Frontiers of Physics
https://doi.org/10.1007/s11467-019-0939-7
Front. Phys. 14(6), 61602 (2019)

RESEARCH ARTICLE

Recovering information in probabilistic quantum teleportation

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Received August 10, 2019; accepted September 26, 2019

In this paper we redesign the probabilistic teleportation scheme considered in *Phys. Rev. A* **61**, 034301 (2000) by Wan-Li Li *et al.*, where the *optimal state extraction protocol* complements the basic teleportation process with a partially entangled pure state channel, in order to transfer the unknown state with fidelity 1. Unlike that scheme, where the information of the unknown state is lost if the state extraction fails, our proposal teleports exactly and optimally an unknown state, and allows to recover faithfully that state when the process has not succeeded. In order to study the resilience of the scheme, we apply it to the teleportation problem through a quantum channel in a mixed state with pure dephasing. We find that a successful process transfers an unfaithful state, namely, the outcome state acquires the decoherence of the channel, but the unknown state is recovered by the sender with fidelity 1 if the teleportation fails. In addition, in this case, the fidelity of the teleported state has quantum features only if the channel has an amount of entanglement different from zero.

Keywords quantum information, teleportation

1 Introduction

Quantum teleportation [1–11] is in the heart of the development of quantum information theory, and also it plays a fundamental role in protocols quantum computation [12–16]. Essential resources for the faithful teleportation of an unknown state to a remote receiver are: a maximally entangled quantum channel, the ability to perform joint local measurements and unitary transformations, and a classical communication channel [1]. The absence of some of these elements turns teleportation inexact, probabilistic, or a combination of both. For instance, if the quantum channel is noisy, the receiver cannot reconstruct exactly the teleported unknown state, but only with an average fidelity $F < 1$ and a probability $p \neq 1$ [17–20]. Otherwise, a pure non-maximally entangled channel does allow to teleport a state with unit fidelity, but with probability different from 1 [21–29]. Specifically, Li *et al.* [23] proposed a probabilistic teleportation protocol which allows exact teleportation through a partially entangled pure state channel by introducing an *optimal state extraction*, a process which is carried out by the receiver. However, there is a finite probability for the protocol to fail and the information is lost. This protocol has been generalized for the N-qubit case [30].

In this article we propose a redesign of the Wan-Li Li *et al.* teleportation protocol [23] in such a way that the unknown state to be teleported is recovered if the protocol is not successful. This way, to attain the teleportation, the protocol can be repetead with other partially entangled channel but without the need of any copy of the unknown state [26, 31]. In order to study the resilience of the protocol, we apply it using a mixed channel with pure-dephasing [32–36], finding that when the teleportation is successful, the decoherence parameter of the channel is transferred to the teleported state. In spite of this, if the protocol fails, the unknown state $|\psi\rangle$ is recovered and can be used to repeat the process with a new channel until it succeeds.

2 Teleportation with state extraction

In this section we briefly discuss the protocol of probabilistic teleportation proposed by Li *et al.* in Ref. [23]. This scheme has the objective of transferring, with fidelity one, the unknown state $|\psi\rangle = \langle 0|\psi\rangle|0\rangle + \langle 1|\psi\rangle|1\rangle$ from laboratory A to laboratory B, through a non-maximally entangled channel, assumed in the normalized pure state

$$|\phi\rangle_{AB} = \alpha|0\rangle_A|0\rangle_B + \sqrt{1-\alpha^2}|1\rangle_A|1\rangle_B, \tag{1}$$

where, without loss of generality, we have assumed that α real. The kets $\{|0\rangle, |1\rangle\}$ are the eigenstates of the Pauli operator σ_z and the subscript $A(B)$ labels the qubit in lab $A(B)$. The entanglement degree of this channel state is given by the concurrence $C_{AB} = 2|\alpha|\sqrt{1-\alpha^2}$ [37, 38].

The teleportation process can be read from the following identity:

$$|\psi\rangle_a|\phi\rangle_{AB}=\sqrt{\frac{p_\phi}{2}}(|\phi^+\rangle_{aA}|\varphi_\phi\rangle+|\phi^-\rangle_{aA}\sigma_z|\varphi_\phi\rangle) + \sqrt{\frac{p_\psi}{2}}(|\psi^+\rangle_{aA}\sigma_x|\varphi_\psi\rangle+|\psi^-\rangle_{aA}\sigma_x\sigma_z|\varphi_\psi\rangle), \quad (2)$$

where $\{|\phi^\pm\rangle_{aA},|\psi^\pm\rangle_{aA}\}$ are the Bell states of qubits aA and where we have defined the following states:

$$|\varphi_\phi\rangle=\frac{\alpha\langle 0|\psi\rangle|0\rangle+\sqrt{1-\alpha^2}\langle 1|\psi\rangle|1\rangle}{\sqrt{p_\phi}}, \quad (3a)$$

$$|\varphi_\psi\rangle=\frac{\sqrt{1-\alpha^2}\langle 0|\psi\rangle|0\rangle+\alpha\langle 1|\psi\rangle|1\rangle}{\sqrt{p_\psi}}, \quad (3b)$$

with

$$p_\phi=\alpha^2|\langle 0|\psi\rangle|^2+(1-\alpha^2)|\langle 1|\psi\rangle|^2, \quad (4a)$$

$$p_\psi=\alpha^2|\langle 1|\psi\rangle|^2+(1-\alpha^2)|\langle 0|\psi\rangle|^2. \quad (4b)$$

In its first stage, the protocol is identical to the standard teleportation scheme of Bennett *et al.* [1]: The sender in A performs a measurement on the Bell basis on its two qubits and communicates the result to the receiver in B through a classical two bits channel. The latter applies one of the gates σ_x, $\sigma_x\sigma_z$, or σ_z, in order to complete the teleportation process. However, since the channel is not maximally entangled, the outcome state after all these steps is not the original $|\psi\rangle$, but one of the states (3a) and (3b), with the respective probabilities (4a) or (4b).

Now, to be able to teleport exactly the state $|\psi\rangle$ it is required to *extract* $|\psi\rangle$ from the resulting state, (3a) or (3b). Such a state extraction process is performed by the receiver by means of a unitary-reduction process on the target qubit B and an ancillary qubit b. The state extraction process involves applying a controlled U gate which depends on which was the teleported state (3), followed by a measurement on the ancilla, assumed initially in a state $|0\rangle_b$. For instance, if the teleported state is $|\varphi_\phi\rangle$ and $\alpha\le 1/\sqrt{2}$, then the joint unitary U becomes

$$U=|0\rangle\langle 0|\otimes I_b+|1\rangle\langle 1|\otimes U_b, \quad (5a)$$

$$U_b|0\rangle_b=\frac{1}{\sqrt{1-\alpha^2}}\left(\alpha|0\rangle_b+\sqrt{1-2\alpha^2}|1\rangle_b\right). \quad (5b)$$

Thus the extraction can be read from the following equation,

$$U|\varphi_\phi\rangle_B|0\rangle_b=\frac{\alpha}{\sqrt{p_\phi}}|\psi\rangle|0\rangle_b+\frac{\sqrt{1-2\alpha^2}\langle 1|\psi\rangle}{\sqrt{p_\phi}}|1\rangle|1\rangle_b, \quad (6)$$

and the probability to recover $|\psi\rangle$ becomes the probability to project the ancillary qubit b to the state $|0\rangle_b$, which is $p=\alpha^2/p_\phi$. Notice that $\alpha^2+(1-2\alpha^2)|\langle 1|\psi\rangle|^2=p_\phi$, which guarantees the normalization of the right side of Eq. (6).

Similarly, if $\alpha\ge 1/\sqrt{2}$, then the joint unitary U becomes

$$U=|1\rangle\langle 1|\otimes I_b+|0\rangle\langle 0|\otimes U_b, \quad (7a)$$

$$U_b|0\rangle_b=\frac{1}{\alpha}\left(\sqrt{1-\alpha^2}|0\rangle_b+\sqrt{2\alpha^2-1}|1\rangle_b\right), \quad (7b)$$

in this case the extraction is read from the next identity,

$$U|\varphi_\phi\rangle|0\rangle_b=\frac{\sqrt{1-\alpha^2}}{\sqrt{p_\phi}}|\psi\rangle|0\rangle_b+\frac{\sqrt{2\alpha^2-1}\langle 0|\psi\rangle}{\sqrt{p_\phi}}|0\rangle|1\rangle_b, \quad (8)$$

and by projecting onto $|0\rangle_b$ the state $|\psi\rangle$ is recovered, which holds with probability $p=(1-\alpha^2)/p_\phi$. Again, $(1-\alpha^2)+(2\alpha^2-1)|\langle 0|\psi\rangle|^2=p_\phi$, so that the right side of Eq. (8) is normalized. Therefore, the probability of extracting $|\psi\rangle$ from the state $|\varphi_\phi\rangle$ can be expressed as $p=\min\{\alpha^2,1-\alpha^2\}/p_\phi$. In consequence, taking into account both outcome states (3) and their probabilities $\{p_\psi,p_\phi\}$, the total success conditional probability of extracting $|\psi\rangle$ is given by

$$p_{ext}=p_\phi\frac{\min\{\alpha^2,1-\alpha^2\}}{p_\phi}+p_\psi\frac{\min\{\alpha^2,1-\alpha^2\}}{p_\psi}, \\ =2\min\{\alpha^2,1-\alpha^2\}, \quad (9)$$

otherwise, as can be seen from Eqs. (6) and (8) the information is lost with total conditional probability $p_f=1-p_{ext}$.

3 Teleportation without loss of information

In this section we present the proposed scheme. We start by describing the probabilistic teleportation protocol with a pure state channel, and then we study its resilience by applying it to a mixed state channel with pure dephasing. Without loss of generality, we assume that α is a positive real number with $\alpha\le 1/\sqrt{2}$, and that the ancilla is in a state $|0\rangle_b$.

3.1 Pure channel

Here we show that if the state extraction process is performed before that the sender measures the qubits aA, the teleportation with fidelity 1 is achieved with optimal probability (9) and the state $|\psi\rangle$ is not lost if the process fails.

By considering the auxiliary qubit and applying the gate (5), the identity (2) goes to

$$U|\psi\rangle|\phi_{AB}\rangle|0\rangle_b \\ =\sqrt{2}\alpha\frac{1}{2}(|\phi^+_{aA}\rangle|\psi\rangle+|\phi^-_{aA}\rangle\sigma_z|\psi\rangle+|\psi^+_{aA}\rangle\sigma_x|\psi\rangle \\ +|\psi^-_{aA}\rangle\sigma_z\sigma_x|\psi\rangle)|0\rangle_b+\sqrt{1-2\alpha^2}|\psi\rangle|1\rangle|1\rangle|1\rangle_b. \quad (10)$$

It is straightforward to verify that the above state is normalized. The proposed protocol can be straighforwardly read from Eq. (10). If the receiver makes a measurement of the ancilla on the basis $\{|0\rangle_b,|1\rangle_b\}$, there is a probability $2\alpha^2$ that it is projected to the state $|0\rangle_b$, case in which standard teleportation process (2) arises. Otherwise, with probability $1-2\alpha^2$ the ancilla is projected on state $|1\rangle_b$, in this case, as follows from Eq. (10), the qubit a is left in state $|\psi\rangle$, which means that the state to be teleported is recovered if the process fails. Notice that, in both cases a classical channel of one bit is required in order that the receiver in the laboratory B can communicate the sender in the laboratory A the result of the measurement on the ancilla, in order that the sender either apply the Bennett protocol, or the probabilistic process is repeated with a new channel.

3.2 Mixed channel

Now we assume that the teleportation is performed using a mixed state channel with pure dephasing, which can be written as

$$\rho_{AB} = \alpha^2|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| + \gamma\alpha\sqrt{1-\alpha^2}(|0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| + |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B|) + (1-\alpha^2)|1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|,$$
$$= |\phi\rangle_{AB}\langle\phi|_{AB} + (\gamma-1)\alpha\sqrt{1-\alpha^2}(|0\rangle_A|0\rangle_B\langle 1|_A\langle 1|_B + |1\rangle_A|1\rangle_B\langle 0|_A\langle 0|_B),$$

where $0 \le \gamma \le 1$ describes the decoherence effect when it is different from 1. The entanglement of this channel is described by its concurrence $C = 2\gamma\alpha\sqrt{1-\alpha^2}$.

By applying gate (5) and hence projecting onto $|0\rangle_b|\phi^\pm\rangle_{aA}$ the initial state $\rho = |\psi\rangle\langle\psi|\rho_{AB}|0\rangle_b\langle 0|_b$ we obtain the normalized outcome states

$$\rho_{0,\phi^\pm} = \langle\phi^\pm|_{aA}\langle 0|_b\rho|0\rangle_b|\phi^\pm\rangle_{aA},$$
$$= \sigma_z^{(1\mp 1)/2}[|\psi\rangle\langle\psi| + (\gamma-1)(\langle 0|\psi\rangle\langle\psi|1\rangle|0\rangle_B\langle 1|_B + \langle\psi|0\rangle\langle 1|\psi\rangle|1\rangle_B\langle 0|_B)]\sigma_z^{(1\mp 1)/2}. \quad (11)$$

Similarly, by projecting onto $|0\rangle_b|\psi^\pm\rangle_{aA}$ the initial state goes to the normalized outcome states

$$\rho_{0,\psi^\pm} = \langle\psi^\pm|_{aA}\langle 0|_b\rho|0\rangle_b|\psi^\pm\rangle_{aA},$$
$$= \sigma_x\sigma_z^{(1\mp 1)/2}[|\psi\rangle\langle\psi| + (\gamma-1)(\langle 1|\psi\rangle\langle\psi|0\rangle|1\rangle_B\langle 0|_B + \langle 0|\psi\rangle\langle\psi|1\rangle|0\rangle_B\langle 1|_B)]\sigma_z^{(1\mp 1)/2}\sigma_x. \quad (12)$$

Clearly, by removing the unitaries from states (11) and (12) the teleported state from qubit a to B becomes

$$\rho_t = |\psi\rangle\langle\psi| + (\gamma-1)(\langle 1|\psi\rangle\langle\psi|0\rangle|1\rangle\langle 0| + \langle 0|\psi\rangle\langle\psi|1\rangle|0\rangle\langle 1|). \quad (13)$$

We note that this is not the unknown state $|\psi\rangle$, since the non-diagonal terms have been affected by the quantum channel decoherence. The fidelity of this state [19, 39, 40], averaged over all the possible initial states, is

$$F = \frac{2}{3} + \frac{\gamma}{3}, \quad (14)$$

which is a linear function of γ, being 2/3 when $\gamma = 0$, that is, the channel is completely incoherent (classical limit), and 1 when $\gamma = 1$, that is, the channel is pure. In addition, since each of the four outcomes (11) and (12) arises with probability $\alpha^2/2$, then the probability of obtaining the teleported state (13) is $2\alpha^2$.

On the other hand, there is a probability $1-2\alpha^2$ that the measurement of the ancilla b project it to the state $|1\rangle_b$, in this case the qubits a, A, B projected to the normalized pure state

$$\rho_f = |\psi\rangle\langle\psi| \otimes |1\rangle_A\langle 1|_A \otimes |1\rangle_B\langle 1|_B, \quad (15)$$

that is, the qubit a is left in the unknown state $|\psi\rangle$, or equivalently, the original information is recovered.

Thus, the proposed protocol, operating with a mixed quantum channel with pure-dephasing, allows to teleport the state $|\psi\rangle$ with a success probability $P_s = 2\alpha^2$, with a fidelity which falls from 1 to 2/3 as γ increases from 0 to 1. Notice that γ affects only the teleportation fidelity, and not the ability of the protocol to recover the original state $|\psi\rangle$.

4 Conclusions

We have redesigned the probabilistic teleportation protocol proposed by Wan-Li Li *et al.* [23], allowing to transfer faithfully an unknown state, in order that the information can be recovered if the protocol fails. To attain this, we have reversed the order in which the two basic protocols are applied: in the first place the receiver performs the *optimal state extraction* and hence the sender applies the Bell-measurement. The cost of this change is the need of an extra classical bit, so that the receiver can communicate to the sender whether the extraction has been successful.

We have studied the resilience of the scheme in the problem of teleportation via a quantum channel in a mixed state with pure-dephasing. We find that a successful process transfers an unfaithful state, namely, the outcome state acquires the decoherence of the channel, but the unknown state is recovered by the sender with fidelity 1 if the teleportation fails. In this case, the fidelity of the teleported state has quantum features when the channel has amount of entanglement different from zero.

Acknowledgements This work was supported by FONDECyT under grant No. 1161631. A. M. thanks to CONICyT.

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