglement as upper bound for the nonlocality of a general two-qubit system, *Phys. Rev. A* 101(4), 042112 (2020)
31. M. Li, M. J. Zhao, S. M. Fei, and Z. X. Wang, Experimental detection of quantum entanglement, *Front. Phys.* 8(4), 357 (2013)
32. Z. R. Zhong, X. Wang, and W. Qin, Towards quantum entanglement of micromirrors via a two-level atom and radiation pressure, *Front. Phys.* 13(5), 130319 (2018)
33. Q. Dong, A. J. Torres-Arenas, G. H. Sun, W. C. Qiang, and S. H. Dong, Entanglement measures of a new type pseudo-pure state in accelerated frames, *Front. Phys.* 14(2), 21603 (2019)
34. P. Zhang, Quantum entanglement in the Sachdev–Ye–Kitaev model and its generalizations, *Front. Phys.* 17(4), 43201 (2022)
35. Y. Y. Yang, W. Y. Sun, W. N. Shi, F. Ming, D. Wang, and L. Ye, Dynamical characteristic of measurement uncertainty under Heisenberg spin models with Dzyaloshinskii–Moriya interactions, *Front. Phys.* 14(3), 31601 (2019)

36. L. Y. Cheng, F. Ming, F. Zhao, L. Ye, and D. Wang, The uncertainty and quantum correlation of measurement in double quantum-dot systems, *Front. Phys.* 17(6), 61504 (2022)
37. Y. Cao, D. Wang, X. G. Fan, F. Ming, Z. Y. Wang, and L. Ye, Complementary relation between quantum entanglement and entropic uncertainty, *Commum. Theor. Phys.* 73(1), 015101 (2021)
38. H. C. Nguyen and T. Vu, Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes, *Europhys. Lett.* 115(1), 10003 (2016)
39. H. C. Nguyen, H. V. Nguyen, and O. Gühne, Geometry of Einstein–Podolsky–Rosen correlations, *Phys. Rev. Lett.* 122(24), 240401 (2019)
40. X. G. Fan, W. Y. Sun, Z. Y. Ding, H. Yang, F. Ming, D. Wang, and L. Ye, Universal complementarity between coherence and intrinsic concurrence for two-qubit states, *New J. Phys.* 21(9), 093053 (2019)
41. S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, Einstein–Podolsky–Rosen steering and the steering ellipsoid, *J. Opt. Soc. Am. B* 32(4), A40 (2015)
42. X. G. Fan, H. Yang, F. Ming, X. K. Song, D. Wang, and L. Ye, Necessary and sufficient criterion of steering for two-qubit T states, arXiv: 2103.04280v1 (2021)
43. R. Horodecki, P. Horodecki, and M. Horodecki, Violating bell inequality by mixed spin-1/2 states: Necessary and sufficient condition, *Phys. Lett. A* 200(5), 340 (1995)
44. P. G. Kwiat, E. Waks, A. G. White, I. Appelbaum, and P. H. Eberhard, Ultrabright source of polarization-entangled photons, *Phys. Rev. A* 60(2), R773 (1999)
45. A. Aiello, G. Puentes, D. Voigt, and J. P. Woerdman, Maximally entangled mixed-state generation via local operations, *Phys. Rev. A* 75(6), 062118 (2007)