Frontiers of Physics
https://doi.org/10.1007/s11467-021-1061-1
Front. Phys. 16(4), 43503 (2021)

RESEARCH ARTICLE

An analytical solution for quantum scattering through a $\mathcal{PT}$-symmetric delta potential

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Received January 18, 2021; accepted February 26, 2021

We employ the Lippmann–Schwinger formalism to derive the analytical solutions of the transmission and reflection coefficients through a one-dimensional open quantum system, in which particle loss or gain on one lattice site located at $x = 0$, or particle loss and gain on the lattice sites located at $x = \pm\frac{L}{2}$ are considered respectively. The gain and loss on the lattice site are modeled by the delta potential with positive and negative imaginary values. The analytical solution reveals the underlying physics that the sum of the transmission and reflection coefficients through an open quantum system (even a $\mathcal{PT}$-symmetric open system) may not be **1**, i.e., qualitatively explains that the number of particles is not conserved in an open quantum system. Furthermore, we find that the resonance states can be formed in the $\mathcal{PT}$-symmetric delta potential, which is similar to the case of real delta potential. The results of our analysis can be treated as the starting point of studying quantum transport problems through a non-Hermitian system using Green's function method, and more general cases for high-dimensional systems may be deduced by the same procedure.

Keywords transmission, non-Hermitian, $\mathcal{PT}$-symmetry, Green function

1 Introduction

Conventional quantum systems are typically described by Hermitian Hamiltonians because Hermiticity guarantees a real energy spectrum and the probability-preserving of the particle in the entire space. However, the Hermiticity is only a sufficient but not necessary condition. Bender and collaborators [1–3] have found that a non-Hermitian Hamiltonian satisfying parity–time ($\mathcal{PT}$) symmetry can still display a real spectrum. In the series of articles by Mostafazadeh [4–6], the underlying reason is shown that $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians actually belong to the class of pseudo-Hermitian Hamiltonian, in which there exists a similarity transformation deforming the non-Hermitian Hamiltonian to Hermitian form. Since such an analog system was first realized experimentally in optical waveguide structures [7], other systems, such as flat microwave cavities [8] and optical cavities [9–11, 16], electronic circuits [13–15] and even single-spin quantum system [16] described by $\mathcal{PT}$-symmetric non-Hermitian effective Hamiltonians, then have been extensively investigated.

*This article can also be found at http://journal.hep.com.cn/fop/EN/10.1007/s11467-021-1061-1.

Generally, the time evolution of the open quantum systems can be determined by solving the master equation like the Lindblad equation [17–20]. However, a short-time evolution can often be approximately described by the effective non-Hermitian Hamiltonians [21, 22]. The $\mathcal{PT}$-symmetric non-Hermitian systems are open systems with balanced gain and loss, which can be realized by a pair of imaginary potentials with the opposite value. Since the imaginary potential can give rise to the non-conservation of the flow of probability [23], a great deal of effort has been devoted to investigating the quantum transport properties of non-Hermitian systems with $\mathcal{PT}$-symmetric imaginary potentials. It has been shown that $\mathcal{PT}$-symmetric imaginary potentials provide a novel method to control and engineer the Fano resonance line shape [24–30]. And a $\mathcal{PT}$-symmetric imaginary potential can give rise to the formation of bound states inside the bandgap in strongly coupled bilayer lattices [31]. In Dirac systems, the non-Hermitian defects affect the spatial distribution of the local density of states and change the frequency dependence of the density of states [32]. In one-dimensional $\mathcal{PT}$-symmetric quasicrystals, the localization transition induced by a non-Hermitian quasiperiodic potential is found. Thus the fundamental consequence of the $\mathcal{PT}$-symmetric potential is to create two different kinds of electronic states: localized and extended states [33, 34].

The nonequilibrium Green's function (NEGF) method is a popular and powerful tool to study quantum electron transport phenomena [35]. However, scattering centers with pure imaginary potential bring new challenges to it. Especially the positive imaginary potential may give rise to the transition of Green function from retarded one to advanced one, which leads to the failure of causality. When transmission and reflection coefficients are calculated, the poles of it may be induced and some unexplainable results physically could be present.

In the present paper, from the Lippmann–Schwinger formalism, we derive the analytical solutions of the transmission and reflection coefficients when free particle moving in a one-dimensional system is scattered by one or two delta functions like imaginary potential. Our obtained analytical solution reveals the underlying physics that the sum of the transmission and reflection coefficients through an open quantum system may not be **1**, i.e., qualitatively explains that the number of particles is not conserved in an open quantum system. Furthermore, we find that the resonance states are formed in the $\mathcal{PT}$-symmetric delta potential, which is similar to the case of real delta potential. The results of our analysis can be treated as the starting point of studying quantum transport problems through the non-Hermitian system using Green's function method.

The rest of the paper is organized as follows. In Section 2, we briefly present the Lippmann–Schwinger formalism, which is one of the most used equations to describe particle scattering in quantum mechanics. As an example, the transmission and reflection coefficients for a delta imaginary potential with positive and negative values are derived in Section 3. Section 4 shows our derived analytical solutions for $\mathcal{PT}$-symmetric non-Hermitian systems, it is found that the resonance states can be formed. Furthermore, we also present what the condition of balanced gain and loss is and how the particle-number conservation is broken in a $\mathcal{PT}$-symmetric non-Hermitian system. At last, the results are summarized in Section 5.

2 Lippmann–Schwinger equation

We start by considering a free particle with mass m moving along a one-dimensional system, the Hamiltonian of which is $\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ ($\hbar$ is Planck constant). When the free particle hits the potential V, the general form of the Schrödinger equation is given by

$$(E - \hat{H}_0)\,|\psi\rangle = V\,|\psi\rangle\,, \tag{1}$$

where $|\psi\rangle$ is the wavefunction of free particle. The Green equation defining the Green operator $\hat{G}$ associated to the Schrödinger's equation above is

$$(E - \hat{H}_0)\hat{G}_0(E) = 1. \tag{2}$$

We can obtain the Green function by inverting $(E - \hat{H}_0)$ as

$$\hat{G}_0(E) = \frac{1}{E - \hat{H}_0}. \tag{3}$$

Supposing the homogeneous solution of the Schröinger equation above $|\psi_0\rangle$, we write the equation

$$(E - \hat{H}_0)\,|\psi_0\rangle = 0, \tag{4}$$

and obtain

$$|\psi\rangle = |\psi_0\rangle + \hat{G}_0(E)V\,|\psi\rangle\,, \tag{5}$$

which is called the Lippmann–Schwinger formalism [36]. In the x-representation, we can rewrite Eq. (5) as

$$\langle x|\psi\rangle = \langle x|\psi_0\rangle + \int \mathrm{d}x'\,\langle x|\,\hat{G}_0(E)\,|x'\rangle\,\langle x'|\,V\,|\psi\rangle\,, \tag{6}$$

where $|x\rangle$ is the eigenvector of the position operator. Here we would like point out that the $|x\rangle$ can form a complete orthonormal basis $\langle x|x'\rangle = \delta(x-x')$, which can constitute an abstract Hilbert space. Thus $\langle x|\,\hat{G}_0(E)\,|x'\rangle$ means that the Green function of position coordinates x and x' is defined in the single particle Hilbert space. Based on the above conditions, we can rewrite the Eq. (2) in the basis $|x\rangle$ as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right)G_0(E;x,x') = \frac{2m}{\hbar^2}\delta(x-x'), \tag{7}$$

where we define $G_0(E;x,x') = \langle x|\,\hat{G}_0\,|x'\rangle$. By solving above equation, the unperturbed Green's function can be given by [37]

$$G_0(E;x,x') = -\frac{\mathrm{i}m}{\hbar^2 k}\mathrm{e}^{\mathrm{i}k|x-x'|}, \tag{8}$$

where the wave vector k is equal to $\frac{\sqrt{2mE}}{\hbar}$ with the incident energy $E > 0$. The Lippmann–Schwinger equation in one dimension is therefore

$$\langle x|\psi\rangle = \langle x|\psi_0\rangle - \frac{\mathrm{i}m}{\hbar^2 k}\int \mathrm{d}x'\mathrm{e}^{\mathrm{i}k|x-x'|}\,\langle x'|\,V\,|\psi\rangle\,, \tag{9}$$

3 Application to single delta imaginary potential

In previous work, the quantum scattering problem of the real δ function has been considered [38]. Now we consider the single δ negative imaginary potential $V(x)$ localized at $x = 0$, which can be written as

$$V(x) = -\mathrm{i}\lambda\delta(x), \tag{10}$$

where i is the imaginary unit and λ is the potential strength. If we take the position eigenkets $|x\rangle$ with a real eigenvalue x as basis vector, the above potential can be written as in the bra-ket formalism [39, 40]

$$V(x) = -\mathrm{i}\lambda\,|x=0\rangle\,\langle x=0|\,. \tag{11}$$

For simplicity, we will abbreviate $|x=0\rangle$ as $|0\rangle$ in the following derivation. After we simply substitute Eq. (11) into Eq. (5) and multiply the result from the left by the bra $\langle x|$, the wave functions of particles in the scattered final-state can be obtained

$$\langle x|\psi\rangle = \langle x|\psi_0\rangle - \mathrm{i}\lambda \langle x| G_0(E) |0\rangle \langle 0|\psi\rangle , \tag{12}$$

where $\langle x|\psi_0\rangle$ is the initial wave function presenting a free plane wave function $\mathrm{e}^{\mathrm{i}kx}$. In the ordinary x-representation, the above relation can be written as follows:

$$\psi(x) = \mathrm{e}^{\mathrm{i}kx} - \mathrm{i}\lambda G_0(E; x, 0)\psi(0). \tag{13}$$

Obviously, if we want to obtain the explicit form for $\psi(x)$, we need to find $\psi(0)$, which can be solved by replacing x by 0 in above equation

$$\psi(0) = \frac{1}{1+\mathrm{i}\lambda G_0(E;0,0)} = \frac{\hbar^2 k}{\hbar^2 k + m\lambda}. \tag{14}$$

Thus we can get the explicit form for $\psi(x)$

$$\psi(x) = \mathrm{e}^{\mathrm{i}kx} - \frac{m\lambda}{\hbar^2 k + m\lambda}\mathrm{e}^{\mathrm{i}k|x|}. \tag{15}$$

We assume that an electron is incident from the left on a delta potential located at $x=0$. Thus the general forms of wave functions $\psi(x)$ for two regions $x<0$ and $x>0$ can be written

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} - \dfrac{m\lambda}{\hbar^2 k + m\lambda}\mathrm{e}^{\mathrm{i}kx}, & x>0, \\ \mathrm{e}^{\mathrm{i}kx} - \dfrac{m\lambda}{\hbar^2 k + m\lambda}\mathrm{e}^{-\mathrm{i}kx}, & x<0. \end{cases} \tag{16}$$

Once the scattered wave functions are solved, we may obtain the analytical solutions of the transmission and reflection coefficients

$$T = \left|1 - \frac{m\lambda}{\hbar^2 k + m\lambda}\right|^2 = \frac{\hbar^4 k^2}{(\hbar^2 k + m\lambda)^2}, \tag{17}$$

and

$$R = \left|-\frac{m\lambda}{\hbar^2 k + m\lambda}\right|^2 = \frac{m^2\lambda^2}{(\hbar^2 k + m\lambda)^2}. \tag{18}$$

It is easy to show that the total probability is

$$T + R = \frac{\hbar^4 k^2 + m^2\lambda^2}{(\hbar^2 k + m\lambda)^2}, \tag{19}$$

which is always less than 1 except for $\lambda = 0$. It can be illustrated that the negative imaginary potential describes the case of particle loss in an open quantum system. While electron incoming from the left terminal hits the negative imaginary potential, it is possible that electron flows out to an external environment and the number of electrons indeed decays.

In physics, the disappearance of incident particle may appear in nuclear reactions where incident particles get absorbed by nuclei. Also the imaginary potential is used to described the dissipative effects in quantum optics. The disappearance or dissipation of particles can be characterized by the electronic absorption coefficient A [41]

$$\begin{aligned} A &= -\frac{J(x=+\infty) - J(x=-\infty)}{J_{in}(x=-\infty)} \\ &= -\frac{J(x=+\infty) - J(x=-\infty)}{\frac{\hbar k}{m}}, \end{aligned} \tag{20}$$

where $J(x) = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$ is the probability flux and $J_{in}(x=-\infty)$ is the incident probability flux. According to Eq. (16), we know the incident wave-function is $\psi_{in}(x) = \mathrm{e}^{\mathrm{i}kx}$, so $J_{in}(x=-\infty) = \frac{\hbar k}{m}$. Therefore, the electronic absorption coefficient for the model is

$$A = \frac{2\hbar^2 k m\lambda}{(\hbar^2 k + m\lambda)^2}. \tag{21}$$

Obviously, transmission, reflection and absorption coefficients satisfy $T + R + A = 1$.

By same procedures, we can derive the transmission and reflection coefficients of the electron hitting the single positive imaginary potential $V(x) = \mathrm{i}\lambda\delta(x)$, which can be present as

$$T = \left|1 + \frac{m\lambda}{\hbar^2 k - m\lambda}\right|^2 = \frac{\hbar^4 k^2}{(\hbar^2 k - m\lambda)^2}, \tag{22}$$

and

$$R = \left|-\frac{m\lambda}{\hbar^2 k - m\lambda}\right|^2 = \frac{m^2\lambda^2}{(\hbar^2 k - m\lambda)^2}. \tag{23}$$

It is obvious that the total probability $T + R$ is always greater than 1, except for $\lambda = 0$. The reason is that the positive imaginary potential describes the case of particle gain, which indicates that more particles can flow into the quantum system from an external environment. Thus the conservation of total probability is broken in the system.

To see the impact of positive/negative delta-like imaginary potential on transmission coefficients more intuitively, in Fig. 1, we plot the transmission coefficients as a function of the wave vector k with different imaginary potential strength λ. It can be seen that the transmission coefficients will somewhat decrease with the increase of potential strength λ for negative imaginary potential, which indicates that the number of the particle may flow out to an external environment when electrons transport through a negative imaginary potential. Intuitively, we think that the transmission coefficient T should be larger than 1 for the positive imaginary potential, because those electrons flow into the quantum system from an external environment. However, in Fig. 1(b), one can see that the transmission coefficients are below unity when the wave vector k is less than $m\lambda/2\hbar^2$. Here the positive imaginary potential has two competing effects: one is to scatter the incident electrons and make the T smaller; the other is to

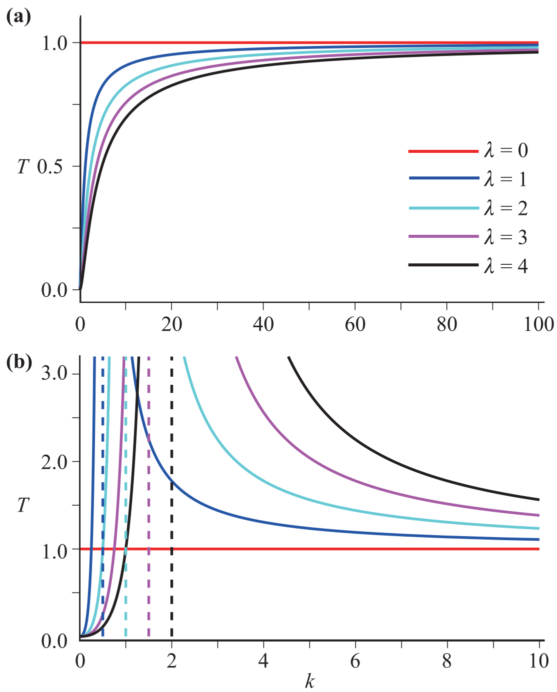

Fig. 1 Transmission coefficients as a function of the wave vector k for different strength of potential while electron transport through single negative imaginary delta potential **(a)** and positive imaginary potential **(b)**. The other system parameters are set as $\hbar = 1$, $m = 1/2$. And the red, blue, cyan, magenta and black lines correspond to the results for $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = 3$, and $\lambda = 4$, and the corresponding dashed lines are the vertical asymptote expressing the infinity, respectively.

inject particles into the system to make the T larger. For the small incident wave vector k, the former dominates, so that the transmission coefficient T is less than 1. As k approaches $m\lambda/\hbar^2$, the transmission coefficients will sharply increase and arrive at the value of infinity, which can be illustrated that the denominator of Eq. (14) is approaching to zero. While k is greater $m\lambda/\hbar^2$, the transmission coefficients will decrease from the value of infinity, but it always exceeds unity 1. Furthermore, with the increasing of positive imaginary potential strength λ, the transmission coefficients increase and more particles flow into the system from an external environment.

4 Application to $\mathcal{PT}$-symmetric potential

With the above two examples as a bedding, now we can consider the $\mathcal{PT}$-symmetric potential described by

$$V(x) = \mathrm{i}\lambda\delta(x + \frac{L}{2}) - \mathrm{i}\lambda\delta(x - \frac{L}{2}), \tag{24}$$

where λ is the potential strength. Here λ can be positive or negative. When the λ is positive, this means that the gain is at the site $-\frac{L}{2}$, and the loss is the site at $\frac{L}{2}$. Otherwise, when the λ is negative, the loss is at $-\frac{L}{2}$ and the gain is at $\frac{L}{2}$. The $\mathcal{PT}$-symmetric potential in Eq. (24) can be written in the bra-ket formalism

$$V(x) = \mathrm{i}\lambda \,|{-\frac{L}{2}}\rangle \langle{-\frac{L}{2}}| - \mathrm{i}\lambda \,|\frac{L}{2}\rangle \langle\frac{L}{2}|\,. \tag{25}$$

Substituting Eq. (25) into Eq. (5) and multiplying the result from left by the bra $\langle x|$, we can obtain the wave function of particle

$$\langle x|\psi\rangle = \langle x|\psi_0\rangle + \mathrm{i}\lambda \langle x|\, G_0(E)\,|{-\frac{L}{2}}\rangle \langle{-\frac{L}{2}}|\psi\rangle - \mathrm{i}\lambda \langle x|\, G_0(E)\,|\frac{L}{2}\rangle \langle\frac{L}{2}|\psi\rangle\,. \tag{26}$$

In the ordinary x-representation, Eq. (26) can be rewritten as

$$\psi(x) = \mathrm{e}^{\mathrm{i}kx} + \mathrm{i}\lambda G_0(E; x, -\frac{L}{2})\psi(-\frac{L}{2}) - \mathrm{i}\lambda G_0(E; x, \frac{L}{2})\psi(\frac{L}{2}). \tag{27}$$

Now we set $x = -\frac{L}{2}$ and $x = \frac{L}{2}$ in above relation, by simple derivation, the relationship between $\psi(-\frac{L}{2})$ and $\psi(\frac{L}{2})$ can be reduced to

$$(1 - \mathrm{i}\lambda G_0(E; -\frac{L}{2}, -\frac{L}{2}))\psi(-\frac{L}{2}) + \mathrm{i}\lambda G_0(E; -\frac{L}{2}, \frac{L}{2})\psi(\frac{L}{2}) = \mathrm{e}^{-\mathrm{i}k\frac{L}{2}}, \tag{28}$$

and

$$(1 + \mathrm{i}\lambda G_0(E; \frac{L}{2}, \frac{L}{2}))\psi(\frac{L}{2}) - \mathrm{i}\lambda G_0(E; \frac{L}{2}, -\frac{L}{2})\psi(-\frac{L}{2}) = \mathrm{e}^{\mathrm{i}k\frac{L}{2}}. \tag{29}$$

This relation can be written in a matrix form

$$\begin{pmatrix} 1 - \mathrm{i}\lambda G_0(E; -\frac{L}{2}, -\frac{L}{2}) & \mathrm{i}\lambda G_0(E; -\frac{L}{2}, \frac{L}{2}) \\ -\mathrm{i}\lambda G_0(E; \frac{L}{2}, -\frac{L}{2}) & 1 + \mathrm{i}\lambda G_0(E; \frac{L}{2}, \frac{L}{2}) \end{pmatrix} \cdot \begin{pmatrix} \psi(-\frac{L}{2}) \\ \psi(\frac{L}{2}) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k\frac{L}{2}} \\ \mathrm{e}^{\mathrm{i}k\frac{L}{2}} \end{pmatrix}. \tag{30}$$

Thus we only need to multiply the inverse of matrix on the left in order to successfully obtain $\psi(-\frac{L}{2})$ and $\psi(\frac{L}{2})$

$$\begin{pmatrix} \psi(-\frac{L}{2}) \\ \psi(\frac{L}{2}) \end{pmatrix} = \begin{pmatrix} 1 - \mathrm{i}\lambda G_0(E; -\frac{L}{2}, -\frac{L}{2}) & \mathrm{i}\lambda G_0(E; -\frac{L}{2}, \frac{L}{2}) \\ -\mathrm{i}\lambda G_0(E; \frac{L}{2}, -\frac{L}{2}) & 1 + \mathrm{i}\lambda G_0(E; \frac{L}{2}, \frac{L}{2}) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k\frac{L}{2}} \\ \mathrm{e}^{\mathrm{i}k\frac{L}{2}} \end{pmatrix}. \tag{31}$$

Then we can obtain the final scattered wave function

$$\psi(x) = \mathrm{e}^{\mathrm{i}kx} + \alpha \mathrm{e}^{\mathrm{i}k|x+\frac{L}{2}|} \frac{\mathrm{e}^{-\mathrm{i}k\frac{L}{2}} + \alpha \mathrm{e}^{-\mathrm{i}k\frac{L}{2}} - \alpha \mathrm{e}^{\mathrm{i}k\frac{3L}{2}}}{1 + \alpha^2(\mathrm{e}^{2\mathrm{i}kL} - 1)}$$

$$-\alpha e^{ik|x-\frac{L}{2}|}\frac{e^{ik\frac{L}{2}}}{1+\alpha^2(e^{2ikL}-1)}, \quad (32)$$

where the dimensionless parameter $\alpha \equiv \frac{m\lambda}{\hbar^2 k}$. For the case of $x < -\frac{L}{2}$, the wave function can be deduced to

$$\psi(x) = e^{ikx} + \frac{\alpha}{1+\alpha^2(e^{2ikL}-1)}\Big[(e^{-ikL}-e^{ikL}) + \alpha(e^{-ikL}-e^{ikL})\Big]e^{-ikx}, \quad (33)$$

and for case $x > \frac{L}{2}$, the wave function is

$$\psi(x) = \frac{1}{1+\alpha^2(e^{2ikL}-1)}e^{ikx}. \quad (34)$$

Then, we can find the reflection and the transmission coefficients

$$R = \frac{4\alpha^2(1+\alpha)^2\sin^2(kL)}{1-4\alpha^2(1-\alpha^2)\sin^2(kL)}, \quad (35)$$

and

$$T = \frac{1}{1-4\alpha^2(1-\alpha^2)\sin^2(kL)}. \quad (36)$$

It is obvious that transmission coefficients are always greater than 1 while the condition satisfies $0 < \alpha^2 < 1$, and transmission coefficients are always less than 1 while the condition satisfies $\alpha^2 > 1$. Besides, for $\alpha^2 = 1$ or $k = \frac{n\pi}{L}$, transmission coefficients will be always 1, which indicate that the quantum system is the balanced state in this case and the gain of electron is equal to the loss electron. This means that the gain and loss are balanced only under special parameters. Finally we can successfully obtain the total probability

$$T + R = \frac{1+4\alpha^2(1+\alpha)^2\sin^2(kL)}{1-4\alpha^2(1-\alpha^2)\sin^2(kL)}. \quad (37)$$

Here we can find a special case of $\alpha = -1$, in which the total probability $T+R$ is always 1 irrespective of the energy. The reason is that by choosing $\alpha = -1$, we can obtain perfect transmission state for given k value.

Now we have derived the analysis results of transmission and reflection coefficients transmitting through a pair of $\mathcal{PT}$-symmetric delta-like potentials, and our obtained results are consistent with previous works by H. F. Jones [42]. By analyzing Eq. (37), we find that the numerator and denominator are always a non-negative value, so $T+R \geq 0$. But $T+R$ could be less than, greater than, or equal to 1, due to the competition of gain and loss. To show the property of $T+R$ more intuitively, in Fig. 2(a) we plot the $T+R$ as function of k while the λ is set to be $0, \pm 2, \pm 4, \pm 8$. As a reference, for $\lambda = 0$, it can be seen that $T+R$ is always be 1 satisfying the conservation law for probability current. While λ is set to be the positive value with $2, 4, 8$, we can see that the $T+R$ shows the periodic oscillation with the increasing of the wave vector k. Except for a few special points, the $T+R$ equals to 1, $T+R$ is always greater than 1, because the incident electron from the left side first hit the gain at $x = -\frac{L}{2}$ and the gain dominates. On the other hand, while λ is set to be the negative value with $-2, -4, -8$, with the increasing of k, the $T+R$ is reduced to less than 1 at the beginning and will be greater than 1 as k continues to increase. For $\alpha < -1$, $0 \leq T+R \leq 1$, whereas $\alpha > -1$, $T+R \geq 1$. Although both quantum systems with λ taking the positive and negative values maintain $\mathcal{PT}$ symmetry, the results of $T+R$ are different under the two conditions. Whether λ is set to be a positive value or negative value, the $T+R$ will not be 1 except for a few points ($k = \frac{n\pi}{L}$ or $\alpha = -1$), which indicates the breaking of the conservation law of probability current in this case. In the previous researches work, people always hope that the $\mathcal{PT}$-symmetric potential can balance gain and loss. However, according

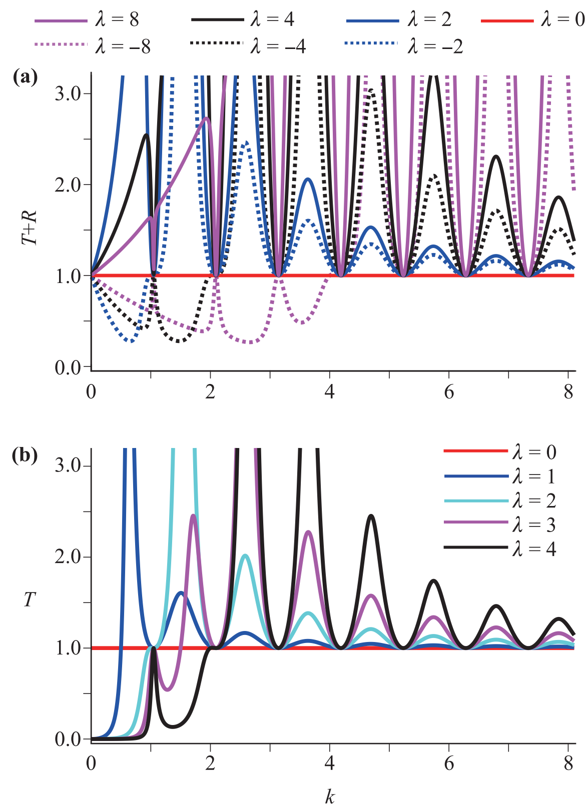

Fig. 2 The sum of the transmission and reflection coefficients **(a)** and transmission coefficients **(b)** as a function of k for different strength of potential while electron transports through the $\mathcal{PT}$-symmetric a pair of delta like potential. The system parameters as set as $\hbar = 1$, $m = 1/2$, and $L = 3$. In figure (a), the dotted magenta line, dotted black line and dotted blue line correspond to the results for $\lambda = -8$, $\lambda = -4$ and $\lambda = -2$, the red line, blue line, black line and magenta line correspond to the results for $\lambda = 0$, $\lambda = 2$, $\lambda = 4$, and $\lambda = 8$. In figure (b), the red line, blue line, cyan line, magenta line and black line correspond to the results for $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = 3$, and $\lambda = 4$.

to our obtained analytical solution Eq. (37), the balance between gain and loss (i.e., $T+R=1$) can only exist at a few special parameters, $k=\frac{n\pi}{L}$ or $\alpha=-1$. In fact, the effects of the incident electron to first hit the gain or to first hit the loss are different. As a result, the balance between gain and loss is destroyed usually.

It is well known that bound states correspond to poles of the scattering matrix, which has been postulated to provide a fundamental physical viewpoint [43]. However, the case of $\mathcal{PT}$-symmetric potentials is an exception[44]. From Fig. 2, one can see that $T+R$ keeps a certain value as a function of k, which indicates $\mathcal{PT}$-symmetric potentials cannot form the bound state and localize the electron completely.

According to Eq. (36), we also plot the transmission coefficients as the function of the wave vector k for a pair of $\mathcal{PT}$-symmetric delta-like potential. In Fig. 2(b), it can be observed that the transmission coefficients are always less than 1 for the conditions satisfying $\alpha^2>1$, and show resonant transmission points for $\lambda=2,3,4$. On the other hand, for $0<\alpha^2<1$, the transmission coefficients are larger than 1, which is consistent with our above analysis. While $\alpha=\frac{\sqrt{2}}{2}$ and $kL=(n+1/2)\pi$, the denominator of Eq. (36) is approaching to zero, the transmission coefficients reach a ∞ value. Furthermore, the transmission coefficients show periodic oscillation behavior with the minimum value of 1 and the amplitudes of oscillation decrease with decreasing of α from $\frac{\sqrt{2}}{2}$, i.e., with increasing of k from $\sqrt{2}\frac{m\lambda}{\hbar^2}$. Generally, we hope that $\mathcal{PT}$-symmetric quantum system is a stationary state with balanced gain and loss of particles. According to our above analysis, we would like to point out that only the condition of $k=\frac{n\pi}{L}$ or $\alpha=-1$ is satisfied, transmission coefficients will be always 1, in this case, this open system is a stationary state with balanced gain and loss of particles.

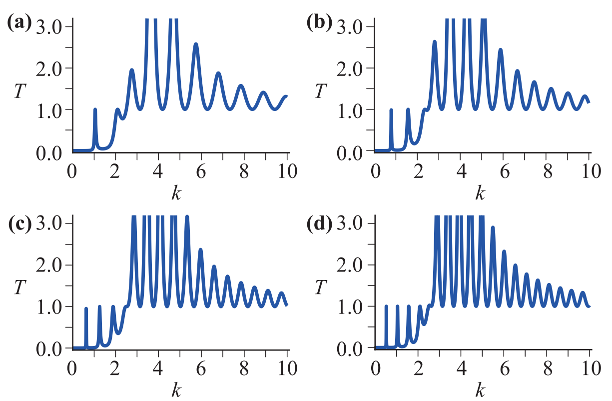

Fig. 3 Transmission coefficients as a function of k for different distance between two potential while electron transport through the $\mathcal{PT}$-symmetric a pair of delta like potential. The distance between two delta potential is set as $L=3$ **(a)**, $L=4$ **(b)**, $L=5$ **(c)** and $L=6$ **(d)**. The system other parameters are $\hbar=1$, $m=1/2$, and $\lambda=5$.

In Fig. 2(b), it has been found that there exists one resonance peak for the case $\lambda=4$. Now it turns to focus our attention on what elements will affect the resonant peaks of transmission coefficients. In Fig. 3 we plot the transmission coefficients as a function of k for the different L. It can be seen that the peaks of the transmission coefficient can be modulated by the two potential separated distance L, the numbers of resonant peaks increase with the increasing length of L, which indicate the more than one resonance states formed in a pair of $\mathcal{PT}$-symmetric delta-like potentials. This point is similar to the case of real value delta function potential, in which there exist the discrete energy spectrum and possible resonance states [45, 46], corresponding to the resonant peaks of the transmission coefficient.

5 Conclusions

In summary, we have employed the Lippmann–Schwinger formalism to derive an analytical solution of the transmission and reflection coefficients through an one-dimensional open quantum system, in which particle loss or gain on one lattice site, or particle loss on one lattice and gain on the other lattice site are considered respectively. The gain and loss on the lattice site are modeled by the delta potential with two opposite imaginary value. According to the analytical solution and computing results, we find that the total probability of $T+R$ is less/greater than 1 for the negative/positive imaginary potential, due to the particular flowing out/into the system to/from an external environment. Furthermore, we investigate the transport property of electron transmitting through a pair of $\mathcal{PT}$-symmetric delta potentials. It is found that $T+R$ is not equal to 1 in usual. This indicates that the balance between gain and loss is usually destroyed in the transport $\mathcal{PT}$-symmetric non-Hermitian system, which is different from the previous expectations. In addition, the resonance states can be formed by $\mathcal{PT}$-symmetric potential and transmission coefficients can be modulated by the strength of potential, which may be below or above 1. The results of our analysis can be treated as the starting point of studying quantum transport problems through a non-Hermitian system, and more general cases for high-dimensional systems may be deduced by the same procedure. And we plan to follow up the research results on the based of our proposed simple δ potential model, study the case of a single finite-range potential like the Poschl–Teller potential by the numerical calculation method in the next research work.

Acknowledgements This work was financially supported by the National Natural Science Foundation of China (Grant Nos. 12074097 and 11921005), the Natural Science Foundation of Hebei Province (Grant No. A2020205013), the National Key R&D Program of China (Grant No. 2017YFA0303301), the Strategic Priority Research Program of Chinese Academy of Sciences (Grant No. DB28000000), and Beijing Municipal Science & Technology Commission (Grant No. Z191100007219013).

References and notes

1. C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having *PT* symmetry, *Phys. Rev. Lett.* 80(24), 5243 (1998)
2. C. M. Bender, D. C. Brody, and H. F. Jones, Complex extension of quantum mechanics, *Phys. Rev. Lett.* 89(27), 270401 (2002)
3. C. M. Bender, Making sense of non-Hermitian Hamiltonians, *Rep. Prog. Phys.* 70(6), 947 (2007)
4. A. Mostafazadeh, Pseudo-Hermiticity for a class of non-diagonalizable Hamiltonians, *J. Math. Phys.* 43(12), 6343 (2002)
5. A. Mostafazadeh, Pseudo-Hermiticity versus *PT*-symmetry (II): A complete characterization of non-Hermitian Hamiltonians with a real spectrum, *J. Math. Phys.* 43(5), 2814 (2002)
6. A. Mostafazadeh, Pseudo-Hermiticity versus *PT*-symmetry (III): Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries, *J. Math. Phys.* 43(8), 3944 (2002)
7. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Observation of parity–time symmetry in optics, *Nat. Phys.* 6(3), 192 (2010)
8. S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, *PT* symmetry and spontaneous symmetry breaking in a microwave billiard, *Phys. Rev. Lett.* 108(2), 024101 (2012)
9. L. Chang, X. Jiang, S. Hua, C. Yang, J. Wen, L. Jiang, G. Li, G. Wang, and M. Xiao, Parity–time symmetry and variable optical isolation in active–passive-coupled microresonators, *Nat. Photonics* 8(7), 524 (2014)
10. L. Feng, Z. J. Wong, R. M. Ma, Y. Wang, and X. Zhang, Single-mode laser by parity–time symmetry breaking, *Science* 346(6212), 972 (2014)
11. H. Hodaei, M. A. Miri, M. Heinrich, D. N. Christodoulides, and M. Khajavikhan, Parity–time-symmetric microring lasers, *Science* 346(6212), 975 (2014)
12. Y. Wu, B. Zhu, S. F. Hu, Z. Zhou, and H. H. Zhong, Floquet control of the gain and loss in a *PT*-symmetric optical coupler, *Front. Phys.* 12(1), 121102 (2017)
13. N. Bender, S. Factor, J. D. Bodyfelt, H. Ramezani, D. N. Christodoulides, F. M. Ellis, and T. Kottos, Observation of asymmetric transport in structures with active nonlinearities, *Phys. Rev. Lett.* 110(23), 234101 (2013)
14. S. Assawaworrarit, X. Yu, and S. Fan, Robust wireless power transfer using a nonlinear parity–time-symmetric circuit, *Nature* 546(7658), 387 (2017)
15. Y. Choi, C. Hahn, J. W. Yoon, and S. H. Song, Observation of an anti-*PT*-symmetric exceptional point and energy-difference conserving dynamics in electrical circuit resonators, *Nat. Commun.* 9(1), 2182 (2018)
16. Y. Wu, W. Liu, J. Geng, X. Song, X. Ye, C. K. Duan, X. Rong, and J. Du, Observation of parity–time symmetry breaking in a single-spin system, *Science* 364(6443), 878 (2019)
17. G. Lindblad, On the generators of quantum dynamical semigroups, *Comm. Math. Phys.* 48(2), 119 (1976)
18. V. Gorini, A. Kossakowski, and E. C. Sudarsahan, Completely positive dynamical semigroups of N-level systems, *J. Math. Phys.* 17(5), 821 (1976)
19. S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, Topology by dissipation in atomic quantum wires, *Nat. Phys.* 7(12), 971 (2011)
20. F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum computation and quantum-state engineering driven by dissipation, *Nat. Phys.* 5(9), 633 (2009)
21. J. Dalibard, Y. Castin, and K. Molmer, Wave-function approach to dissipative processes in quantum optics, *Phys. Rev. Lett.* 68(5), 580 (1992)
22. H. J. Carmichael, Quantum trajectory theory for cascaded open systems, *Phys. Rev. Lett.* 70(15), 2273 (1993)
23. L. Jin and Z. Song, Physics counterpart of the *PT* non-Hermitian tight-binding chain, *Phys. Rev. A* 81(3), 032109 (2010)
24. A. E. Miroshnichenko, Nonlinear fano-Feshbach resonances, *Phys. Rev. E* 79(2), 026611 (2009)
25. L. Jin, and Z. Song, Hermitian scattering behavior for a non-Hermitian scattering center, *Phys. Rev. A* 85(1), 012111 (2012)
26. G. Zhang, X. Q. Li, X. Z. Zhang, and Z. Song, Transmission phase lapse in the non-Hermitian Aharonov–Bohm interferometer near the spectral singularity, *Phys. Rev. A* 91(1), 012116 (2015)
27. B. G. Zhu, R. Lü, and S. Chen, PT symmetry in the non-Hermitian Su–Schrieffer–Heeger model with complex boundary potentials, *Phys. Rev. A* 89(6), 062102 (2014)
28. L. L. Zhang, G. H. Zhan, Z. Z. Li, and W. J. Gong, Effect of PT symmetry in a parallel double-quantum-dot structure, *Phys. Rev. A* 96(6), 062133 (2017)
29. L. L. Zhang, and W. J. Gong, Transport properties in a non-Hermitian triple-quantum-dot structure, *Phys. Rev. A* 95(6), 062123 (2017)
30. L. L. Zhang, Z. Z. Li, G. H. Zhan, G. Y. Yi, and W. J. Gong, Eigenenergies and quantum transport properties in a non-Hermitian quantum-dot chain with side-coupled dots, *Phys. Rev. A* 99(3), 032119 (2019)
31. K. L. Zhang, X. M. Yang, and Z. Song, Quantum transport in non-Hermitian impurity arrays, *Phys. Rev. B* 100(2), 024305 (2019)
32. P. O. Sukhachov and A. V. Balatsky, Non-Hermitian impurities in Dirac systems, *Phys. Rev. Research* 2(1), 013325 (2020)
33. Y. Liu, X. P. Jiang, J. Cao, and S. Chen, Non-Hermitian mobility edges in one-dimensional quasicrystals with parity–time symmetry, *Phys. Rev. B* 101(17), 174205 (2020)
34. C. Wang and X. R. Wang, Level statistics of extended states in random non-Hermitian Hamiltonians, *Phys. Rev. B* 101(16), 165114 (2020)

35. S. Datta, Quantum Transport: From Atoms to Transistors, Cambridge, New York: Cambridge University Press, 2005
36. E. N. Economou, Greens Functions in Quantum Physics, 3rd Ed., Springer-Verlag, Germany, 2006
37. Here we use the form of Green′s function $G_0(x,x') = -\frac{im}{\hbar^2 k}e^{ik|x-x'|}$. While the energy E expands into the complex energy $E \to E \pm i0$, in fact there are two form solutions for the Green′s function $G_0^{\pm}(x,x') = -\frac{\mp im}{\hbar^2 k}e^{\pm ik|x-x'|}$. Here we only choose $G_0^{+}(x,x')$ as our solution because it can promise only the scattering waves traveling toward the positive direction exist in the limite $x \to \infty$.
38. D. Boese, M. Lischka, and L. E. Reichl, Resonances in a two-dimensional electron waveguide with a single δ-function scatterer, *Phys. Rev. B* 61(8), 5632 (2000)
39. F. Erman, M. Gadella, and H. Uncu, One-dimensional semirelativistic Hamiltonian with multiple Dirac delta potentials, *Phys. Rev. D* 95(4), 045004 (2017)
40. F. Erman, M. Gadella, and H. Uncu, On scattering from the one-dimensional multiple Dirac delta potentials, *Eur. J. Phys.* 39(3), 035403 (2018)
41. P. Molinàs-Mata and P. Molinàs-Mata, Electron absorption by complex potentials: One-dimensional case, *Phys. Rev. A* 54(3), 2060 (1996)
42. H. F. Jones, Scattering from localized non-Hermitian potentials, *Phys. Rev. D* 76(12), 125003 (2007)
43. J. Mehra and H. Rechenberg, The Historical Development of Quantum Theory, Vol. 6, Part 2, Springer-Verlag, New York, 2001; G. Chew, The Analytic S-Matrix, W.A. Benjamin, New York, 1966
44. J. G. Muga, J. P. Palao, B. Navarro, and I. L. Egusquiza, Complex absorbing potentials, *Phys. Rep.* 395(6), 357 (2004)
45. R. Zavin and N. Moiseyev, One-dimensional symmetric rectangular well: From bound to resonance via self-orthogonal virtual state, *J. Phys. Math. Gen.* 37(16), 4619 (2004)
46. F. Erman, M. Gadella, S. Tunalı, and H. Uncu, A singular one-dimensional bound state problem and its degeneracies, *Eur. Phys. J. Plus* 132(8), 352 (2017)