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PERSPECTIVE

Bohm's approach to quantum mechanics: Alternative theory or practical picture?

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Since its inception Bohmian mechanics has been generally regarded as a hidden-variable theory aimed at providing an *objective* description of quantum phenomena. To date, this rather narrow conception of Bohm's proposal has caused it more rejection than acceptance. Now, after 65 years of Bohmian mechanics, should still be such an interpretational aspect the prevailing appraisal? Why not favoring a more pragmatic view, as a legitimate picture of quantum mechanics, on equal footing in all respects with any other more conventional quantum picture? These questions are used here to introduce a discussion on an alternative way to deal with Bohmian mechanics at present, enhancing its aspect as an efficient and useful picture or formulation to tackle, explore, describe and explain quantum phenomena where phase and correlation (*entanglement*) are key elements. This discussion is presented through two complementary blocks. The first block is aimed at briefly revisiting the historical context that gave rise to the appearance of Bohmian mechanics, and how this approach or analogous ones have been used in different physical contexts. This discussion is used to emphasize a more pragmatic view to the detriment of the more conventional hidden-variable (ontological) approach that has been a leitmotif within the quantum foundations. The second block focuses on some particular formal aspects of Bohmian mechanics supporting the view presented here, with special emphasis on the physical meaning of the local phase field and the associated velocity field encoded within the wave function. As an illustration, a simple model of Young's two-slit experiment is considered. The simplicity of this model allows to understand in an easy manner how the information conveyed by the Bohmian formulation relates to other more conventional concepts in quantum mechanics. This sort of pedagogical application is also aimed at showing the potential interest to introduce Bohmian mechanics in undergraduate quantum mechanics courses as a working tool rather than merely an alternative interpretation.

Keywords Bohmian mechanics, quantum phase, velocity field, interference, Young's two-slit experiment

1 Introduction

In 1917 Einstein settled down the theoretical foundations for the maser and the laser with the introducing of the notion of stimulated radiation emission [1]. By the end of the same year, David Bohm, Einstein's "spiritual son", was born in Wilkes-Barre, Pennsylvania. About 35 years later, Bohm proposed what is now widely known as Bohmian mechanics [2, 3], an alternative way to look at quantum phenomena that would have a strong influence on John Bell and his celebrated theorem – which led to new research areas in quantum mechanics, such as the quantum information theory and the so-called quantum technologies. But, just when this work on the reformulation of quantum mechanics has turned 65, how much of its pedagogical value to describe, understand and explain quantum phenomena is actually known out of the quantum foundations community? Indeed, one may even wonder whether it would not be worth taking it out of such a community, and incorporate it into the curricular scopes of quantum mechanics courses, in the same way we learn, for instance, to operate with the Schrödinger and Heisenberg formulations.

Since the 1930s there has been an active discussion around the question of whether it is possible to find a compromise between the internal consistency of the quantum theory and its apparently statistical or probabilistic nature, quite distant from the classical causality we are used to. In 1952, moved by the need to understand and provide a clear answer to this fundamental question, Bohm proposed [2, 3] what he called an "interpretation of the

quantum theory in terms of "hidden" variables". Several decades later, this interpretation started being referred to as *Bohmian mechanics* and became one of the warhorse issues within the still open debate on the interpretation of the quantum realm, with a strong influence on the perception and understanding of Reality and the Universe we live in [4, 5].

Leaving aside such deep matters and remaining at a bare formal level – i.e., taking on a more pragmatic viewpoint – what Bohmian mechanics does is to provide us with a precise mathematical language to tackle quantum problems [6–11], complementary to the mathematical language involved in other more widely known quantum pictures [12], such as those proposed by Schrödinger, Heisenberg, Wigner and Moyal, or Feynman, for instance (for the interested reader, an enlightening perspective on the different formulations of quantum mechanics is provided by Styer *et al.* in "Nine formulations of quantum mechanics" [13]). Such a language has a strong reminiscence of classical hydrodynamics, offering an alternative way to represent quantum phenomena in terms of swarms of time-evolving streamlines or trajectories. In spite of this neat single-event dynamics, the results obtained from a statistical analysis of the behavior of these swarms of trajectories is in direct correspondence with the expectation value associated with quantum observables in those other quantum pictures.

The main goal of this work is to introduce a discussion on an updated view of Bohmian mechanics beyond its more widely known ontological side. This is done taking advantage of the gradual transition in the perception of this approach, from its origins as a nonlocal hidden-variable model to the current wider view as a quantum picture. This discussion is thus guided by two intertwined questions:

- After 65 years of Bohmian mechanics, should its interpretational aspect still be its prevailing appraisal?
- Is Bohmian mechanics a legitimate picture of quantum mechanics, on equal footing in all respects with any other quantum picture?

Accordingly, the work has been organized around two complementary building blocks. The first block, Section 2, enters a discussion on the origin and role of Bohmian mechanics as a hidden-variable model [14], introducing the issue through two of its most common criticisms and concluding it with a discussion on how the idea of monitoring waves (not necessarily quantum ones) by means of streamlines or trajectories has been used in the literature with other purposes than fundamental ones. Because it covers both fundamental and computational aspects, this block offers a wider perspective on questions that do not usually appear together in the context of the quantum foundations in spite of the relevance of a joint view. The second block, Section 3, addresses in a simplified manner the more basic formal aspects of Bohmian mechanics, useful to interpret quantum outcomes obtained from Schrödinger's equation on a hydrodynamic basis with the aid of Bohmian trajectories. Other Bohmian-related quantities, such as the local quantum phase field, the quantum velocity, or the quantum potential are also discussed. As an illustration of these concepts, and particularly with the purpose to explicitly show how Bohmian mechanics operates or what kind of information it may provide us, this block also includes a revision and discussion of a simple implementation of Young's two-slit experiment. Finally, in Section 4 a series of remarks summarize the main points stressed in this work.

2 The general "picture"

2.1 What's wrong with Bohmian mechanics?

Different criticisms can be risen against Bohmian mechanics [15]. Among them, there are two that are particularly relevant to the discussion in this work. The first criticism is very common: because the outputs rendered by Bohmian mechanics are equivalent to those provided by other quantum pictures, this approach is redundant and, therefore, unnecessary. Trivial, but acceptable. Although it is also unfair, unless the same is rigorously applied to other pictures. Of course, this leads us to the absurd situation that all pictures except one are also redundant and unnecessary, which is not the case. The Schrödinger and Heisenberg pictures, for instance, are conceptually different, not only in their origin, but also concerning their mathematical languages. However, both have proven to be very useful to tackle quantum problems, providing us with different perspectives and/or strategies on the same issue.

In general, once a quantum picture is accepted, even if it makes the same predictions as other available pictures, there are at least two intertwined aspects that set a difference:

(1) A *physical aspect*. This is analogous to what happens in classical mechanics, where each (classical) picture allows us to approach the same physical problem from a different perspective, highlighting a particular feature or concept, and thus providing us with alternative insights – think, for instance, of Newton's forces at a distance versus Faraday's fields. This aspect, connected to the way how we explain and understand physical phenomena, has an intrinsic interest (interpretational) at the fundamental level.

(2) A *practical aspect*. It is associated with the potentiality of the picture as a resource to generate and develop new numerical tools based on it, aimed at solving in a more efficient way quantum problems. This important feature allows us to recreate (simulate) quantum phenomena and, therefore, to extend the theory beyond the limitations of analytical treatments, which is the case in most cases of practical

> interest. Usually this is precisely the aspect behind the fact that a given picture will eventually become or not widely accepted (or, at least, it has an important weight on this fact).

It is by virtue of these two aspects that we do not need to chose between one approach and the rest. Same happens with Bohmian mechanics. In this particular case, apart from its intrinsic physical interest, the practical side is also backed by the number and variety of numerical algorithms appeared since the end of the 1990s [16], the so-called *quantum trajectory methods* [6]. These methods take advantage of an earlier reformulation of Schrödinger's equation in a hydrodynamic form, proposed in 1926 by Madelung [17], which establishes a link between quantum mechanics and classical hydrodynamics, and hence an effective transfer of numerical resources from the latter to the former.

The second criticism is related to the above physical aspect, specifically the fact that Bohmian mechanics is based on the concept of trajectory, incompatible in principle with quantum mechanics – Feynman's path integral approach also found an analogous opposition from Bohr (see Ref. [18], pp. 245–248). The apparent "harm" here comes from a direct association of Bohm's trajectories, understood as "hidden variables" (not experimentally accessible), with "real" paths followed by the quantum system [19] (quantum particle). This association arises from the first of the two papers that Bohm published [2] in 1952, where he states:

> The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, viz., that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The only way of investigating the truth of this assumption is by trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the types that can now be carried out. In this paper [...] an interpretation of the quantum theory in terms of just such "hidden" variables is suggested.

This identification has been used to raise Bohmian mechanics to the level of ontological quantum theory, where appealing to the action of an external observer to cause the collapse of the wave function is not necessary – a quantum theory without observers [20, 21] However, apart from just being a matter of interpretation, there is no formal or empirical evidence for such a connection. Bohmian trajectories only serve to monitor the flux or diffusion of the quantum system throughout the corresponding configuration space [22].

2.2 Quantum mechanics and hidden variables

The original contextual background of Bohm's approach is the debate on the interpretation of the wave function [23–25], specifically the quest for a realistic description of quantum phenomena – an *ontological interpretation* of the wave function – led by Einstein in the 1930s. It is thus worth getting back to the 5th International Solvay Conference [26], "Electrons et photons", held in Brussels in 1927, probably the most famous one within this conference series. The attendees to this conference (see Fig. 1) addressed the problem of the interpretation of quantum mechanics, an issue that was somehow settled by Bohr (with no little opposition from Einstein — "God does not play dice") and eventually gave rise to the so-called Copenhagen or orthodox interpretation [27]. According to this interpretation, it does not make any sense to ask what or where a quantum system is until a measurement is performed. This way to conceive the quantum realm is tightly associated with the rather abstract matrix formulation of quantum mechanics, formulated by Heisenberg, Born, and Jordan in 1925 [28]. No need to say that this position did not favor at all the hypothesis presented in the same conference by de Broglie on the possibility of a pilot wave guiding the quantum system, strongly contested by Pauli and other participants [29].

Copenhagen left little room for a different view than the one posed by Bohr: unlike classical mechanics, quantum mechanics only provides us with probabilistic information on the system, formally accessible through the squared modulus of the wave function describing the actual state of such a system. This was opposite to Einstein's view: the wave function, by its own, cannot completely specify the state of the system, hence a set of additional parameters or variables are needed to unambiguously specify the state of the system, and therefore to provide of a full physical sense to the statistical outcomes rendered by the wave function. In brief, the wave function description is

Fig. 1 Attendees to the 5th International Solvay Conference (Brussels, 1927).

incomplete. These variables are referred to as *hidden variables*, because in principle they would be "hidden" to the experimentalist (not experimentally accessible).

At a formal level, von Neumann showed (see p. 210 in Ref. [30]) that "an introduction of hidden parameters is certainly not possible without a basic change in the present theory" (quantum mechanics). To illustrate the role played by the hidden variables, von Neumann recalls the probabilistic way how the kinetic theory of gases operates, where the set of Newtonian positions and momenta that uniquely specify the state of each gas atom or molecule is replaced by only two statistical parameters: pressure and temperature. However, unlike this classical picture, von Neumann proves (see pp. 323–325 in Ref. [30]) that quantum mechanics cannot be re-derived as a statistical approximation of an underlying causal classical-like theory based on a set of additional variables. It is this incompatibility what leads to the impossibility to reach a more complete specification of the state of a quantum system than the one provided by the associated wave function.

Twenty years after von Neumann's proof on the impossibility of hidden variables, Bohm found [2, 3] a counterexample after recasting Schrödinger's equation in the form of two real-valued partial differential equations (see Section 3). This model was formally identical to the hydrodynamic one proposed in 1926 by Madelung [17], although interpretively in the spirit of de Broglie's pilot wave theory [31]. According to Bohm, a causal explanation to the quantum outcomes observed is possible in terms of statistical distributions of initial conditions in configuration space for quantum systems. He associated these initial conditions with the non-observable hidden variables referred to by von Neumann, thus proving a way to recast quantum mechanics in terms of a set of hidden variables compatible with all its predictions. Causality and probability could then be unified within quantum mechanics without the formulation of a totally new theory, as claimed by von Neumann.

Bohm's proposal could reproduce all the predictions of quantum mechanics, although it also introduced a disturbing element into play: *nonlocal* hidden variables. Classical systems are ruled by the *principle of locality*, i.e., no influence on a given physical system can travel faster than the speed of light. Therefore, any action on these systems must come from a certain neighborhood. In quantum mechanics, though, things work differently and actions at remote distances may cause important disturbances on the system without violating the special theory of relativity, as it was pointed out in 1935 by Einstein, Podolsky and Rosen [32] as well as by Schrödinger [33, 34]. This phenomenon was possible by virtue of the property of *entanglement*. Accordingly, as noticed by Schrödinger, as soon as two quantum systems interact, they will remain strongly correlated independently of the distance between them [33]; any measurement performed on one of the systems will instantaneously determine the outcome of the same measurement on the other one. This puzzling behavior, with no classical counterpart, started a longstanding debate on the incompleteness of the wave function and the necessity to explain such a "spooky action at distance" in terms of hidden variables. Within this scenario, Bohm's work did not satisfy either those against hidden variables or those in favor. His contribution was considered to be wrong and, after a few years, it was essentially relegated to oblivion.

In the 1960s Bell got back to those fundamental questions [35, 36] (precisely after coming across Bohm's papers [37]), reformulating the problem of hidden variables and changing the landscape of quantum mechanics by laying the grounds for what is now known as the Second Quantum Revolution [38]. Specifically, what Bell found is that no physical theory of local hidden variables can ever reproduce all the predictions of quantum mechanics (see p. 542 in Ref. [39]), because quantum mechanics is *intrinsically nonlocal*. Bohm was not wrong indeed; in spite of Einstein's reluctancy against the idea of quantum mechanics being nonlocal, Bohm provided a neat model that explained this quantum feature with the presence of a nonlocal potential, the so-called quantum potential (see Section 3). This potential is implicit in Schrödinger's equation and becomes explicit when it is recast in a Hamilton–Jacobi form by means of a nonlinear transformation from complex to real field variables. Because von Neumann's theorem implicitly assumes non-contextuality (von Neumann had in mind ensembles in classical-like variables), it can only be correctly applied to *local* hidden-variable models, leaving aside a distinctive aspect of quantum mechanics, namely its nonlocality, which is made more apparent through Bohm's formulation. The story that follows Bell's discovery, particularly after its experimental confirmation in 1981 and 1982 by Alain Aspect and co-workers [40–42] is well known [43]: the appearance and fast development of a series of new areas of quantum mechanics with very important technological applications, such as quantum teleportation, quantum computing, quantum cryptography, quantum imaging or quantum sensors.

Apart from inspiring Bell, Bohm himself went back in the 1970s to his former works from 1952 in collaboration with his colleague Hiley, who had already produced with PhD students Chris Dewdney [44] and Chris Philippidis two seminal works based on Bohm's approach, one about Young's two-slit experiment [45] and another one about scattering of square potential wells and barriers [46]. They started working on the implementation of the physics behind Bohm's suggested interpretation [47], stressing the role of the quantum potential as an interpretational tool to explain well-known quantum phenomena in a causal fashion. A number of papers followed along the 1980s and early 1990s by the members of this group as well as by other authors, where trajectories were produced for different quantum problems, proving the correctness of Bohm's ideas. Much of that work is summarized in the monograph

The Quantum Theory of Motion [15], published in 1993 by Peter Holland; other more fundamental interpretational aspects are discussed in *The Undivided Universe* [48], published in 1993 by Bohm and Hiley.

By the beginning of the 1990s, another Bohmian school arose around Detlef Dürr, Sheldon "Shelly" Goldstein and Nino Zhangì, where Bohmian mechanics was approached through a statistical mechanical perspective instead of the usual quantum potential one, introducing the notion of the quantum equilibrium hypothesis [49]. According to this idea, the quantum states we observe are states at equilibrium, i.e., states that result from the action of a sort of subquantum medium on arbitrary states that evolve or "thermalize" into the usual wave functions that satisfy Schrödinger's equation. It is at this point where Bohmian mechanics enters as a natural way to explain how such a thermalization process would take place, in analogy to how the thermalization of classical ensembles produces the statistical Boltzmann distributions we observe at equilibrium.

From those early days to date the use of Bohmian mechanics has remarkably increased to tackle different quantum mechanical problems [6–11, 50, 51]. Figure 2 may help to get an idea on the acceptance of Bohmian mechanics. In this figure we observe the time-evolution of the amount of works citing Bohm's seminal works from 1952 with black line: dashed line for paper I, dotted line for paper II, and solid line for the total number of citations (papers I & II). Data have been taken from the ISI Web of Science. Although there might be some other works dealing with or based on Bohmian mechanics that do not cite these two papers, they still constitute a reliable reference of the trend over time, which is what matters at this point. The case of de Broglie's pilot wave model has not been considered here because, given that there are several possible references (published in different periods), the follow-up is more complicated and probably not that much reliable. As can be seen, since the 1990s the interest in Bohmian mechanics has remarkably increased up to date, when it is more broadly considered and accepted among its users as a computational and interpretive working tool, to tackle a wide variety of problems in different areas of physics [6–11], than as a hidden-variable theory, i.e., an ontological interpretational model to understand the quantum realm, as it was essentially the case from the 1950s to the 1980s.

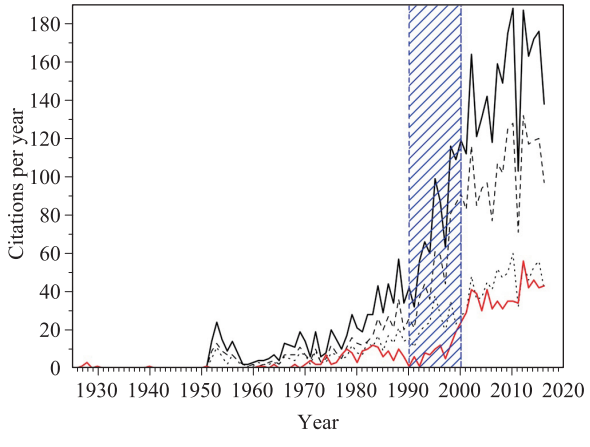

Fig. 2 Number of citations per year of the two papers published by Bohm in 1952 (black line) and the paper published by Madelung in 1926 (red line). The black dashed and dotted lines make reference, respectively, to the first and second Bohm papers separately. The vertical blue band serves to denote a kind of transition period where the role of Bohmian mechanics as a working tool (both computational and interpretive) started gaining importance to the detriment of other more fundamental (hidden-variable related) aspects.

Due to its interest in the development of the ideas implicit in Bohmian mechanics (the use of streamlines as a computational and interpretive resource), the evolution of the citations to the paper published by Madelung in 1926 is also displayed in Fig. 2 with red solid line. Nonetheless, it should be noticed that there might be many more works based on quantum hydrodynamics that those citing Madelung's paper, since it is an important approach in areas such as plasma physics, particle physics, condensed matter physics, superfluidity, etc., used and promoted, among others, by Landau or London concerning the modeling of liquid helium in the 1940s [52–55]. In these latter cases, neither there is a direct link to Bohmian mechanics, and still the essence of such formulations are exactly the same.

Although Madelung's hydrodynamic formulation was neglected until the 1970s, from this time onwards it started being considered as a practical tool to visualize the evolution in real time of intermediate stages of processes with interest in Chemistry, from molecular reactions [56–58] to the development and evolution of vortex dynamics [59–63] (with the concept of quantum vortex dating back to the works on superfluid helium from the 1940s and 1950s). The idea behind these simulations was to provide a full and deeper understanding of the processes investigated without the need to use interpretations based on classical trajectory computations (quantum-classical correspondence). This idea run through the 1980s and 1990s, and was further extended to the analysis of magnetism in molecular systems [64–70]. However, the most noticeable boost arises by the end of the 1990s, when Courtney Lopreore and Robert "Bob" Wyatt proposed [16] the first *quantum trajectory method* [6], and Madelung's approach merges with Bohmian mechanics (except, perhaps, in the area of the quantum foundations, where there are still some objections to such a unification), explaining the rapidly increasing number of citations from the 1990s onwards in Fig. 2.

This section would be incomplete without a mention to the experiments with classical vibrating fluids performed by Yves Couder and Immanuel Fort [71–74] at the Université Paris Diderot, and John Bush [75–77] at the MIT,

which show a nice classical counterpart of the behavior devised by de Broglie with respect to quantum systems through his pilot-wave theory: a fluid droplet bounces and generates a self-sustained wave that, at the same time, guides the motion of the droplet [78]. Based on these experiments and de Broglie's former double-solution program, Thomas Durt has recently proposed [79] a model to explain them that assumes the existence of a nonlinear self-focusing gravitational-like potential that prevents the particle from undergoing spreading (it is kept as a peaked soliton).

3 Quantum mechanics within the Bohmian picture

3.1 Basic formal aspects

The basic equations of Bohmian mechanics can be obtained in different ways. The usual one departs from recasting the wave function in polar form [2, 15, 17],

$$\Psi(\boldsymbol{r},t)=\sqrt{\rho(\boldsymbol{r},t)}\mathrm{e}^{\mathrm{i}S(\boldsymbol{r},t)/\hbar}. \tag{1}$$

This is just a nonlinear transformation from the complex field variables Ψ and Ψ^* to the real field variables ρ and S (from now on the explicit dependence on $\boldsymbol{r}$ and t will be omitted for simplicity). In particular,

$$\rho=\Psi\Psi^* \tag{2}$$

is the usual probability density, and

$$S=\frac{\hbar}{2i}\ln\left(\frac{\Psi}{\Psi^*}\right) \tag{3}$$

provides the local value of the system's quantum phase. Physically, the probability density represents the statistical distribution of possible realizations (detections) of a given quantum system on a certain region $d\boldsymbol{r}$ of the configuration space at a given time t, e.g., a statistically meaningful number of counts registered by a detector positioned at a certain angle from a target for a given exposure time, for instance. Because it can be measured (intensity) and has an associated operator (the density operator), we call it a *quantum observable*. Regarding the quantum phase S, it provides us with information about the local variations of the system quantum phase, which is not a quantum observable, because it can only be inferred through indirect measurements [80], e.g., an interference pattern.

After substitution of the polar ansatz (1) into the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi, \tag{4}$$

we obtain the equations of motion that describe the time-evolution of the real-valued field variables,

$$\frac{\partial\rho}{\partial t}+\nabla\left(\rho\frac{\nabla S}{m}\right)=0, \tag{5}$$

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}+V+Q=0, \tag{6}$$

with

$$\begin{aligned}Q&=-\frac{\hbar^2}{8m}\left[2\left(\frac{\nabla^2\rho}{\rho}\right)-\left(\frac{\nabla\rho}{\rho}\right)^2\right]\\&=-\frac{\hbar^2}{2m}\left\{\mathrm{Re}\left(\frac{\nabla^2\Psi}{\Psi}\right)+\left[\mathrm{Im}\left(\frac{\nabla\Psi}{\Psi}\right)\right]^2\right\}\end{aligned} \tag{7}$$

being the so-called *quantum potential*. Eq. (5) is readily identified with the usual continuity equation; Eq. (6) is regarded as a quantum Hamilton–Jacobi equation due to its close resemblance with its classical analog. Notice that the quantum phase S has the same dimensions as the classical mechanical action, [energy]×[time], which is a reminiscence of the Hamiltonian analogy considered by Schrödinger to derive his equation. Regarding the quantum potential Q, it has typically assigned the role of a potential function; Q and V thus form a global effective potential acting on the trajectories. However, Q actually comes from the action of the Laplacian operator, ∇^2, on the wave function in Eq. (4) and therefore it is closer to a kinetic-like energy than to a potential function. This question will be revisited again in Section 3.2.

Once a Hamilton–Jacobi equation is introduced, as it is done in classical mechanics, the concept of trajectory arises in a natural way if we take into account that such trajectories are characteristic solutions obtained after integrating the quantum version of the Jacobi law,

$$\dot{\boldsymbol{r}}=\frac{\nabla S}{m}, \tag{8}$$

with initial condition $\boldsymbol{r}_0$. This simple idea is precisely what Bohm suggested in 1952 and, at the same time, the way how the hidden-variable controversy comes into play, since after this postulate one feels very much tempted to think (and identify) these trajectories with the actual paths followed by a quantum particle. Notice that this is not in contradiction with Heisenberg's uncertainty relations, because uncertainty comes from the fact that, in practice, it is impossible to accurately determine x_0; initial conditions are randomly distributed, with the ensemble statistics being described by ρ at $t=0$.

In general, the Bohmian set of equations has been approached or considered in the literature in two different ways, each one with a different purpose. According to Wyatt [6], these two approaches can be denoted as *analytic* and *synthetic schemes*, which closely resemble the homologous analytic-synthetic distinction or dichotomy in Philosophy to primarily classify (logical) propositions or judgments, one of the pillars of Kant's philosophical system [81]. In the first case, first the Schrödinger equation is

solved and, once we get the wave function, the trajectories are computed and subsequently used to analyzed, interpret, understand and explain the system under study. In the second case, the calculation of the trajectories is independent of the wave function, thus being the optimal route considered in the design of new quantum propagation methods based on Bohmian mechanics.

In order to reach Eq. (8) and hence to associate or compute trajectories that monitor the evolution of a quantum system, it is not necessary introducing an additional postulate in quantum mechanics, since in its conventional Schrödinger formulation we already have all the necessary elements. Specifically, we have the probability density and also a current density [82],

$$\boldsymbol{J} = \frac{1}{m}\mathrm{Re}\left\{\Psi^{*}\hat{\boldsymbol{p}}\Psi\right\}, \tag{9}$$

with $\hat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$ being the usual momentum operator. This quantity allows us to introduce in a natural way a velocity field variable, $\boldsymbol{v} = \boldsymbol{J}/\rho$, which formally coincides with the Bohmian velocity given above,

$$\boldsymbol{v} = \frac{\boldsymbol{J}}{\rho} = \frac{1}{m}\mathrm{Re}\left\{\frac{\hat{\boldsymbol{p}}\Psi}{\Psi}\right\} = \frac{\nabla S}{m} = \dot{\boldsymbol{r}}, \tag{10}$$

although without appealing to a hidden-variable scenario. The fact that all quantities to the right of the first and second equalities are well-defined within the conventional version of quantum mechanics, and hence the use of the quantities to the right of the third and fourth equalities, namely S and $\boldsymbol{r}$, should also be legitimately considered, leaving aside further interpretational issues. Actually, taking into account the definition of $\hat{\boldsymbol{p}}$, the quantum potential can be recast as

$$Q = \frac{1}{m}\left\{\frac{1}{2}\mathrm{Re}\left(\frac{\hat{\boldsymbol{p}}^{2}\Psi}{\Psi}\right) + \frac{1}{2}\left[\mathrm{Im}\left(\frac{\hat{\boldsymbol{p}}\Psi}{\Psi}\right)\right]^{2}\right\}, \tag{11}$$

which further stresses the dynamical origin of Q mentioned above.

To conclude this section, it is worth stressing that analogous approaches to what we now know as Bohmian mechanics have been used in the literature to solve different problems. For example, in Optics we find models based on the Poynting vector and the Maxwell equations already in the 1950s [83, 84], and only much later the approaches of Bohm and Madelung were considered [85–92]. In Acoustics the use of streamlines was also suggested in the 1980s as a visualization and analysis tool [93–97], although there was not a direct link with Madelung or Bohm in this direction until recently [98]. Finally, we also find different models aimed at treating dissipation in quantum systems, such as the one proposed by Kostin [99], which is based on the idea of adding nonlinear contributions to the Schrödinger equation, which, interestingly, are given in terms of the phase of the wave function, and therefore the Bohmian momentum (which is identified with the dissipative term that appears in the classical Newton equations of motion with friction).

3.2 A simple illustration: interference from two mutually coherent sources

Quantum interference is one of the distinctive traits of quantum mechanics, providing a rather convenient environment to investigate fundamental questions by devising different types of interferometric experimental setups [100–103]. Therefore, it is not strange that interference has been a recurrent topic in the Bohmian literature since the former numerical simulation published by Philippidis, Dewdney and Hiley in the case of Young's two-slit experiment [45]. Thus, for a proper understanding of the role of Bohmian mechanics within the usual context of quantum mechanics, we are now going to consider a simple working model on Young's two-slit experiment. This model combines in a simple way analyticity and numerics with the purpose to be easily considered in the classroom. To start with, we consider paraxial conditions [104], so that interference is assumed to happen essentially along the transverse direction (i.e., parallel to the screen where the two slits are supposed to be), here denoted by the x coordinate. Accordingly, only this (transverse) component of the wave function is considered. Furthermore, we want to focus on the phenomenology of the experiment, namely the appearance of interference by coalescence of two waves coming from *two mutually coherent sources*, so the time-evolution will start right behind the two slits. To avoid edge-related diffraction effects, the initial state is described by a coherent superposition of two Gaussian wave packets,

$$\Psi(x,0) \sim \mathrm{e}^{-(x-d/2)^{2}/(4\sigma_{0}^{2})} + \mathrm{e}^{-(x+d/2)^{2}/(4\sigma_{0}^{2})}, \tag{12}$$

which in the far field provides us with a neat interference pattern only affected by a Gaussian envelope (compare this with the sinc function affecting square waves [105]). The time-evolution of (12) is fully analytical [9]; at a time t it is easy to show that the wave function reads as

$$\Psi(x,t) \sim \mathrm{e}^{-(x+d/2)^{2}/(4\sigma_{0}\tilde{\sigma}_{t})} + \mathrm{e}^{-(x-d/2)^{2}/(4\sigma_{0}\tilde{\sigma}_{t})}, \tag{13}$$

with

$$\tilde{\sigma}_{t} = \sigma_{0} + \mathrm{i}\left(\frac{\hbar}{2m\sigma_{0}}\right)t. \tag{14}$$

Notice in both (12) and (13) that the global phase and normalizing factors have been neglected. This is only to simplify the analysis, since they are irrelevant from a quantum dynamical viewpoint, as can be seen below.

The appearance of interference can be readily understood with this simple model by just noticing in (13) the appearance of the time-developing complex factor (14) in the argument of both exponentials. Taking this factor into account, these exponential functions can be recast as

$$\mathrm{e}^{-(x\pm d/2)^{2}/(4\sigma_{0}\tilde{\sigma}_{t})} = \mathrm{e}^{-(x\pm d/2)^{2}/(4\sigma_{t}^{2})}\mathrm{e}^{\mathrm{i}(\hbar t/8m\sigma_{0}^{2})(x\pm d/2)^{2}/(4\sigma_{t}^{2})}, \tag{15}$$

with

$$\sigma_t \equiv |\tilde{\sigma}_t| = \sigma_0 \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2}, \tag{16}$$

The first exponential function on the right-hand side of Eq. (15) describes the widening undergone by the wave packets as time proceeds; the second one accounts for the development of a phase factor depending on time and, more importantly, the x coordinate. This factor induces the appearance of a phase field, which assigns a value to the (*local*) phase at each space point and time. Eventually, this leads to the appearance of interference features, even in the very case that there is no any relative phase between the two wave packets at $t = 0$. This is more apparent through the associated probability density,

$$\rho(x,t) \sim \mathrm{e}^{-(x+d/2)^2/(2\sigma_t^2)} + \mathrm{e}^{-(x-d/2)^2/(2\sigma_t^2)} + 2\mathrm{e}^{-[x^2+(d/2)^2]/(2\sigma_t^2)} \cos\left[\left(\frac{\hbar t d}{4m\sigma_0^2\sigma_t^2}\right) x\right]. \tag{17}$$

This result shows that the total intensity recorded is not the bare addition of partial intensities, as in classical statistical mechanics. Nonetheless, more interestingly, this model also shows that interference features are present since the very beginning through the third oscillatory factor, even if the initial distance between both wave packets is too large compared to their widths (i.e., $d \gg \sigma_0$). The appearance of this term as soon as $t \neq 0$, no matter how relevant it is, is what allows us to talk about and formally describe as *coherence* when dealing with the superposition of waves.

The time-evolution of the probability density (17) is displayed in Fig. 3(a) in terms of surface and contour plots. The numerical values in this simulation are $\hbar = 1$, $m = 1$, $\sigma_0 = 0.5$, and $d = 10$, and the evolution been halted at $t = 10$, enough to observe all the dynamical regimes ruling the behavior of the quantum system [22]. These particular values have been chosen, because they produce a characteristic time ruling the wave-packet spreading dynamics of $\tau = 2m\sigma_0^2/\hbar = 0.5$ (at $t = \tau$, $\sigma_t = \sqrt{2}\sigma_0$), which means that at the time when interference fringes start appearing both wave packets already display a linear widening [22]. As it can be noticed, until $t \approx 2$, corresponding to the Huygens-Ehrenfest and Fresnel regimes that characterize the early stages of a wave function evolution, the wave packets seem to evolve independently of each other. Then,

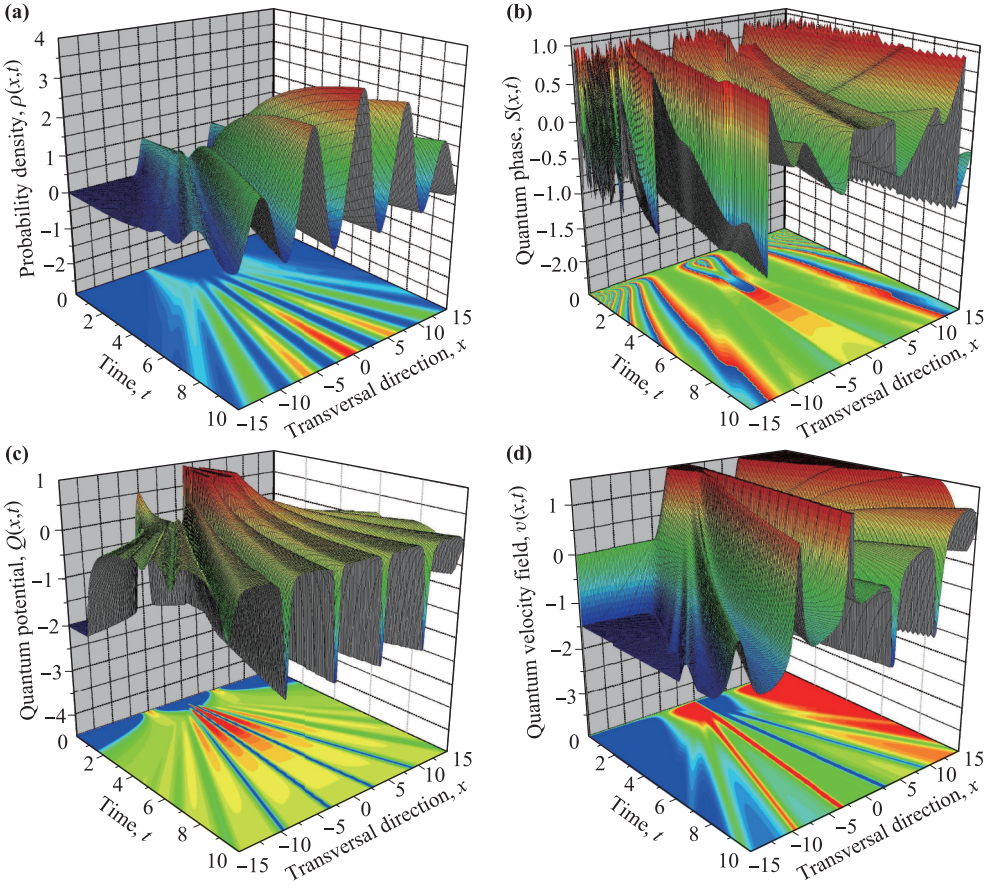

Fig. 3 Numerical simulation illustrating different aspects of Young's two-slit experiment [106]. The contour-plots illustrate the evolution in time of the probability density **(a)**, the quantum phase **(b)**, Bohm's quantum potential **(c)**, and the quantum velocity field determined by the wave function **(d)**. The black solid lines in all plots represent ensembles of Bohmian trajectories leaving the slits with different initial conditions and, accordingly, ending at different points of a scanning screen (detector).

for $2 < t < 4$, a fringed structure starts developing until $t \approx 5$, when such a structure remains stationary, with nodes that spread out both sides linearly with time. This latter stage corresponds to the Fraunhofer regime, where we can neatly observe the characteristic fringed pattern of Young's experiment. This long-time limit can be easily described with our working model. Notice that asymptotically, for $t \gg \tau$, Eq. (16) can be approximated by $\sigma_t \approx \hbar t/(2m\sigma_0)$. The probability density (17) then reads as

$$\rho(x,t) \sim \mathrm{e}^{-(x+d/2)^2/[2(v_s t)^2]} + \mathrm{e}^{-(x-d/2)^2/[2(v_s t)^2]} + 2\mathrm{e}^{-[x^2+(d/2)^2]/[2(v_s t)^2]} \cos\left[\left(\frac{md}{\hbar}\right)\frac{x}{t}\right], \tag{18}$$

which is modulated by an even simpler phase factor, and where $v_s \equiv \hbar/(2m\sigma_0)$ is the spreading rate or velocity at which Gaussian wave packets widen asymptotically [107]. According to the phase factor in (18), the distance between adjacent nodes or maxima increases nearly linearly with time as

$$\Delta x = \left(\frac{2\pi\hbar}{md}\right) t. \tag{19}$$

In our case, this linearity condition is satisfied precisely for $t \approx 5$ ($\gg \tau$). In particular, for $t = 10$ we have $\Delta x = 2\pi$, in compliance with the results displayed in Fig. 3(a).

The probability density conveys information on the chance to find the particle within a certain region of the configuration space after it (its associated wave function) has passed through the slits. At this level, where the value of the probability density is locally analyzed, there is no clue on whether there are or there are not phase-related effects, such as interference traits; a global view of the topology exhibited by the probability density is necessary to get such an information, i.e., to observe the well-known fringe structure. This does not mean that it is not possible to get phase information on equal footing; it is only that in quantum mechanics it is not common to directly focus on the quantum phase, because it is a quantum observable quantity (according to the conventional definition of quantum observable), while the probability density is directly related to intensities, experimentally accessible by accumulating events (detections) for a given time. As seen above, local phase information (not to be confused with global phase factors) is encoded in the phase field S, defined by (3), which can be determined from the wave function. In the recreation of the two-slit experiment here analyzed, the time-evolution of this field along the transverse coordinate is readily determined by substituting the wave function (13) into Eq. (3), and then assigning values for x and t, as shown in Fig. 3(b). For a better visualization, given the continuous increase of the function as $|x|$ increases (at a given time), a $(-\pi, \pi]$ representation has been chosen (the vertical axis is given in units of π for clarity), which explains the sharp "jumps" from π to $-\pi$ around $|x| \approx 12.5$, for instance. In any case, and neglecting the modulo operation to bound the value of S for clarity purposes, it is apparent that the topology exhibited by the phase field resembles that of the probability density. Or is it just that the evolution of the latter is strongly influenced by the former?

A quick answer can be given to such a question by invoking the so-called Hamiltonian analogy, which links Fermat's and Huygen's optics principles with Jacobi's mechanistic formulation of Maupertuis' principle, and was reconsidered later on by Schrödinger to derive his wave equation by merging de Broglie's relation with the Hamilton–Jacobi formulation of classical mechanics. Although it is often neglected (probably, always), this is actually in conventional quantum mechanics, which provides a natural way to understand the relationship between the probability density (ρ) and the associated phase field (S). Additionally, taking the Bohmian perspective (picture) of quantum mechanics, the issue can be tackled even in a simple manner by appealing to two distinctive elements of this view, namely the quantum potential (Q) and the local velocity field (v). The numerical evolution of these two fields is represented in Figs. 3(c) and (d), respectively.

In the case of the quantum potential [see Fig. 3(c)], the near field is essentially characterized by two inverted parabolas, which correspond to the measure of the curvature of the wave function in the neighborhood of the slits. However, as time proceeds, Q passes from displaying a rapidly increasing "dynamic barrier" in the middle region between the two slits (truncated red 3D surface representation in the figure, starting around $t \approx 2$), typical of the Fresnel regime, to an alternate structure of plateaus and dips or "canyons" as we move further away towards the Fraunhofer regime. The central barrier in the Fresnel regime represents the effective meeting of the two wave packets, that is, the point at which the single slit diffraction process ends because the two waves start coalescing, giving rise to the appearance of incipient interference features. This process takes some time until a sort of "equilibrium" is reached, typical of the Fraunhofer regime – a regime where the overall shape of the wave or, in this case, the quantum potential does not change, but spreads linearly with time [22] (or, equivalent, the distance from the two slits).

Comparing Figs. 3(a) and (c), we note the direct relationship between both ρ and Q. Actually, according to Eq. (7), the quantum potential is just a sort of measure of the local curvature of the probability density, so that the maxima of the probability density lie on the plateaus of Q, while the minima do it on the dips. Dynamically speaking, this means that nodes of the probability density are associated with regions relatively unstable regions of the quantum potential, while non-vanishing probability gives rise to regions of relative stability. This is precisely an important issue when designing quantum trajectory methods [6], since regions with low densities generate numerical instabilities, eventually translating into a source of inaccuracy of the method.

The problem of interpreting the quantum potential as a potential function is that it depends directly on the probability density, thus providing us with redundant information. That is, it does not matter whether we look at the quantum potential or the probability density to understand that two initial ensembles of trajectories leaving each slit (see discussion below) will eventually split up into different sub-ensembles. Nonetheless, there is an interesting difference worth mentioning and also worth being taught to explain and understand interference at a more intuitive level (even if one regards the quantum potential itself as not intuitive at all). The probability density describes the chance to find trajectories along some particular directions, avoiding others. It only gives us statistical information. On the other hand, the quantum potential, appealing to a Newtonian-like view, provides us with a certain kind of mechanistic information, specifying that the trajectories evolve along those directions, because the corresponding quantum force along them is negligible ($\nabla Q \approx 0$), while they avoid the regions where this potential undergoes remarkable fast variations (intense quantum forces), as can be seen in Fig. 3(c).

So, probability density and quantum potential are face and tail of the same coin. Now, given that the quantum potential is directly connected to the probability density, one may wonder if there is an alternative way to interpret and explain the dynamics observed. To answer this question, remember that Q comes from the Laplacian (kinetic) operator and therefore contains (and conveys) nonlocal dynamical information unlike classical potential functions. Consequently, this should manifest in some way. This is the point where the quantum phase and its associated velocity field come into play. As seen in the previous section, the quantum phase is linked to the concept of quantum flux, which in classical statistics is related to the evolution of swarms of particles. Let us then try this way and compute the velocity field arising from the gradient of the phase field, which is plotted in Fig. 3(d). This field gives us an idea on the local variations undergone by the quantum phase field [Fig. 3(b)]. As time proceeds, within the Fraunhofer regime, the velocity field displays a series of relative maxima and minima that lie on a nearly positive-increasing slope (for a given time). The maxima and minima have the shape of localized spikes (positive and negative, respectively), which evolve precisely along the minima of the probability density and denote regions where the velocity field is relatively intense, i.e., trajectories will cross those regions with a rather high speed (see Fig. 4). Regarding the overall slope mentioned before, it is interesting to note that it develops from a full horizontal position at $t = 0$ to an inclined (positive) angle as time proceeds, from the Fresnel to the Fraunhofer regime [108]. This behavior is governed by the presence of a strong correlation between the dynamics exhibited by the two diffracted beams since $t = 0$. It is the physical manifestation of what we call *quantum coherence*, which makes a clear distinction between a bare superposition of solutions and its physical consequences: although at a practical level the probability density looks the same independently of whether one first propagates one wave packet and then the other (according to the usual *superposition principle*), physically both wave packets must leave the slits at the same time, because it is the phase field associated with both simultaneously propagating what originates the particular dynamics exhibited by quantum systems [107] (compared to classical systems).

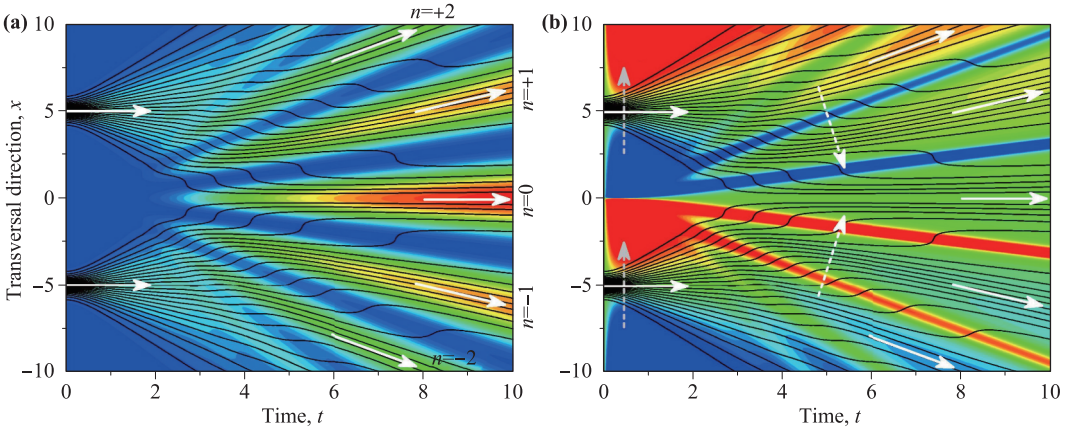

Fig. 4 Numerical simulation illustrating the usual Bohmian interpretation of Young's two-slit experiment [106] by means of Bohmian quantum trajectories (black solid lines). For the sake of clarity, these trajectories are superimposed on the contour plots of the probability density **(a)** and the velocity field **(b)**. The trajectories represent the evolution in time [according to the Bohmian prescription, given by the guidance equation (8)] of a single element of quantum fluid, from any of the two slits to some final position where a scanning screen (detector) would be allocated. White arrows indicate the main (evolution) direction of the flow initially and asymptotically, which shows how from zero transversal flow the dynamics turns into a set of components traveling with different transverse speed. In part (b), the gray dashed arrows indicate the direction in which the velocity field increases at the early stages of the evolution, while the white dashed arrows show the trend motion followed by the different sub-ensembles of trajectories (downwards in the upper half of the graph; upwards in the lower half).

As a result of the global nature of the quantum phase field and, through it, the associated velocity field, the trajectories or streamlines that the Bohmian picture renders have nothing to do with those one typically consider in classical mechanics. Actually, this nonlocal or global behavior is typical of ensembles of streamlines mapping any kind of wave, regardless of its nature [91, 98]. Specifically, in the case here considered, the trajectories tend to move to regions with lower values of the modulus of the velocity field (more stable, dynamically speaking), where they display what could be regarded as a nearly classical-like uniform motion ($v \approx$ constant), as can be seen in Fig. 4(b). On the contrary, near sharp changes of the quantum phase, and therefore large absolute values of the velocity field, the trajectories undergo the effects of a sudden acceleration, which leads them from one stability region to the neighboring one. Notice that the changes of the velocity allow to explain in a better way the motion displayed by the trajectories than the quantum potential. Specifically, the quantum potential does not explain, for instance, why the motion should be symmetric with respect to $x = 0$ — apart from extra symmetry arguments —, while by looking at the sign of the velocity field this effect can be perfectly understood [see arrows in Fig. 4(b)].

Regarding the physics described by the velocity field (10), it is worth stressing that its value coincides with the value rendered after performing a "weak measurement" following the usual or traditional view of the quantum theory, according to Wiseman [109]. This result was experimentally confirmed precisely from a realization of Young's experiment with photons [110]. Specifically, this experiment showed that the trajectories obtained from Eq. (10) are compatible with the data rendered by weak measurements of the average velocity associated with a swarm of identically prepared photons. These experiments have motivated the search of Bohmian trajectories in different systems [111, 112]. Of course, this does not mean that we know the specific path chosen by a particle in Young's renowned two-slit experiment [113], but only that the average flow of a large number of such particles, described by the same single-particle wave function, is laminar, which cannot easily be inferred a priori from other pictures, although it is a perfectly reachable result through the flux operator [107].

4 Concluding remarks

In the centennial of David Bohm's birth anniversary, when his "suggested interpretation" of quantum phenomena is turning 65, it is probably time to start thinking in more natural terms this approach, showing and teaching it from a different perspective, closer to the conception Bell had about it [114]. This has been the goal of this work, tackled through the two questions posed in the introductory part. Accordingly, an alternative view of Bohmian mechanics has been presented, as a complementary picture of quantum mechanics on equal footing with other more widely known and used pictures. To this end, the work has been split up into two independent but complementary blocks, one dealing with a brief historical perspective and another one about specific aspects of Bohmian mechanics and its application to the description of quantum interference.

Combining both blocks, we find that there is no need for appealing to the concept of *hidden variable* when talking about Bohmian mechanics at present. Rather, this is just another alternative and complementary picture of quantum mechanics, which provides us with hydrodynamic-like information of quantum systems that does not contradict the kind of information rendered by other pictures, as mentioned above. Actually, because we have at hand this kind of information in terms of trajectories, it is possible to apply techniques typical of classical mechanics and hydrodynamics for its analysis, thus unraveling interesting properties and determining alternative descriptions and explanations for the evolution of quantum systems, particularly useful to understand the role played in them by the quantum phase and quantum correlations (entanglement) [115]. This is illustrated here by means of a simple realization of Young's two-slit experiment, which combines both analytical and numerical aspects feasible to be taught and developed in a standard elementary course on quantum mechanics. The fact that such evolution can be followed by means of trajectories does not contradict at all our understanding of the quantum theory, but is in compliance with it provided we do not assign any reality to such trajectories (or a direct link between them and the actual motion displayed by a real quantum particle, as hidden-variable models do). Notice there is no experimental evidence that allows us to establish it – recent experiments only confirm that the average flux is compatible with these trajectories [110].

Acknowledgements This work intends to be a posthumous tribute to David Bohm on the occasion of the centennial of his birthday. He was willing to look at quantum mechanics with different eyes in times where all eyes looked on the same direction. And, in so doing, he and his works served as an inspiration to forthcoming generations of physicists. The author is grateful to Profs. Basil Hiley and José Luis Sánchez Gómez for enjoyable hours of conversations on different aspects concerning Bohmian mechanics, hidden variables and the foundations of the quantum theory. The author would also like to acknowledge fruitful feedback from Profs. Hrvoje Nicolić and Chris Dewdney. This work has benefitted from financial support from the Spanish MINECO (Grant No. FIS2016-76110-P).

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