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RESEARCH ARTICLE

# Hyperentanglement-assisted hyperdistillation for hyper-encoding photon system

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In quantum information processing, the quality of photon system is decreased by the inevitable interaction with environment, which will greatly reduce the efficiency and security of quantum information processing. In this paper, we propose hyperentanglement-assisted hyperdistillation schemes to guarantee the quality of hyper-encoding photon system based on the method of quantum hyper-teleportation, which can increase the success probability of hyperdistillation and reduce the resource consumption. First, we propose a hyperentanglement-assisted single-photon hyperdistillation (HASPHD) scheme for polarization and spatial qubits to get rid of the vacuum state component caused by transmission loss, whose success probability can achieve the optimal one by increasing the efficiency of quantum hyper-teleportation. Subsequently, we present two hyperentanglement-assisted hyperentanglement distillation (HAHED) schemes for photon system to protect hyperentanglement from both transmission loss and quantum channel noise, which can recover the less-entangled mixed state to maximally hyperentangled state for known-parameter and unknown-parameter cases with high success probability and low resource consumption. In these hyperdistillation schemes, the influence of imperfect effects of optical elements can be largely decreased by the quantum hyper-teleportation method. These characters make the hyperentanglement-assisted hyperdistillation schemes have potential application prospects in practical quantum information processing.

**Keywords** hyperdistillation, transmission loss, quantum channel noise, quantum communication, quantum information

## 1 Introduction

Quantum information process, e.g., quantum communication process or quantum computation process, is an amazing information process by using quantum mechanics theories to improve its security and speed. Photon system is an important information carrier in quantum communication [1], due to its high-speed, manipulability, and high-capacity [2] properties, and it can be used in quantum key distribution [3–6], quantum teleportation [7], quantum dense coding [8, 9], quantum secret sharing [10, 11], quantum secure direct communication [12–18], and some other schemes. Photons have multiple degrees of freedom (DOFs), containing polarization DOF, momentum DOF, time-bin DOF, frequency DOF and orbital angular momentum DOF, and they can be entangled in multiple DOFs simultaneously, which refers to hyperentanglement and has been experimentally demonstrated in many types [19–25]. In the practical long-distance quantum communication processing, the efficiency and distance of communication and the security and integrity of information will be limited and reduced by the transmission loss and quantum channel noise. For instance, the transmission loss caused by the optical fiber attenuation and imperfect photon detectors may lead to vacuum state component, and the quantum channel noise will transform the maximally (hyper)entangled state to partially (hyper)entangled state or mixed (hyper)entangled state.

In order to overcome the transmission loss problem, Ralph and Lund [26] first introduced the concept of heralded amplification based on quantum teleportation model. Since then, a lot of theoretical works [27, 28] and experimental works [29, 30] have been presented for heralded amplification, including heralded amplification for photon qubit. The qubit amplifier can be used to overcome the problem of channel loss in device independent quantum key distribution (DI-QKD) [27, 28, 31] and

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device-independent quantum secure direct communication (DI-QSDC) [32], which can increase the communication distance of DI-QKD and DI-QSDC. The noiseless linear amplification can also be used to protect single-photon entanglement from photon transmission loss [33–36]. Moreover, the hyperdistillation schemes based on amplification [37–40] have also been proposed for protecting spatial-mode-polarization hyperentangled state from photon loss.

In order to overcome the quantum channel noise problem, (hyper)entanglement purification and (hyper)entanglement concentration are introduced to improve the fidelity of the (hyper)entangled system degraded by the quantum channel noise. Entanglement (hyperentanglement) purification is introduced to distill high-fidelity (hyper)entangled state from mixed (hyper)entangled state [41–52]. Entanglement (hyperentanglement) concentration is introduced to distill maximally (hyper)entangled state from partially (hyper)entangled state [53]. Since the first entanglement concentration protocol was proposed by Bennet *et al.* using collective measurement [53], many entanglement concentration protocols have been proposed for photon system using linear optical elements [54–58] and nonlinear optical elements [59–63], and some of the entanglement concentration protocols have been demonstrated experimentally using linear optical elements [54, 55]. Hyperentanglement concentration protocols [64–72] have also attracted much attention for recovering the partially hyperentangled state to maximally hyperentangled state with certain success probability.

In this paper, we propose hyperentanglement-assisted hyperdistillation schemes for protecting hyper-encoding photon system from transmission loss and quantum channel noise using quantum hyper-teleportation method, which can increase the success probability of hyperdistillation and reduce the resource consumption. First, we propose a hyperentanglement-assisted single-photon hyperdistillation (HASPHD) scheme for polarization and spatial qubits to get rid of the vacuum state component caused by transmission loss, where the success probability of HASPHD scheme can reach the optimal one by enhancing the efficiency of quantum hyper-teleportation. Then, we show that the quantum circuit of HASPHD scheme can be directly used in hyperentanglement-assisted hyperentanglement distillation (HAHED) scheme for less-entangled mixed state with known parameters to increase the success probability of recovering the less-entangled mixed state to maximally hyperentangled state. Moreover, the HAHED scheme for less-entangled mixed state with unknown parameters can also be implemented easily using quantum hyper-teleportation method with high success probability and low resource consumption. In these hyperentanglement-assisted hyperdistillation schemes, the quantum hyper-teleportation method can depress the imperfect effects of optical elements on the final quantum states of these schemes. All these characters make these schemes have potential application prospects in quantum information process and quantum network.

## 2 Hyperentanglement-assisted single-photon hyperdistillation for polarization and spatial qubits

In quantum communication process, the photon is initially prepared in the state $|\varphi\rangle = (\alpha|H\rangle + \beta|V\rangle)_A(\gamma|a_1\rangle + \delta|a_2\rangle)$, where the four parameters are real for simplicity and satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$. The states $|H\rangle$ and $|V\rangle$ represent the horizontal polarization component and vertical polarization component of a photon, and the states $|a_1\rangle$ and $|a_2\rangle$ represent the two spatial modes of the photon $A$. If Alice sends the photon to a remote party Bob by the lossy quantum channel, the state of the photon may decay to $\rho_0 = (1 - P_0)|vac\rangle\langle vac| + P_0|\varphi\rangle\langle\varphi|$ by the inevitable transmission loss. Here $|vac\rangle$ represents the vacuum state caused by the transmission loss, and $P_0$ represents the fidelity of the state $|\varphi\rangle$ in mixed state $\rho_0$.

In order to increase the fidelity of the state $|\varphi\rangle$, a HASPHD scheme for polarization and spatial qubits is introduced using quantum hyper-teleportation method. Here we choose a four-photon system $BCDE$ in the hyperentangled Greenberger–Horne–Zeilinger (GHZ) state

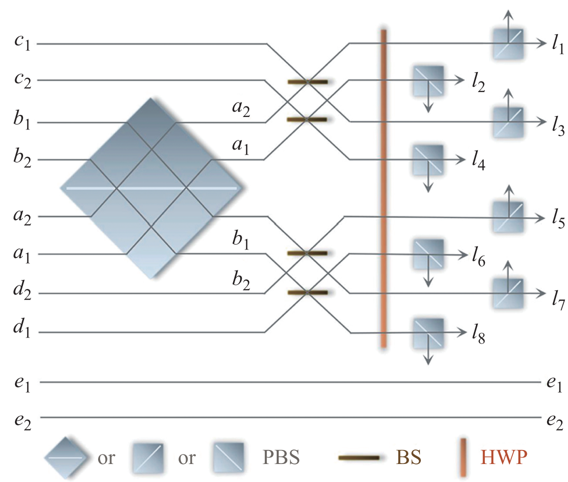


**Fig. 1** The schematic diagram of hyperentanglement-assisted single-photon hyperdistillation scheme. BS represents a 50:50 beam splitter. PBS represents polarization beam splitter, and it is used to transmit horizontal polarization component $|H\rangle$ and reflect vertical polarization component $|V\rangle$ of a photon, where the spatial modes of the photons transmitted through PBS are supposed to be invariant and the spatial modes of the photons reflected by PBS are supposed to be changed. HWP represents half-wave plate, which can perform Hadamard operation on the polarization DOF of the photon.

$[|\phi\rangle_{BCDE} = \frac{1}{2}(|HHHH\rangle + |VVVV\rangle)_{BCDE}(|b_1c_1d_1e_1\rangle + |b_2c_2d_2e_2\rangle)]$ as an auxiliary to implement the HASPHD scheme. (Other type of hyperentangled state as auxiliary is discussed in Section 4.) The setup of HASPHD assisted by hyperentangled GHZ state is shown in Fig. 1. It has ten input ports (marked by $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$, $d_2$, $e_1$ and $e_2$) and two output ports (marked by $e_1$ and $e_2$), and it consists of polarization beam splitter (PBS), 50:50 beam splitters (BSs), half-wave plates (HWPs), and photon detectors. PBS can perform the polarization parity-check measurement on the two photons $A$ and $B$, where the spatial modes of the photons transmitted through PBS are supposed to be invariant and the spatial modes of the photons reflected by PBS are supposed to be changed. BS can perform Hadamard operation on the spatial DOF of the photon. HWP can perform Hadamard operation on the polarization DOF of the photon. Photon detectors can select the successful cases.

The initial state of the five-photon system $ABCDE$ can be described as $\rho_0 \otimes |\phi\rangle_{BCDE}\langle\phi|$. Bob first lets the wave packets in spatial modes $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$ and $|b_2\rangle$ pass through PBS, and he postselects the cases that the polarization state of the photon system $AB$ is in the even-parity mode ($|HH\rangle$ or $|VV\rangle$) and the spatial states of two-photon systems $AC$ and $BD$ are both in even-parity mode ($|a_1c_1\rangle$ or $|a_2c_2\rangle$, $|b_1d_1\rangle$ or $|b_2d_2\rangle$), where four photon detectors are triggered (i.e., only one photon can be detected in one of the spatial modes $|l_m\rangle$ and $|l_n\rangle$, $mn = 13, 24, 57, 68$). The postselected state of five-photon system $ABCDE$ is

$$|\Phi_1\rangle = (\alpha|HHHHH\rangle + \beta|VVVVV\rangle)(\gamma|a_1b_1c_1d_1e_1\rangle + \delta|a_2b_2c_2d_2e_2\rangle). \tag{1}$$

Then, Bob lets the wave packets in spatial modes $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$, $|c_1\rangle$, $|c_2\rangle$, $|d_1\rangle$ and $|d_2\rangle$ pass through four BSs, respectively, which can make

$$\begin{aligned}
&|k\rangle_{a_1} \rightarrow \tfrac{1}{\sqrt{2}}(|k\rangle_{l_2} - |k\rangle_{l_4}), |k\rangle_{a_2} \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_{l_1} - |k\rangle_{l_3}),\\
&|k\rangle_{b_1} \rightarrow \tfrac{1}{\sqrt{2}}(|k\rangle_{l_5} + |k\rangle_{l_7}), |k\rangle_{b_2} \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_{l_6} + |k\rangle_{l_8}),\\
&|k\rangle_{c_1} \rightarrow \tfrac{1}{\sqrt{2}}(|k\rangle_{l_1} + |k\rangle_{l_3}), |k\rangle_{c_2} \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_{l_2} + |k\rangle_{l_4}),\\
&|k\rangle_{d_1} \rightarrow \tfrac{1}{\sqrt{2}}(|k\rangle_{l_6} - |k\rangle_{l_8}), |k\rangle_{d_2} \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_{l_5} - |k\rangle_{l_7}),
\end{aligned} \tag{2}$$

where $k$ represents the polarization mode ($H$ or $V$) of a photon. After the wave packets in spatial modes pass through four BSs, the postselected state is changed to

$$\begin{aligned}|\Phi_2\rangle = \frac{1}{4}\{&(\alpha|HHHHH\rangle + \beta|VVVVV\rangle)[\gamma(|l_2\rangle - |l_4\rangle)\\
&(|l_5\rangle + |l_7\rangle)(|l_1\rangle + |l_3\rangle)(|l_6\rangle - |l_8\rangle)|e_1\rangle + \delta(|l_1\rangle\\
&-|l_3\rangle)(|l_6\rangle + |l_8\rangle)(|l_2\rangle + |l_4\rangle)(|l_5\rangle - |l_7\rangle)|e_2\rangle]\}.\end{aligned} \tag{3}$$

Finally, Bob lets the photons $ABCD$ pass through eight HWPs $[|H\rangle \rightarrow \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle),\ |V\rangle \rightarrow \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)]$, respectively. Then, the postselected state of five-photon system $ABCDE$ will be changed to

$$\begin{aligned}|\Phi_3\rangle = \frac{1}{16}\{&[\alpha(|H\rangle + |V\rangle)(|H\rangle + |V\rangle)(|H\rangle + |V\rangle)(|H\rangle\\
&+|V\rangle)|H\rangle + \beta(|H\rangle - |V\rangle)(|H\rangle - |V\rangle)(|H\rangle - |V\rangle)\\
&(|H\rangle - |V\rangle)|V\rangle][\gamma(|l_2\rangle - |l_4\rangle)(|l_5\rangle + |l_7\rangle)(|l_1\rangle\\
&+|l_3\rangle)(|l_6\rangle - |l_8\rangle)|e_1\rangle + \delta(|l_1\rangle - |l_3\rangle)(|l_6\rangle + |l_8\rangle)\\
&(|l_2\rangle + |l_4\rangle)(|l_5\rangle - |l_7\rangle)|e_2\rangle]\}.\end{aligned} \tag{4}$$

Bob measures the photons $ABCD$ in the eight output modes ($|l_1\rangle$, $|l_2\rangle$, $|l_3\rangle$, $|l_4\rangle$, $|l_5\rangle$, $|l_6\rangle$, $|l_7\rangle$ and $|l_8\rangle$).

According to the postselection rules, the successful cases of the HASPHD scheme are shown in Table 1. There are four groups in Table 1. In the first group, the polarization states of two-photon systems $AC$ and $BD$ are in the same parity mode, and the spatial states of two-photon systems $AC$ and $BD$ are in the same parity mode. Here, the polarization modes $|HH\rangle$ and $|VV\rangle$ ($|HV\rangle$ and $|VH\rangle$) are defined as even (odd) parity mode, and the spatial modes $|l_1l_2\rangle$, $|l_5l_6\rangle$, $|l_3l_4\rangle$ and $|l_7l_8\rangle$ ($|l_1l_4\rangle$, $|l_3l_2\rangle$, $|l_5l_8\rangle$ and $|l_7l_6\rangle$) are defined as even (odd) parity mode. After the photons $ABCD$ are detected, the state of photon $E$ is transformed to $|\varphi_E\rangle = (\alpha|H\rangle + \beta|V\rangle)(\gamma|e_1\rangle + \delta|e_2\rangle)$, which is the initial photon state without vacuum state. In the second group, the polarization states of two-photon systems $AC$ and $BD$ are in different parity modes, and the spatial states of two-photon systems $AC$ and $BD$ are in the same parity mode. After the photons $ABCD$ are detected, the state of photon $E$ is transformed to $|\varphi_E^1\rangle = (\alpha|H\rangle - \beta|V\rangle)(\gamma|e_1\rangle + \delta|e_2\rangle)$, and a polarization phase-flip operation $\sigma_z^P = |H\rangle\langle H| - |V\rangle\langle V|$ is required to perform on photon $E$ to obtain the state $|\varphi_E\rangle$. In the third group, the polarization states of two-photon systems $AC$ and $BD$ are in the same parity mode, and the spatial states of two-photon systems $AC$ and $BD$ are in different parity modes. After the photons $ABCD$ are detected, the state of photon $E$ is transformed to $|\varphi_E^2\rangle = (\alpha|H\rangle + \beta|V\rangle)(\gamma|e_1\rangle - \delta|e_2\rangle)$, and a spatial phase-flip operation $\sigma_z^S = |e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|$ is required to perform on photon $E$ to obtain the state $|\varphi_E\rangle$. In the fourth group, the polarization states of two-photon systems $AC$ and $BD$ are in different parity modes, and the spatial states of two-photon systems $AC$ and $BD$ are in different parity modes. After the photons $ABCD$ are detected, the state of photon $E$ is transformed to $|\varphi_E^3\rangle = (\alpha|H\rangle - \beta|V\rangle)(\gamma|e_1\rangle - \delta|e_2\rangle)$, and polarization phase-flip operation $\sigma_z^P$ and spatial phase-flip operation $\sigma_z^S$ are required to perform on photon $E$ to obtain the state $|\varphi_E\rangle$.

If only three photon detectors are triggered or the photons $ABCD$ are detected in the other cases, the HASPHD process is failed, and these cases should be removed. Therefore, the success probability of the HASPHD scheme assisted by hyperentangled GHZ state is $\frac{P_0}{4}$.

**Table 1** The relationship between the successful measurement results and conditional operations on photon $E$ in HASPHD scheme (or HAHED scheme for less-entangled mixed state with known parameters).

| Parity detection of polarization state ($AC$ and $BD$) | Parity detection of spatial state ($AC$ and $BD$) | Operation ($E$) |
|---|---|---|
| even ($AC$), even ($BD$); odd ($AC$), odd ($BD$) | even ($AC$), even ($BD$); odd ($AC$), odd ($BD$) | $I$ |
| even ($AC$), odd ($BD$); odd ($AC$), even ($BD$) | even ($AC$), even ($BD$); odd ($AC$), odd ($BD$) | $\sigma_z^P$ |
| even ($AC$), even ($BD$); odd ($AC$), odd ($BD$) | even ($AC$), odd ($BD$); odd ($AC$), even ($BD$) | $\sigma_z^S$ |
| even ($AC$), odd ($BD$); odd ($AC$), even ($BD$) | even ($AC$), odd ($BD$); odd ($AC$), even ($BD$) | $\sigma_z^P$,$\sigma_z^S$ |

## 3 Hyperentanglement-assisted polarization-spatial hyperentanglement distillation

In long distance quantum communication, the maximally hyperentangled photon systems (e.g., hyperentangled state $|\psi\rangle_{AA'} = \frac{1}{2}(|HH\rangle + |VV\rangle)(|a_1a_1'\rangle + |a_2a_2'\rangle))$ are prepared locally by Alice. In order to establish the quantum channel, Alice will transmit one photon of hyperentangled photon system (e.g., photon $A$) to the remote party Bob through the lossy channels with the transmission coefficients of $T_1$ and $T_2$ for two spatial modes, respectively ($0 < T_1 \leq 1$ and $0 < T_2 \leq 1$). Therefore, the maximally hyperentangled photon system may be influenced by the decoherence caused by transmission loss and channel noise, and the maximally hyperentangled state $|\psi\rangle_{AA'}$ would deteriorate to less-entangled mixed state $\rho_2 = P_1|\psi_1\rangle\langle\psi_1| + (1 - P_1)\rho_1$, where

$$
\begin{aligned}
|\psi_1\rangle &= (\alpha|HH\rangle + \beta|VV\rangle) \otimes (\gamma|a_1a_1'\rangle + \delta|a_2a_2'\rangle),\\
\rho_1 &= (\alpha^2|H\rangle\langle H| + \beta^2|V\rangle\langle V|)[\frac{1-T_1}{(1-T_1)+(1-T_2)}\\
&\quad |a_1'\rangle\langle a_1'| + \frac{1-T_2}{(1-T_1)+(1-T_2)}|a_2'\rangle\langle a_2'|]\\
&\quad \otimes |vac\rangle_A\langle vac|,\\
P_1 &= \frac{T_1+T_2}{2}. \qquad (5)
\end{aligned}
$$

Here, the parameters $\gamma$ and $\delta$ are $\sqrt{\frac{T_1}{T_1+T_2}}$ and $\sqrt{\frac{T_2}{T_1+T_2}}$, respectively. The correlation between the transmission coefficient $T_i$ and the transmission length $s_i$ can be described by $T_i = 10^{-\alpha_0 s_i/10}$ ($\alpha_0$ =0.2 dB/km, $i = 1, 2$). If we know the length of each quantum channel, we could calculate the values of $T_1$ and $T_2$. In order to depress the effects of transmission loss and channel noise on hyperentangled photon system, we introduce two hyperentanglement-assisted hyperentanglement distillation (HAHED) schemes in this section, which can recover the less-entangled mixed state $\rho_2$ to maximally hyperentangled state with high success probability and low resource consumption.

### 3.1 HAHED scheme for less-entangled mixed state with known parameters

We first introduce a HAHED scheme for less-entangled mixed state with parameters $\alpha$, $\beta$, $\gamma$ and $\delta$ known to the remote users Alice and Bob, where the four parameters can be obtained by measuring a large number of sample states [56–58]. In order to obtain the maximally hyperentangled state $|\psi\rangle_{AA'}$ from the mixed hyperentangled state $\rho_2$, a four-photon system $BCDE$ in the partially hyperentangled GHZ state $|\psi'\rangle_{BCDE}$ is chosen as an auxiliary, which can increase the success probability and depress the resource consumption of HAHED scheme in linear optics. Here,

$$
\begin{aligned}
|\psi'\rangle_{BCDE} &= (\beta|HHHH\rangle + \alpha|VVVV\rangle)(\delta|b_1c_1d_1e_1\rangle\\
&\quad +\gamma|b_2c_2d_2e_2\rangle). \qquad (6)
\end{aligned}
$$

The HAHED scheme for less-entangled mixed state with known parameters can also be implemented by using the quantum circuit shown in Fig. 1. The initial state of six-photon system $A'ABCDE$ can be written as $\rho_2 \otimes |\psi'\rangle_{BCDE}\langle\psi'|$. Bob first lets the spatial modes $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$ and $|b_2\rangle$ pass through PBS, and he postselects the cases that the polarization state of the photon system $AB$ is in the even-parity mode and the spatial states of two-photon systems $AC$ and $BD$ are both in even-parity mode with four photon detectors triggered (i.e., only one photon can be detected in one of the spatial modes $|l_m\rangle$ and $|l_n\rangle$, $mn = 13, 24, 57, 68$). The postselected state of six-photon system $A'ABCDE$ is

$$
\begin{aligned}
|\Psi_1\rangle &= \frac{1}{2}(|HHHHHH\rangle + |VVVVVV\rangle)\\
&\quad (|a_1'a_1b_1c_1d_1e_1\rangle + |a_2'a_2b_2c_2d_2e_2\rangle). \qquad (7)
\end{aligned}
$$

Subsequently, Bob lets the spatial modes $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$, $|c_1\rangle$, $|c_2\rangle$, $|d_1\rangle$, and $|d_2\rangle$ pass through BSs and HWPs. The postselected state of six-photon system evolves to

$$
\begin{aligned}
|\Psi_2\rangle &= \frac{1}{32}\{[(|H\rangle + |V\rangle)(|H\rangle + |V\rangle)(|H\rangle + |V\rangle)(|H\rangle\\
&\quad +|V\rangle)|HH\rangle + (|H\rangle - |V\rangle)(|H\rangle - |V\rangle)(|H\rangle\\
&\quad -|V\rangle)(|H\rangle - |V\rangle)|VV\rangle] \otimes [(|l_2\rangle - |l_4\rangle)(|l_5\rangle
\end{aligned}
$$

$$+|l_7\rangle)(|l_1\rangle+|l_3\rangle)(|l_6\rangle-|l_8\rangle)|a_1'e_1\rangle+(|l_1\rangle -|l_3\rangle)(|l_5\rangle-|l_7\rangle)(|l_2\rangle+|l_4\rangle)(|l_6\rangle+|l_8\rangle) |a_2'e_2\rangle]\}. \quad (8)$$

At last, Bob measures the photons $ABCD$ in the eight output modes ($|l_1\rangle$, $|l_2\rangle$, $|l_3\rangle$, $|l_4\rangle$, $|l_5\rangle$, $|l_6\rangle$, $|l_7\rangle$ and $|l_8\rangle$). The successful cases of this HAHED scheme is the same as the HASPHD scheme as shown in Table 1. In the first group, the two-photon system $A'E$ can be projected to the maximally hyperentangled state $|\psi\rangle_{A'E}=\frac{1}{2}(|HH\rangle+|VV\rangle)(|a_1'e_1\rangle+|a_2'e_2\rangle)$. In the other groups, polarization phase-flip operation $\sigma_z^P$ or spatial phase-flip operation $\sigma_z^S$ has to perform on photon $E$ to obtain the maximally hyperentangled state $|\psi\rangle_{A'E}$.

If only three photon detectors are triggered or the photons $ABCD$ are detected in the other cases, the HAHED process is failed, and these cases should be removed. Therefore, the success probability of the HAHED scheme assisted by partially hyperentangled GHZ state $|\psi'\rangle_{BCDE}$ is $4P_1|\alpha\beta\gamma\delta|^2$.

### 3.2 HAHED scheme for less-entangled mixed state with unknown parameters

If the parameters $\alpha$, $\beta$, $\gamma$ and $\delta$ are unknown to the two remote users Alice and Bob, another nonlocal photon pair $B'C'$ in the less-entangled mixed state $\rho_2'=P_1|\psi_1'\rangle\langle\psi_1'|+(1-P_1)\rho_1'$ is required to complete the HAHED scheme. Here,

$$|\psi_1'\rangle=(\alpha|HH\rangle+\beta|VV\rangle)\otimes(\gamma|b_1'c_1'\rangle+\delta|b_2'c_2'\rangle),$$
$$\rho_1'=(\alpha^2|H\rangle\langle H|+\beta^2|V\rangle\langle V|)\otimes\left[\frac{1-T_1}{(1-T_1)+(1-T_2)}|b_1'\rangle\langle b_1'|+\frac{1-T_2}{(1-T_1)+(1-T_2)}|b_2'\rangle\langle b_2'|\right]\otimes|vac\rangle_{C'}\langle vac|). \quad (9)$$

The two photons $B'$ and $C'$ are obtained by the two remote users Alice and Bob respectively. The quantum circuit of HAHED for less-entangled mixed state with unknown parameters is shown in Fig. 2, where the amplification and concentration processes are performed simultaneously to increase the success probability and only one auxiliary hyperentangled GHZ state ($|\phi\rangle_{BCDE}=\frac{1}{2}(|HHHH\rangle+|VVVV\rangle)_{BCDE}(|b_1c_1d_1e_1\rangle+|b_2c_2d_2e_2\rangle)$) is required to depress the resource consumption. The initial state of eight-photon system $A'B'C'ABCDE$ is $\rho_2\otimes\rho_2'\otimes|\phi\rangle_{BCDE}\langle\phi|$.

First, Alice and Bob perform the bit-flip operations on the polarization and spatial states of two photons $B'$ and $C'$ respectively as shown in Fig. 2. Then, Alice puts the spatial modes $|a_1'\rangle$, $|a_2'\rangle$, $|b_1'\rangle$ and $|b_2'\rangle$ into PBS in Fig. 2(a),

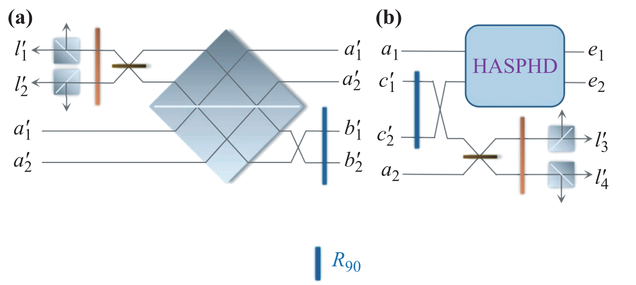


**Fig. 2** The schematic diagram of hyperentanglement-assisted hyperentanglement distillation for less-entangled mixed state with unknown parameters. **(a)** The operations performed by Alice. **(b)** The operations performed by Bob. HASPHD represents hyperentanglement-assisted single-photon hyperdistillation shown in Fig. 1. $R_{90}$ represents a half-wave plate, which is used to perform a polarization bit-flip operation $\sigma_x^P=|H\rangle\langle V|+|V\rangle\langle H|$ on the photon.

and Bob puts the spatial modes $|a_1\rangle$ and $|c_2'\rangle$ into the PBS of HASPHD in Fig. 2(b), where the principle of HASPHD is introduced in Section 2. Here $|c_2'\rangle$ is put into HASPHD by the input port $a_2$ in Fig. 1. Alice postselects the cases that the polarization state of the photon system $A'B'$ is in the even-parity mode (one photon can be detected in one of the spatial modes $|l_1'\rangle$ and $|l_2'\rangle$), and Bob postselects the successful cases of the HASPHD (the polarization state of the photon system $AB$ is in the even-parity mode and the spatial states of two-photon systems $AC$ and $BD$ are both in even-parity mode with four photon detectors triggered) and the cases that one photon can be detected in one of the spatial modes $|l_3'\rangle$ and $|l_4'\rangle$. The postselected state of eight-photon system $A'B'C'ABCDE$ is

$$|\Psi_1'\rangle=\frac{1}{2}(|HHHHHHHH\rangle+|VVVVVVVV\rangle)\otimes(|a_1'b_1'c_1'a_1b_1c_1d_1e_1\rangle+|a_2'b_2'c_2'a_2b_2c_2d_2e_2\rangle). \quad (10)$$

Subsequently, Alice and Bob put the photons into BSs (BSs in Fig. 2 and HASPHD), which can make

$$|k\rangle_{a_1}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_2}-|k\rangle_{l_4}),\quad |k\rangle_{a_2}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_3'}-|k\rangle_{l_4'}),$$
$$|k\rangle_{c_1'}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_3'}+|k\rangle_{l_4'}),\quad |k\rangle_{c_2'}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_1}-|k\rangle_{l_3}),$$
$$|k\rangle_{b_1'}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_1'}-|k\rangle_{l_2'}),\quad |k\rangle_{b_2'}\rightarrow\frac{1}{\sqrt{2}}(|k\rangle_{l_1'}+|k\rangle_{l_2'}). \quad (11)$$

The postselected state of eight-photon system $A'B'C'ABCDE$ is changed to

$$|\Psi_2'\rangle=\frac{1}{16}\{(|HHHHHHHH\rangle+|VVVVVVVV\rangle) [|a_1'e_1\rangle(|l_2\rangle-|l_4\rangle)(|l_1'\rangle-|l_2'\rangle)(|l_3'\rangle+|l_4'\rangle) (|l_5\rangle+|l_7\rangle)(|l_1\rangle+|l_3\rangle)(|l_6\rangle-|l_8\rangle)+|a_2'e_2\rangle$$

**Table 2** The relationship between the successful measurement results and conditional operations on photon $E$ in HAHED scheme for less-entangled mixed state with unknown parameters.

| Parity detection of polarization state | | Parity detection of spatial state | | Operation ($E$) |
|---|---|---|---|---|
| $De$ | $AC$ ($BD$) | $De$ | $AC$ ($BD$) | |
| even | even (even), odd (odd) | even | even (even), odd (odd) | $I$ |
| even | even (even), odd (odd) | odd | even (odd), odd (even) | |
| odd | even (odd), odd (even) | even | even (even), odd (odd) | |
| odd | even (odd), odd (even) | odd | even (odd), odd (even) | |
| even | even (odd), odd (even) | even | even (even), odd (odd) | $\sigma_z^P$ |
| even | even (odd), odd (even) | odd | even (odd), odd (even) | |
| odd | even (even), odd (odd) | even | even (even), odd (odd) | |
| odd | even (even), odd (odd) | odd | even (odd), odd (even) | |
| even | even (even), odd (odd) | even | even (odd), odd (even) | $\sigma_z^S$ |
| even | even (even), odd (odd) | odd | even (even), odd (odd) | |
| odd | even (odd), odd (even) | even | even (odd), odd (even) | |
| odd | even (odd), odd (even) | odd | even (even), odd (odd) | |
| even | even (odd), odd (even) | even | even (odd), odd (even) | $\sigma_z^P$,$\sigma_z^S$ |
| even | even (odd), odd (even) | odd | even (even), odd (odd) | |
| odd | even (even), odd (odd) | even | even (odd), odd (even) | |
| odd | even (even), odd (odd) | odd | even (even), odd (odd) | |

$$(|l'_3\rangle-|l'_4\rangle)(|l'_1\rangle+|l'_2\rangle)(|l_1\rangle-|l_3\rangle)(|l_6\rangle+|l_8\rangle)(|l_2\rangle+|l_4\rangle)(|l_5\rangle-|l_7\rangle)]\}. \tag{12}$$

Then, Alice and Bob let the photons pass through HWPs (HWPs in Fig. 2 and HASPHD), respectively. The postselected state of eight-photon system $A'B'C'ABCDE$ is changed to

$$\begin{aligned}|\Psi'_3\rangle = \frac{1}{128}\{&[|HH\rangle(|H\rangle+|V\rangle)(|H\rangle+|V\rangle)(|H\rangle+|V\rangle)\\&(|H\rangle+|V\rangle)(|H\rangle+|V\rangle)(|H\rangle+|V\rangle)+|VV\rangle\\&(|H\rangle-|V\rangle)(|H\rangle-|V\rangle)(|H\rangle-|V\rangle)(|H\rangle-|V\rangle)\\&(|H\rangle-|V\rangle)(|H\rangle-|V\rangle)][|a'_1e_1\rangle(|l_2\rangle-|l_4\rangle)\\&(|l'_1\rangle-|l'_2\rangle)(|l'_3\rangle+|l'_4\rangle)(|l_5\rangle+|l_7\rangle)(|l_1\rangle+|l_3\rangle)\\&(|l_6\rangle-|l_8\rangle)+|a'_2e_2\rangle(|l'_3\rangle-|l'_4\rangle)(|l'_1\rangle+|l'_2\rangle)\\&(|l_1\rangle-|l_3\rangle)(|l_6\rangle+|l_8\rangle)(|l_2\rangle+|l_4\rangle)(|l_5\rangle-|l_7\rangle)]\}.\end{aligned} \tag{13}$$

Finally, Alice and Bob measure the photons in the twelve output modes ($|l'_1\rangle$, $|l'_2\rangle$, $|l'_3\rangle$ and $|l'_4\rangle$ in Fig. 2, and $|l_1\rangle$, $|l_2\rangle$, $|l_3\rangle$, $|l_4\rangle$, $|l_5\rangle$, $|l_6\rangle$, $|l_7\rangle$ and $|l_8\rangle$ in HASPHD). According to the postselection rules, the successful cases of this HAHED scheme is shown in Table 2. Here, the spatial states $|l'_1l'_3\rangle$ and $|l'_2l'_4\rangle$ ($|l'_1l'_4\rangle$ and $|l'_2l'_3\rangle$) are defined as even (odd) parity mode, and $De$ represents the measurement result of the spatial modes $|l'_1\rangle$, $|l'_2\rangle$, $|l'_3\rangle$ and $|l'_4\rangle$. There are four groups in Table 2. In the first group, the state of two-photon system $A'E$ is transformed to the maximally hyperentangled state $|\varphi_{A'E}\rangle = \frac{1}{2}(|HH\rangle+|VV\rangle)(|a'_1e_1\rangle+|a'_2e_2\rangle)$. In the other groups, polarization phase-flip operation $\sigma_z^P$ or spatial phase-flip operation $\sigma_z^S$ has to perform on photon $E$ to obtain the maximally hyperentangled state $|\varphi_{A'E}\rangle$.

If only three or four photon detectors are triggered in Fig. 2(b) or the photons $B'C'ABCD$ are detected in the other cases, the HAHED process is failed, and these cases should be removed. Therefore, the success probability of the HAHED scheme assisted by hyperentangled GHZ state $|\phi\rangle_{BCDE}$ is $P_1^2|\alpha\beta\gamma\delta|^2$.

## 4 Discussion and conclusion

In this paper, we present hyperentanglement-assisted hyperdistillation schemes for protecting hyper-encoding photon system form transmission loss and quantum channel noise using quantum hyper-teleportation method. In fact, hyperentangled Bell state is the simplest resource for quantum hyper-teleportation, with which the vacuum state component caused by transmission loss can be eliminated by single-photon-trigger case in the hyperentangled Bell state measurement process of quantum hyper-teleportation. So far, the efficiency of quantum hyper-teleportation using hyperentangled Bell state is not high enough in experiment [73]. Therefore, we choose hyperentangled GHZ state, which has potential to be generated in experiment [74, 75] and can increase the success probability of quantum hyper-teleportation to 25% in linear optics, to implement the HASPHD scheme with the succuss probability of $\frac{P_0}{4}$. If the efficiency of quantum hyper-teleportation is increased to 100% [76], the succuss probability of HASPHD scheme can achieve $P_0$, which is the optimal succuss probability to eliminate the vacuum state component caused by transmission loss.

In HASPHD scheme, the transmission coefficients of lossy quantum channel for two spatial modes are supposed to be $T_1 = T_2 = T$. If the transmission coefficients are $T_1 \neq T_2$, the state of the single photon may decay to $\rho_0' = (1 - P_0')|vac\rangle\langle vac| + P_0'|\varphi'\rangle\langle\varphi'|$ by the inevitable transmission loss, where

$$|\varphi'\rangle = (\alpha|H\rangle + \beta|V\rangle)_A\Big(\sqrt{\frac{\gamma^2 T_1}{\gamma^2 T_1 + \delta^2 T_2}}|a_1\rangle + \sqrt{\frac{\delta^2 T_2}{\gamma^2 T_1 + \delta^2 T_2}}|a_2\rangle\Big),$$
$$P_0' = \gamma^2 T_1 + \delta^2 T_2. \tag{14}$$

If the transmission coefficients $T_1$ and $T_2$ are known, the HASPHD scheme can be completed by changing the parameters of auxiliary hyperentangled GHZ state as shown in Section 3.1. If the transmission coefficients $T_1$ and $T_2$ are unknown, the HASPHD scheme can also be completed by the method shown in Section 3.2.

The HASPHD process is important for HAHED schemes. In the previous hyperentanglement distillation scheme with only hyperentanglement concentration process [64], if the transmission loss is happened in quantum channel, the wight of vacuum state will be increased after this hyperentanglement distillation process, so the final state of the scheme will become mixed hyperentangled state instead of maximally hyperentangled state. The fidelity of hyperentangled state after the hyperentanglement distillation with only hyperentanglement concentration process (for the case with unkonwn parameters) is expressed as

$$P = \frac{T_2(T_1 + T_2)}{T_2(T_1 + T_2) + T_2(2 - T_1 - T_2) + 4(1 - T_1)(T_1 + T_2)}. \tag{15}$$

The relationships of the fidelities of hyperentangled states after (before) the hyperentanglement distillation processes vs transmission coefficient $T$ ($T_1 = T_2 = T$) are shown in Fig. 3. $P_1$ represents the fidelity of hyperentangled state in the initial state $\rho_2$ before hyperentanglement distillation. $P$ represents the fidelity of hyperentangled state in final state after hyperentanglement distillation with only hyperentanglement concentration process. $P'$ represents the fidelity of hyperentangled state in final state after HAHED process. In Fig. 3, it is obvious that the fidelity of hyperentangled state in final state is significantly decreased after hyperentanglement distillation with only hyperentanglement concentration process. Therefore, it is necessary to use HASPHD to remove the vacuum state, which will improve the fidelity of hyperentangled state. In HAHED scheme, the vacuum state can be completely removed together with the hyperentanglement concentration process, so the fidelity of hyperentangled state in final state is $P' = 1$, which means the less-entangled mixed state is recovered to maximally hyperentangled state.

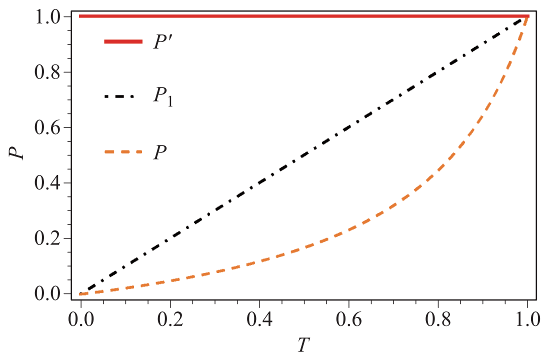


**Fig. 3** The relationships of the fidelities of hyperentangled states after (before) the hyperentanglement distillation processes vs transmission coefficient $T$ ($T_1 = T_2 = T$). $P_1$ represents the fidelity of hyperentangled state in the initial state $\rho_2$ before hyperentanglement distillation. $P$ represents the fidelity of hyperentangled state in final state after hyperentanglement distillation with only hyperentanglement concentration process. $P'$ represents the fidelity of hyperentangled state in final state after HAHED process.

In the hyperentanglement distillation scheme with only hyperentanglement concentration process [64], the success probabilities are $4|\beta\gamma|^2$ ($|\beta| < |\alpha|$ and $|\gamma| < |\delta|$ with parameter-splitting method) and $4|\alpha\beta\gamma\delta|^2$ for known-parameter and unknown-parameter cases, where the vacuum state component can not been removed. In order to remove the vacuum state component simultaneously, HAHED schemes are introduced with lower success probabilities $S_1 = 4P_1|\alpha\beta\gamma\delta|^2$ and $S_2 = P_1^2|\alpha\beta\gamma\delta|^2$ for known-parameter and unknown-parameter cases, where the decrease of success probabilities is caused by the process of removing the vacuum state component. The HAHED scheme is not the simple combination of HASPHD and hyperentanglement concentration. For instance, in the previous hyperdistillation protocol for amplification [37], the vacuum state component in hyperentangled mixed state can be completely eliminated with the success probability of $\frac{P_1}{4}$, but the channel noise effect can not be depressed. In order to depress the channel noise effect and recover the less-entangled state to maximally hyperentangled state, the hyperentanglement concentration process should be performed after the amplification process, and the success probabilities are $S_3 = P_1|\beta\gamma|^2$ ($|\beta| < |\alpha|$ and $|\gamma| < |\delta|$ with parameter-splitting method [64]) and $S_4 = P_1^2|\alpha\beta\gamma\delta|^2/4$ for known-parameter and unknown-parameter cases, respectively, where the vacuum state component caused by the inefficient detection in parameter-splitting method can not be removed and more resources are required for re-

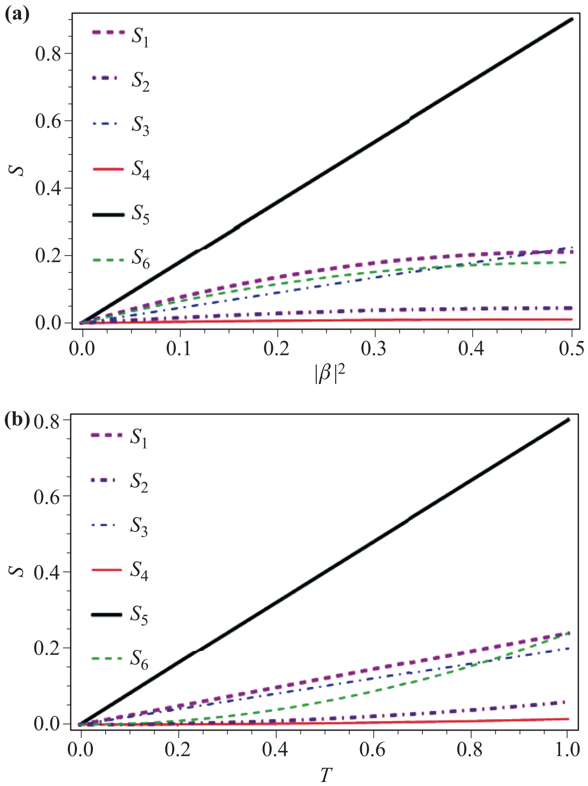


**Fig. 4** The relationships of the success probabilities of hyperdistillation schemes vs parameters of less-entangled state. **(a)** The success probabilities of hyperdistillation schemes vs $|\beta|^2$ with $T_1 = 0.9$ and $T_2 = 0.8$. **(b)** The success probabilities of hyperdistillation schemes vs transmission coefficient $T_1 = T_2 = T$ with $|\beta|^2 = 0.4$.

moving vacuum state components of two copies of less-entangled state in unknown-parameter case. In our HAHED schemes, the HASPHD and hyperentanglement concentration processes are performed simultaneously using quantum hyper-teleportation method, which has largely increased the success probability of recovering the less-entangled state to maximally hyperentangled state and reduced the resource consumption. The relationships of the success probabilities of hyperdistillation schemes vs parameters of less-entangled state are shown in Fig. 4. It is obvious that the success probabilities of recovering the less-entangled state to maximally hyperentangled state can achieve $S_5 = 4P_1|\beta\gamma|^2$ ($|\beta| < |\alpha|$ and $|\gamma| < |\delta|$ with parameter-splitting method) and $S_6 = 4P_1^2|\alpha\beta\gamma\delta|^2$ for known-parameter and unknown-parameter cases with the efficiency of quantum hyper-teleportation increased to 100%. In order to achieve the optimal succuss probabilities and reduce resource consumption, the HASPHD and hyperentanglement concentration processes should be performed simultaneously using quantum hyper-teleportation method as introduced in HAHED schemes instead of amplification-concentration method.

In the HAHED schemes, we suppose the maximally hyperentangled photon system is prepared locally by Alice, where only one photon of hyperentangled photon system will be transmitted to the remote party Bob through the lossy and noisy channel. If the hyperentangled photon source is prepared by Charlie, who is located in the middle point of Alice and Bob, the two photons of hyperentangled photon system will both be transmitted to Alice and Bob by the lossy and noisy channels to establish the quantum channel. Suppose that the transmission coefficients of the channels for spatial modes $a_1$, $a_2$, $a_1'$, and $a_2'$ are $T_1$, $T_2$, $T_3$, $T_4$ ($0 < T_j < 1$, $j = 1, 2, 3, 4$), respectively. Then the state $|\psi_{AA'}\rangle$ will become a mixed hyperentangled state containing the vacuum state component,

$$\rho_3 = \eta_1|\Upsilon_1\rangle_{AA'}\langle\Upsilon_1| + \eta_2\rho_4 + \eta_3\rho_5 + \eta_4|vac\rangle_{AA'}\langle vac|, \tag{16}$$

where

$$\begin{aligned} |\Upsilon_1\rangle_{AA'} &= (\alpha|HH\rangle + \beta|VV\rangle) \otimes (\varepsilon|a_1a_1'\rangle + \theta|a_2a_2'\rangle), \\ \rho_4 &= (\alpha^2|H\rangle + \beta^2|V\rangle)[\vartheta^2|a_1\rangle\langle a_1| + \iota^2|a_2\rangle\langle a_2|] \\ &\quad \otimes |vac\rangle_{A'}\langle vac|, \\ \rho_5 &= (\alpha^2|H\rangle + \beta^2|V\rangle)[\kappa^2|a_1'\rangle\langle a_1'| + \mu^2|a_2'\rangle\langle a_2'|] \\ &\quad \otimes |vac\rangle_A\langle vac|. \end{aligned} \tag{17}$$

The parameters are $\varepsilon = \sqrt{\frac{T_1T_2}{T_1T_2+T_3T_4}}$, $\theta = \sqrt{\frac{T_3T_4}{T_1T_2+T_3T_4}}$, $\vartheta = \sqrt{\frac{T_1(1-T_2)}{T_1(1-T_2)+T_3(1-T_4)}}$, $\iota = \sqrt{\frac{T_3(1-T_4)}{T_1(1-T_2)+T_3(1-T_4)}}$, $\kappa = \sqrt{\frac{(1-T_1)T_2}{(1-T_1)T_2+(1-T_3)T_4}}$, $\mu = \sqrt{\frac{(1-T_3)T_4}{(1-T_1)T_2+(1-T_3)T_4}}$, $\eta_1 = \frac{1}{2}(T_1T_3 + T_2T_4)$, $\eta_2 = \frac{1}{2}[(1 - T_3)T_1 + T_2(1 - T_4)]$, $\eta_3 = \frac{1}{2}T_3(1 - T_1) + T_4(1 - T_2)]$ and $\eta_4 = \frac{1}{2}[(1 - T_1)(1 - T_3) + (1 - T_2)(1 - T_4)]$, respectively, and the coefficients satisfy the normalization relation as $\eta_1 + \eta_2 + \eta_3 + \eta_4 = 1$. In this situation, the HAHED schemes can also be used to recover the less-entangled state $\rho_3$ to maximally hyperentangled state, where Alice has to perform an additional HASPHD (in Fig. 1) on photon $A'$ to remove the vacuum state component of photon $A'$ besides the operations in Fig. 1 and Fig. 2, and the success probabilities will become $\eta_1|\alpha\beta\varepsilon\theta|^2$ and $\eta_1^2|\alpha\beta\varepsilon\theta|^2/4$ for known-parameter and unknown-parameter cases. If the efficiency of quantum hyper-teleportation is increased to 100%, the success probabilities of recovering the less-entangled state to maximally hyperentangled state can achieve $4\eta_1|\beta\gamma|^2$ ($|\beta| < |\alpha|$ and $|\gamma| < |\delta|$ with parameter-splitting method) and $4\eta_1^2|\alpha\beta\gamma\delta|^2$ for known-parameter and unknown-parameter cases.

In hyperdistillation schemes, the efficiencies of linear optical elements (e.g., PBSs, BSs and HWPs) are always assumed to be perfect (approximate to 100%). However, in a practical condition, the linear optical elements do not work in an ideal condition, and the imperfect linear optical elements will affect the final result of hyperdistillation protocol. In our scheme, hyperdistillation protocol is based on quantum hyper-teleportation, and there is no operation performed on the photon $E$ except conditional operations at last. Therefore, the imperfect linear optical elements cause little effects on the final result of our protocol, and the vacuum state can be completely removed.

In summary, we proposed hyperentanglement-assisted hyperdistillation schemes to protect the hyper-encoding photon system from transmission loss and channel noise using quantum hyper-teleportation method, which can increase the success probability of hyperdistillation and reduce the resource consumption. In HASPHD scheme, the vacuum state component casued by transmission loss can be eliminated completely with high success probability by enhancing the efficiency of quantum hyper-teleportation. In the two HAHED schemes, the less-entangled mixed state can be recovered to maximally hyperentangled state with high success probability and low resource consumption. As the quantum hyper-teleportation method is adopted to depress the imperfect effects of optical elements on the final result, these hyperentanglement-assisted hyperdistillation schemes have potential application prospects in the field of practical quantum information processing.

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