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RESEARCH ARTICLE

Equivariant PT-symmetric real Chern insulators

Y. X. Zhao[1,2,†]

[1] *National Laboratory of Solid State Microstructures and Department of Physics, Nanjing University, Nanjing 210093, China*
[2] *Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China*
E-mail: † *zhaoyx@nju.edu.cn*
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It was understood that Chern insulators cannot be realized in the presence of PT symmetry. In this paper, we reveal a new class of PT-symmetric Chern insulators, which has internal degrees of freedom forming real representations of a symmetry group with a complex endomorphism field. As a generalization to the conventional $2n$-dimensional Chern insulators with integer $n \geq 1$, these PT-symmetric Chern insulators have the n-th complex Chern number as their topological invariant, and have a $\mathbb{Z}$ classification given by the equivariant orthogonal K theory. Thus, in a fairly different sense, there exist ubiquitously Chern insulators with PT symmetry. By generalizing the Thouless charge pump argument, we find that, for a PT-symmetric Chern insulator with Chern number ν, there are equally many ν flavors of coexisting left- and right-handed chiral modes. Chiral modes with opposite chirality are complex conjugates to each other as complex representations of the internal symmetry group, but are not isomorphic. For the physical dimensionality $d = 2$, the PT-symmetric Chern insulators may be realized in artificial systems including photonic crystals and periodic mechanical systems.

Keywords topological insulator, Chern insulator

1 Introduction

The prosperity of topological phases of quantum matter started with the integer quantum Hall effect [1, 2], and the field of topological band theory is established by its generalization to systems without external magnetic field [3–5], later termed as Chern insulators, for which the topological invariants are identified as the valence-band Chern numbers taking integer values $\mathbb{Z}$, and the experimental realization has been done [6, 7]. Since the Chern number in band theory is odd under time-reversal, a Chern insulator cannot have time-reversal T or space-time-reversal PT symmetry with P being space inversion, though there are T- or PT-invariant topological insulators with $\mathbb{Z}_2$ classification, whose topology are not characterized by the Chern number. As T-symmetric topological band theories, such as two- and three-dimensional topological insulators, and various T-invariant superconductors, have been well studied during the last decade [8, 9], PT-symmetric band theory has recently been attracting more and more attention [10, 11], in particular systems with $(\hat{P}\hat{T})^2 = 1$, which is in part because that the Dirac nodal-line phase can be realized with the PT symmetry [12–16]. Such PT-symmetric crystals have essentially real band structures, since $\hat{P}\hat{T}$ symmetry is anti-unitary, and operates locally in momentum space without the Kramers degeneracy. It is noteworthy that PT symmetry with $(\hat{P}\hat{T})^2 = 1$ is more natural for artificial systems such as photonic crystals [17, 18] or periodic mechanical systems [19–21] than for fermionic systems, as fermionic systems satisfying $(\hat{P}\hat{T})^2 = 1$ require fermions being either spinless or having full spin-rotation symmetry, which is easily broken by spin–orbit coupling.

In this work, we study the PT-symmetric band theory with internal symmetry based on the equivariant orthogonal K theory or KO_G theory [22–24], and reveal a new class of PT-symmetric Chern insulators, which is in contrast to the commonly accepted understanding that Chern insulators cannot exist with PT symmetry. The internal symmetry group G can be either a Lie group or a finite group. While internal symmetries of Lie groups are a common topic in high energy physics, they may be realized as integer spin degrees of freedom in non-relativistic systems, such as the polarizations of photons and phonons, or may be engineered artificially. The PT-symmetric Chern insulator with an internal symmetry group is characterized by a complex Chern number with the imaginary unit given by an appropriate real representation of the internal symmetry, namely irreducible real representations with a

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complex endomorphism field. The physical meaning of the Chern number is elucidated by generalizing the Thouless charge argument [25–28]. If the Chern number is ν for a real irreducible representation W, for a positively oriented edge there are ν flavors of right-moving modes with each being an irreducible represenation V and ν flavors of left-moving modes with each being $\overline{V}$, where V and $\overline{V}$ are mutually conjugate complex irreducible representations with $W \cong V \oplus \overline{V}$ as a complex representation, but are not isomorphic to each other.

2 Basics about real representations

We start by introducing some basics about real representations of a group G [29, 30], which is either a compact Lie group or a finite group. Recall that Schur's Lemma, which is one of key results in complex group representation theory, states that a homomorphism between two irreducible complex representations of a group is either zero or an isomorphism $\lambda 1$ with λ being a complex number and 1 the identity matrix. So, the endomorphisms of an irreducible representation, namely the homomorphisms from an irreducible representation to itself, are just a complex field $\mathbb{C}$. This just reflects the fact that a matrix that commutes with all matrices representing group elements of G can only be $\lambda 1$. This well-known result is significantly changed in the context of real representations. For a real irreducible representation, it still holds that a nonzero endomorphism matrix that commutes with all real matrices in the irreducible representation is invertible, namely endomorphisms form a division ring over $\mathbb{R}$, but the real linear space of endomorphisms may be more than one dimensional. Specifically, for a real irreducible representation, at most three independent real matrices can be found that commutes with all representation matrices. When only one such matrix exists, the field formed by endomorphisms is of course the real numbers $\mathbb{R}$, which resembles the complex case. But when there are two or three, the corresponding endomorphism field is the complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$, respectively. The defining representation of $SO(2)$ is an example for $\mathbb{C}$, and details can be seen in Appendix A.

We now discuss how to determine the endomorphism field of a real irreducible representation. For a real irreducible representation W, group elements are represented by real invertible matrices, which cannot be simultaneously further block-diagonalized by similarity transformations of real matrices. However, after performing the complexification for W, namely regarding it as a complex representation, the resultant complex representation cW is not necessarily irreducible, and can be diagonalized into at most two irreducible blocks, namely $cW \cong V$ or $cW \cong V \oplus \overline{V}$, where $\overline{V}$ is the complex conjugate of V. Then the aforementioned three isomorphism fields of W can be identified as the following. (i) If cW is irreducible as a complex representation V, then the isomorphism field is $F_W \cong \mathbb{R}$. (ii) If $cW \cong V \oplus \overline{V}$ and V is not isomorphic to $\overline{V}$, $F_W \cong \mathbb{C}$. (iii) If $cW \cong V \oplus \overline{V}$ and $V \cong \overline{V}$, $F_W \cong \mathbb{H}$. The explicit form of the endomorphism field for each case in terms of real matrices are worked out systematically in Appendix B.

3 PT symmetry and equivariant orthogonal real K theory

We consider a quadratic many-particle Hamiltonian whose PT symmetry is represented by the relation $(\hat{P}\hat{T})^2 = 1$, which occurs for spinless fermion systems or effectively for fermions with full spin rotation symmetry, and more naturally for boson systems, such as photonic crystals and phonons for periodic mechanical systems [17, 19]. It is emphasized in the first place that we require no T or P throughout this work, but merely the combination PT. So it is possible that both P and T are broken, but PT is preserved. Probably the most significant consequence of PT symmetry with $(\hat{P}\hat{T})^2 = 1$ for band theory is that each band at any momentum corresponds to a real energy eigenstate $|a, \boldsymbol{k}\rangle \in \mathbb{R}$ for a Hamiltonian $\mathcal{H}(\boldsymbol{k})$ in momentum space with a being the band label. To understand this, it is first recalled that in momentum space P inverses $\boldsymbol{k}$ and so does T as an anti-unitary symmetry. Therefore their combination PT is represented by an anti-unitary operator acting locally in momentum space. Due to the absence of the Kramers degeneracy in the present case of $(\hat{P}\hat{T})^2 = 1$, space-time inversion PT can always be represented as $\hat{P}\hat{T} = \hat{\mathcal{K}}$ under an appropriate basis, where $\hat{\mathcal{K}}$ is the complex conjugate [10, 11]. Thus, the study of a PT symmetric band theory is equivalent to that of a real band theory, which assigns a real vector space at each momentum $\boldsymbol{k}$, namely we shall consider real vector bundles.

We now further consider an internal symmetry group G in such a PT symmetric system, which is independent of PT symmetry, namely $[\hat{P}\hat{T}, \hat{g}] = 0$ for every $g \in G$. In such a real band theory, G has a real representation at each momentum $\boldsymbol{k}$, which as seen previously is quite different from ordinary complex representations. For a fully gapped system (insulator for fermion systems), PT-symmetric bands below the energy gap form a real vector bundle (which are valence bands for fermion systems) representing G by real numbers at each $\boldsymbol{k}$, and can be topologically classified by the equivalent KO theory or KO_G theory over the first Brillouin zone (BZ). Let us recall that the classification of $2n$D Chern insulators with only $U(1)$ charge-conservation symmetry is given by the K group of complex vector bundles over a $2n$D sphere with integer $n \geq 0$ [31, 32], $K(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}$, where the first $\mathbb{Z}$ accounts for the dimensionality of a vector bundle and the second *stable* topological configurations regardless of dimensionality. Since only topological configurations are important and adding trivial bands does not affect topological properties, $2n$D Chern insulators have their classification given by the reduced K group $\widetilde{K}(S^{2n}) \cong \mathbb{Z}$ that is the second component of $K(S^{2n})$. Since $K(S^{2n+1}) \cong \mathbb{Z}$ or $\widetilde{K}(S^{2n+1}) \cong 0$, there is no topological insulator in odd dimensions if no symmetry is assumed.

Note that as internal symmetry, G acts trivially on the BZ. For a base manifold X with the trivial G action, the corresponding KO_G group can be decomposed into three components as

$$KO_G(X) \cong (KO(X) \otimes R(G;\mathbb{R})) \oplus (K(X) \otimes R(G;\mathbb{C})) \oplus (KSp(X) \otimes R(G;\mathbb{H})), \quad (1)$$

where $KO(X)$, $K(X)$ and $KSp(X)$ are K groups for real, complex and quaternionic vector bundles over X, respectively. Here $R(G;\mathbb{R})$ is the representation ring or character ring generated by real irreducible representations of G, for which the addition and multiplication are, respectively, given by the direction sum and tensor product of irreducible representations. Analogously, $R(G;\mathbb{C})$ is the ring of complex representations and $R(G;\mathbb{H})$ the ring of quaternionic representations. The three components correspond exactly to the afore-discussed three cases of real irreducible representations of G.

We now illustrate this point by a monopole in a 3D momentum space as illustrated in Fig. 1. The monopole is surrounded by a sphere S^2. The real Hamiltonian $\mathcal{H}(\boldsymbol{k})$ is assumed to be gapped on S^2. One may regard $\mathcal{H}(\boldsymbol{k})$ with $\boldsymbol{k} \in S^2$ as a Hamiltonian of a 2D gapped system. Accordingly, the classification of gapped systems is the same as that of monopoles in three dimensions, and is given by $KO_G(S^d)$. We now address the three components of $KO_G(S^2)$. Consider a real vector bundle on S^2, where each fiber as a real vector vector gives a real irreducible representation of G. Then on the whole S^2 all irreducible representations are isomorphic to a real irreducible representation W. To coordinate the S^2, we divide it into north and south hemispheres, D_N^2 and D_S^2, and

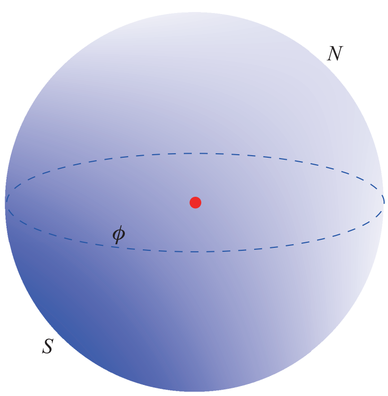

Fig. 1 A monopole surrounded by a sphere. The monopole is denoted by the red point at the origin, and the sphere is covered by north and south hemisphere, which are glued along the equator parametrized by ϕ.

glue them together along the equator $S^1 = \partial D_N^2 = \partial D_S^2$ parametrized by the azimuth angle $\phi \in [0, 2\pi)$. Accordingly we need to glue fibers on the edges of north and south hemispheres as well, which corresponds to a transition function f from S^1 to the automorphism group Auto(W) of W, which consists of invertible elements of End(W). However, as seen $\mathbb{F} = \text{End}(W)$ is one of the fields, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, which justifies the three components of $KO_G(S^2)$. If each fiber is a N-tuplet W^N of W, then an endomorphism is a matrix of $\mathbb{F}$, namely End$(W^N) \cong M_N(\mathbb{F})$, and therefore automorphisms are given by invertible matrices $GL(N, \mathbb{F})$, or Auto$(W^N) \cong GL(N, \mathbb{F})$. Then these vector bundles are topologically classified by the homotopy group $\pi_1(GL(N, \mathbb{F}))$. If N is sufficiently large, $\pi_1(GL(N, \mathbb{F}))$ gives $\widetilde{KO}(S^2)$, $\widetilde{K}(S^2)$ and $\widetilde{KSp}(S^2)$, respectively, for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

In the present case, $\widetilde{KO}(S^2) \cong \mathbb{Z}_2$, $\widetilde{K}(S^2) \cong \mathbb{Z}$ and $\widetilde{KSp}(S^2) \cong 0$, since $\pi_1(GL(N, \mathbb{R})) \cong \mathbb{Z}_2$ for $N \geq 3$, $\pi_1(GL(N, \mathbb{C})) \cong \mathbb{Z}$ for $N \geq 1$, and $\pi_1(GL(N, \mathbb{H})) \cong 0$. Although irreducible representations with endomorphism fields $\mathbb{R}$ and $\mathbb{H}$ give no nontrivial results noting that real bundles without symmetry are also classified by $\widetilde{KO}(S^2) \cong \mathbb{Z}_2$ [11], the second component $K(S^2) \otimes R(G, \mathbb{C})$ of $KO_G(S^2)$ implies that Chern insulators exist with $\mathbb{Z}$ classification for each real irreducible representation with endomorphism field being $\mathbb{C}$, in contrast to the well-known result that no Chern insulator exists with PT symmetry. From Eq. (1), such Chern insulators based on the complex endomorphism field of a real irreducible representation exist in any even dimensions $d = 2n$ with $n \geq 1$, since $\widetilde{K}(S^{2n}) \cong \mathbb{Z}$. As we shall see, such a PT-symmetric Chern insulator has an intimate topological relation to the conventional Chern insulator, which has already been reflected in that they both stem from $K(S^{2n})$, and furthermore the bulk-boundary phenomenology of PT-symmetric Chern insulators can also be derived by referring to that of conventional Chern insulators. For the clarity of narrative, and considering that $d = 2$ is physical for condensed matter physics, we focus on the 2D case, and its generalization to other even dimensions is straightforward.

4 *PT* symmetric Chern insulators

From the discussion above, we now start with a real irreducible representation $W_{\mathbb{C}}$ of G with endomorphism field being $\mathbb{C}$ and the PT operator $\hat{P}\hat{T} = \hat{\mathcal{K}}$ to construct a PT symmetric Chern insulator with internal symmetry G. Note that for any $\hat{P}\hat{T}$ with $(\hat{P}\hat{T})^2 = 1$, there exists a unitary transformation transforming it into $\hat{\mathcal{K}}$. First consider the Chern number of a conventional Chern insulator, $\nu_{\text{Ch}} = -\frac{1}{2\pi}\int \mathrm{d}^2k \ \text{tr i}\mathcal{F}_{xy}(\boldsymbol{k})$, where the Berry curvature $\mathcal{F}_{xy} = \partial_{k_x}\mathcal{A}_y(\boldsymbol{k}) - \partial_{k_y}\mathcal{A}_x(\boldsymbol{k})$. The Berry connection of valence bands is defined as $\mathcal{A}_{j,\alpha\beta}(\boldsymbol{k}) = \langle \alpha, \boldsymbol{k}|\partial_{k_j}|\beta, \boldsymbol{k}\rangle$, where α and β label valence bands, i is the imaginary unit, and

j is x or y. Now we have an irreducible real representation $W_{\mathbb{C}}$ whose endomorphism field has a natural imaginary unit J_W satisfying $J_W^2 = -1_W$ with 1_W being the identity matrix for $W_{\mathbb{C}}$. Thus, performing the replacement $\mathrm{i} \to J_W$, the Chern number for W follows as

$$\nu_{\mathrm{Ch}}^W = -\frac{1}{2\pi|W|}\int \mathrm{d}^2k\ \mathrm{tr} J_W \mathcal{F}_{xy}^W(\boldsymbol{k}), \tag{2}$$

where the Berry curvature $\mathcal{F}_{xy}^W(\boldsymbol{k}) = \partial_{k_x}\mathcal{A}_y^W(\boldsymbol{k}) - \partial_{k_y}\mathcal{A}_x^W(\boldsymbol{k})$, and $\mathcal{A}_{j,ab}^W(\boldsymbol{k}) = \langle a,\boldsymbol{k}|\partial_{k_j}|b,\boldsymbol{k}\rangle$ with a and b labeling real valence bands. Here $|W|$ is the dimensionality of W. Note that real bands form bunches, where each is of $|W|$-fold degeneracy in accordance with the representation W. One may express J_W more explicitly in the formula above as $J_W \otimes 1_N$ with N being the number of bunches.

As inspired by the construction of real Chern number from the complex Chern number, we expect that a PT symmetric Chern insulator for W with an arbitrary real Chern number ν can be constructed from a Hamiltonian $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$ of a conventional Chern insulator with Chern number ν. To do this, we first decompose the Hamiltonian $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$ into real and imaginary parts, $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k}) = \mathcal{H}_R(\boldsymbol{k}) + \mathrm{i}\,\mathcal{H}_I(\boldsymbol{k})$, where both $\mathcal{H}_R(\boldsymbol{k})$ and $\mathcal{H}_I(\boldsymbol{k})$ are real matrices for each $\boldsymbol{k}$, and under the transpose of matrix, $\mathcal{H}_R^T(\boldsymbol{k}) = \mathcal{H}_R(\boldsymbol{k})$ and $\mathcal{H}_I^T(\boldsymbol{k}) = -\mathcal{H}_I(\boldsymbol{k})$. Now the corresponding Hamiltonian of a PT-symmetric Chern insulator with Chern number ν is given by

$$\mathcal{H}_{\mathrm{Ch}PT}^W(\boldsymbol{k}) = 1_W \otimes_{\mathbb{R}} \mathcal{H}_R(\boldsymbol{k}) + J_W \otimes_{\mathbb{R}} \mathcal{H}_I(\boldsymbol{k}). \tag{3}$$

It can be verified by straightforward derivations that the Chern number of Eq. (3) is equal to that of $\mathcal{H}_{\mathrm{Ch}}$, and details can be found in Appendix C.

5 Thouless pump and bulk-boundary correspondence

For a momentum-space Hamiltonian $\mathcal{H}_{\mathrm{Ch}}(k_x, k_y)$ of a conventional Chern insulator, if we regard k_x as the momentum of a 1D insulator and k_y a periodic external parameter varying adiabatically, then it is well known as Thouless charge pump that the Chern number is just the quanta of the quantized charge pump in an adiabatic period of the 1D insulator, and the Thouless charge pump can be used to derive the chiral edge modes of the Chern insulator. We now adapt this picture to the PT symmetric Chern insulator with G being represented by W.

For a Hamiltonian $\mathcal{H}_{\mathrm{Ch}PT}^W(k_x, k_y)$ with a nontrivial Chern number ν of Eq. (2), there exist no eigenfunctions that are globally well defined in the whole BZ with the symmetries being preserved. However, we can choose symmetric eigenfunctions that are periodic for $k_x \in [-\pi, \pi)$ for any k_y. Now for valence bands $|\alpha, w, \boldsymbol{k}\rangle$ with w being the indexes of W and α labeling bunches, we introduce the quantity

$$P_W(k_y) = \frac{1}{|W|}\oint \frac{\mathrm{d}k_x}{2\pi} \sum_{\alpha}\langle\alpha,\boldsymbol{k}|J_W\partial_{k_x}|\alpha,\boldsymbol{k}\rangle, \tag{4}$$

where $|\alpha,\boldsymbol{k}\rangle$ is a vector of dimension $|W|$. $P_W(k_y)$ is analogous to the charge polarization of a 1D insulator with i being replaced by J_W. For the purpose of classification, we have required wave functions of the PT-symmetric band theory to be real, while of course, wave functions are complex in quantum mechanics. As mentioned above, the irreducible real representation W with endomorphism field $\mathbb{C}$ is isomorphic to $V \oplus \overline{V}$ with $V \ncong \overline{V}$ if it is regarded as a complex representation. This implies that there exists a unitary transformation U, such that $UJ_WU^\dagger = \mathrm{diag}(\mathrm{i}1_{|W|/2}, -\mathrm{i}1_{|W|/2})$. Thus, we introduce the centers of Wannier functions for V and $\overline{V}$ [33], respectively, as $P_V(k_y) = (2/|W|)\oint(\mathrm{d}k_x/2\pi)\sum_\alpha\langle\alpha,+,\boldsymbol{k}|\mathrm{i}\partial_{k_x}|\alpha,+,\boldsymbol{k}\rangle$ and $P_{\overline{V}}(k_y) = (2/|W|)\oint(\mathrm{d}k_x/2\pi)\sum_\alpha\langle\alpha,-,\boldsymbol{k}|\mathrm{i}\partial_{k_x}|\alpha,-,\boldsymbol{k}\rangle$, where $|\alpha,+,\boldsymbol{k}\rangle$ is a vector consisting of first $|W|/2$ components of $U|\alpha,\boldsymbol{k}\rangle$ and $|\alpha,-,\boldsymbol{k}\rangle$ the rest. Then, $2P_W(k_y) = P_V(k_y) - P_{\overline{V}}(k_y)$. It is clear that $P_V(k_y) + P_{\overline{V}}(k_y) = 0$, since $P_V(k_y) + P_{\overline{V}}(k_y) = (2/|W|)\oint(\mathrm{d}k_x/2\pi)\sum_a\langle a,\boldsymbol{k}|\partial_{k_x}|a,\boldsymbol{k}\rangle = 0$, but $|a,\boldsymbol{k}\rangle$ are real normalized vectors due to PT symmetry, therefore $P_W(k_y) = P_V(k_y) = -P_{\overline{V}}(k_y)$. Due to the periodicity of wave functions along k_x, the Chern number ν of Eq. (2) implies $P_W(\pi) - P_W(-\pi) = \nu$. Thus in a period through a point of the 1D insulator, ν charges of V is pumped from left to right, and ν charges of $\overline{V}$ is pumped from right to left.

We now infer topological edge modes corresponding to the bulk topological invariant of Eq. (2) from the charge pumps of 1D subsystems. Consider a strip geometry, for which the translational symmetry along y-axis is preserved while two edges are opened perpendicular to x-direction, so that $k_y \in [-\pi, \pi)$ is still a good quantum number. Then varying k_y in a period adiabatically pushes ν charges of V from the bulk to the right edge, which therefore have to be pumped out from the valence bands by appropriate edge bands, and pulls ν charges of V from the left edge into the

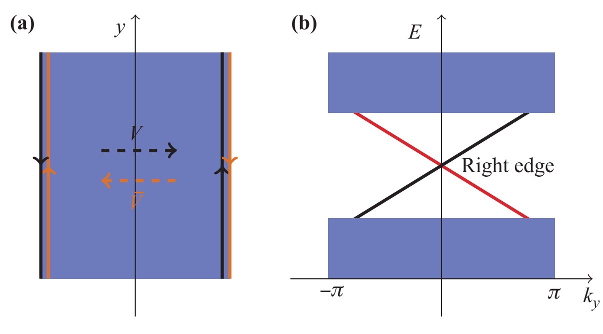

Fig. 2 **(a)** Chiral edge modes of PT symmetric Chern insulator with $\nu_{\mathrm{Ch}}^W = 1$ in a strip geometry with translational symmetry along the y-direction being preserved. **(b)** Schematic band structure of the right edge.

bulk, which have to be supplied by charges in the left edge. Thus, there are ν right-moving chiral bands of V connecting valence and conduction bands in the right edge, and ν left-moving ones in the left edge, as illustrated in Fig. 2. In contrast, for $\overline{V}$ there are ν left-moving modes in the right edge and ν right-moving ones in the left edge, since in the bulk $\overline{V}$ charges are pumped along the counter-direction of V charges. It is noted that these chiral edge modes of V and $\overline{V}$ are consistent with PT symmetry. In real space PT symmetry is anti-unitary space-time inversion, which maps right-moving (left-moving) modes of V ($\overline{V}$) in the right edge to the right-moving (left-moving) modes of $\overline{V}$ (V) in the left edge, since space-time inversion preserves chirality but complex conjugate exchanges V and $\overline{V}$. It is worth noting that the counter-moving modes of the equivariant PT-symmetric Chern insulators are fundamentally different from the edge modes of the quantum spin Hall systems in their topological origins [34, 35]. The quantum spin Hall system is protected by anti-unitary T symmetry with $T^2 = -1$, and the edge currents are intrinsically counter-moving Kramers' doublet without canonical decomposition, while the counter-moving modes in our model are conjugate representations of internal unitary symmetries. Moreover, the quantum spin Hall phase has nontrivial $\mathbb{Z}_2$ topological invariant, but our topological phases are classified by the $\mathbb{Z}$-valued real Chern invariant.

6 Discussion

To generalize the 2D Chern insulator to $2n$ dimensions, the Hamiltonian can still be constructed from Eq. (3) using a $2n$D conventional Chern insulator $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$. The corresponding Chern number is formulated by substituting the imaginary unit i by J_W in the nth Chern number. The bulk-boundary correspondence can be obtained straightforwardly by noting that right-(left-) moving modes is just the manifestation of right-(left-) handed chiral modes in two dimensions. For the case of endomorphism field $\mathbb{H}$, all physical dimensions $d = 0, 1, 2, 3$ are topologically trivial, and the first nontrivial dimension is $d = 4$ with $\widetilde{KSp}(S^4) \cong \mathbb{Z}$, which may be of interest for high energy physics. Finally, we propose that the internal symmetry group G, which may be a subgroup C_n of $SO(2)$, would be realized in some artificial systems, such as photonic and phononic crystals, with internal artificial micro-structures preserving G. Beside the advantage that their periodic structures can be flexibly designed, such systems has PT-symmetry naturally satisfies the required symmetry relation $(PT)^2 = 1$, since their quasi-particles have integer spins, rather than electronic systems have the relation only when the spin–orbit couplings are negligible.

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Appendix A Endomorphism field of the defining representation of $SO(2)$

A rotation by ϕ is represented as $\rho(\phi) = \cos\phi\, I + \sin\phi J$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A1}$$

or in the explicit form

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{A2}$$

So, there are two matrices, I and J, commuting with all $\rho(\phi)$ for $\phi \in [0, 2\pi)$, and a general endomorphism is represented as $f = aI + bJ$ with $a, b \in \mathbb{R}$. Since $J^2 = -I$, the endomorphisms form the complex field $\mathbb{C}$, where the complex conjugate is implemented by the matrix conjugate of

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A3}$$

Namely $f = aI + bJ \mapsto z = a + \mathrm{i}b$, and $K \mapsto \hat{\mathcal{K}}$ with $\hat{\mathcal{K}}$ being the standard complex conjugate.

Appendix B Explicit forms of endomorphism fields

While the first case is trivial, we now show how to work out the explicit form of isomorphism fields of $\mathbb{C}$ and $\mathbb{H}$ for irreducible real representations. In the case of $cW \cong V \oplus \overline{V}$ and $V \ncong \overline{V}$, the explicit form of the complex field is derived as

$$\mathbb{C}_W = \{\mathrm{Re}[T^{-1}\mathrm{diag}(\lambda 1_V, \mu 1_{\overline{V}})T], \lambda, \mu \in \mathbb{C}\}, \tag{B1}$$

where T is the matrix for the block diagonalization of cW, namely $\mathrm{diag}(\phi_V(g), \phi_{\overline{V}}(g)) = T\phi_{cW}(g)T^{-1}$ for $g \in G$. To understand this formula, it is helpful to recall that the ismorphism field of $V \oplus \overline{V}$ is $\mathbb{C} \oplus \mathbb{C}$. Then quite similarly, when $V \cong \overline{V}$, the quaternion field has the explicit form,

$$\mathbb{H}_W = \left\{ \mathrm{Re}\left[T^{-1} \begin{pmatrix} \lambda_1 1_V & \lambda_2 1_V \\ \lambda_3 1_V & \lambda_4 1_V \end{pmatrix} T \right], \lambda_i \in \mathbb{C} \right\}, \tag{B2}$$

where the similarity transformation of T block-diagonalizes cW as $\phi_{cW}(g) = T\mathrm{diag}(\phi_V(g), \phi_V(g))T^{-1}$ for $g \in G$, recalling that the isomorphism field of $V \oplus V$ consists of 4×4 matrices, $M_4(\mathbb{C})$.

Appendix C The equality of two Chern numbers

In this section, we verify that the PT symmetric Chern insulator $\mathcal{H}^W_{\mathrm{Ch}PT}(\boldsymbol{k})$ with the real irreducible representation W has a Chern number of Eq. (2) in the main text, which is equal to the Chern number of $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$. Recall that $\mathcal{H}^W_{\mathrm{Ch}PT}(\boldsymbol{k})$ is constructed from $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$, using the imaginary unit of the endomorphism field of W. For $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$, let $\Pi(\boldsymbol{k})$ be the projector onto valence states for each $\boldsymbol{k}$, and make the decomposition

$$\Pi(\boldsymbol{k}) = \Pi_R(\boldsymbol{k}) + \mathrm{i}\Pi_I(\boldsymbol{k}), \tag{C1}$$

where

$$\Pi_R^T(\boldsymbol{k}) = \Pi_R(\boldsymbol{k}), \quad \Pi_I^T(\boldsymbol{k}) = -\Pi_I(\boldsymbol{k}) \tag{C2}$$

due to the hermiticity of $\Pi(\boldsymbol{k})$. Then the corresponding projector for $\mathcal{H}^W_{\mathrm{Ch}PT}(\boldsymbol{k})$ is

$$\Pi^W(\boldsymbol{k}) = 1_W \otimes_{\mathbb{R}} \Pi_R(\boldsymbol{k}) + J_W \otimes_{\mathbb{R}} \Pi_I(\boldsymbol{k}). \tag{C3}$$

The Berry connections for $\mathcal{H}_{\mathrm{Ch}}(\boldsymbol{k})$ and $\mathcal{H}^W_{\mathrm{Ch}PT}(\boldsymbol{k})$ are just the Levi–Civitá connections, respectively,

$$\begin{aligned}\mathcal{A}_j(\boldsymbol{k}) &= \Pi(\boldsymbol{k})\partial_{k_j}\Pi(\boldsymbol{k}),\\ \mathcal{A}^W_j(\boldsymbol{k}) &= \Pi^W(\boldsymbol{k})\partial_{k_j}\Pi^W(\boldsymbol{k}).\end{aligned} \tag{C4}$$

Then,

$$\begin{aligned}\mathcal{A}_j(\boldsymbol{k}) =& (\Pi_R\partial_{k_j}\Pi_R - \Pi_I\partial_{k_j}\Pi_I)\\ &+ \mathrm{i}(\Pi_I\partial_{k_j}\Pi_R + \Pi_R\partial_{k_j}\Pi_I)\end{aligned} \tag{C5}$$

and

$$\begin{aligned}\mathcal{A}^W_j(\boldsymbol{k}) =& 1_W \otimes (\Pi_R\partial_{k_j}\Pi_R - \Pi_I\partial_{k_j}\Pi_I)\\ &+ J_W \otimes (\Pi_I\partial_{k_j}\Pi_R + \Pi_R\partial_{k_j}\Pi_I).\end{aligned} \tag{C6}$$

The Berry curvature

$$\begin{aligned}\mathcal{F}_{12} &= \partial_{k_1}\mathcal{A}_2 - \partial_{k_2}\mathcal{A}_1\\ &= \mathrm{i}([\partial_{k_1}\Pi_I, \partial_{k_2}\Pi_R] + [\partial_{k_1}\Pi_R, \partial_{k_2}\Pi_I]),\end{aligned} \tag{C7}$$

since

$$(\Pi_R\partial_{k_j}\Pi_R - \Pi_I\partial_{k_j}\Pi_I) = \partial_{k_j}\Pi_R/2 \tag{C8}$$

due to $\Pi_R^2 - \Pi_I^2 = \Pi_R$ from $\Pi^2 = \Pi$. Similarly,

$$\mathcal{F}^W_{12} = J_W \otimes ([\partial_{k_1}\Pi_I, \partial_{k_2}\Pi_R] + [\partial_{k_1}\Pi_R, \partial_{k_2}\Pi_I]). \tag{C9}$$

Then it is clear that

$$\begin{aligned}\nu_{\mathrm{Ch}} &= \nu^W_{\mathrm{Ch}}\\ &= (1/2\pi)\int \mathrm{d}^2k\ \mathrm{tr}([\partial_{k_1}\Pi_I, \partial_{k_2}\Pi_R] + [\partial_{k_1}\Pi_R, \partial_{k_2}\Pi_I]),\end{aligned} \tag{C10}$$

noting that $\mathrm{tr}(A \otimes B) = \mathrm{tr}A\ \mathrm{tr}B$ for any square matrices A and B.

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