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PERSPECTIVE

The nuclear shell model: Simplicity from complexity

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The shell model of atomic nuclei has been in intensive use since the middle of the previous century. This simple model of very complex nuclei, offers a quantitative description of its many features. Other features follow from small deviations from the extreme picture. Our friend and colleague Akito Arima made seminal contributions to this field starting with his famous paper with Horie on the magnetic moments of nuclei [*Prog. Theor. Phys.* 11, 509 (1954)]. In the following, a detailed description of a simple example is considered. It is the $1f_{7/2}$ shell of the neutrons in the nuclei between ^{40}Ca and ^{48}Ca and of the protons in the nuclei between ^{48}Ca and ^{56}Ni. The results demonstrate the power and elegance of the shell model. They show how simplicity arises out of complexity. It is also shown how small deviations from the simple shell model lead to effects, in which valence neutrons act as if they carry electric charge.

Keywords shell model, effective interactions, seniority, binding energies, effective charges, nuclear radii

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1 Introduction

Atomic nuclei are very complex systems. They are composed of many strongly interacting fermions, protons, and neutrons. The quark internal structure of the latter is important in determination of their short range mutual interactions, but it may not be relevant at low energies. Still, even if it is not explicitly taken into account and its effect is limited to its contribution to an interaction between nucleons, nuclear states and energies cannot be exactly calculated. As in other fields of physics, simpler models that can be solved, replace the actual systems. Successful models are sufficiently simple but still posses the important features of the system which they represent.

In spite of their complexity, nuclei exhibit some simple regularities which are the ingredients of nuclear models. Once the neutron was discovered and it was suggested that protons and neutrons are the constituents of nuclei [2], it was noted that certain nuclei are more stable than others. The numbers of protons and neutrons of such nuclei, turned out to be equal to those of fully occupied single fermion orbits in a central potential well, with $l = 0$ and $l = 1$. Such a situation was familiar to the physicists from atomic spectra and it was suggested [3] that protons and neutrons in nuclei actually occupy single nucleon orbits in an average potential well due to the interaction of all the nucleons. This was the very beginning of the shell model development, which was first called the quasi-atomic model. Shells in nuclei, like in atoms, are due to bunching of single nucleon orbits. The latter are similar but not identical to the situation in a three-dimensional harmonic oscillator potential well.

The shell model was investigated in more detail in a series of studies by Elsasser [4] and others. More magic numbers were found and interpreted as numbers of protons or neutrons in shells with all the orbits closed (completely filled). In addition to the $1s$ and $1p$ orbits, with magic numbers 2 and $2 + 6 = 8$, respectively, the next shell contains the $1d$ and $2s$ orbits, adding $10+2 = 12$ to 8, yielding the magic number 20. These shells were easy to understand but higher ones, 50, 82, and 126, were attributed to rather strange bunching of l-orbits.

In 1936, the model was heavily attacked by Niels Bohr himself. In a paper published in *Nature* [5] he explained that nuclear reactions show that the excitation energies

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of nuclei are shared between many nucleons. Hence, a method based on single particle orbits, working so well for atoms, "loses any validity" in nuclei. Mature theorists like Wigner and Hund continued their work on the shell model but usually for nuclei lighter than ^{40}Ca. Younger people left the field. Racah, who developed powerful theoretical methods [6], applied them to atomic spectroscopy. Only later did his students apply those to the nuclear shell model.

The modern (current) version of the shell model was sugguested by Maria Mayer [7], in which, on the basis of numerous experimental data, she concluded that the magic numbers discovered by Elsasser are indeed real and robust. As a result, two papers were published, by Feenberg and Hammack [8] and by Nordheim [9] in which they attempted to construct shells by bunching groups of l-orbits, very much *a la* Elsasser. Maria Mayer gave a talk at the University of Chicago and was asked by Fermi if there was evidence for spin-orbit interaction in nuclei. As she recounted later, she thought and then answered "yes and it explains everything". In a short paper [10] she suggested that the potential well of the shell model contains a strong spin-orbit interaction. The latter splits an l-orbit ($l > 0$) into two j-orbits, one with $j = l - 1/2$ and a lower one with $j = l + 1/2$. This lower orbit for the highest l in a harmonic oscillator shell is actually pushed down into a lower shell, giving rise to closed shells with proton or neutron numbers 50, 82 and 126. The $1f_{7/2}$ orbit of the $1f$ $2p$ oscillator shell is not sufficiently shifted. It forms a shell of its own and makes 28 a magic number. The same idea, of obtaining shell closures due to a strong spin-orbit interaction, occurred independently to Jensen *et al.* [11].

Ground states of nuclei with closed shells have well-defined $J = 0$ wave functions. If a nucleon is added to such a nucleus, the ground state and several excited states, also have well-defined states with $J = j$. Here, j is the angular momentum (spin) of one of the single nucleon orbits in the next shell. States of nuclei where a single nucleon is missing from closed shells (single hole states) are also well defined. Once there are 2 or more nucleons outside closed shells (valence nucleons) or holes, there are more possible states whose order and spacings are not fixed by the potential well of the model. In a detailed paper [12], Mayer introduced coupling rules based on the experimental data. The first rule states that in an even-even nucleus the ground state spin is zero, $J = 0$. In an odd-even nucleus with n valence j-nucleons (j^n configuration) the ground state is a $J = j$ state.

Although there are no observed exceptions to the first rule, Mayer found exceptions to the second one. She tried to find a justification of her rules and reported some calculations, realizing that energy differences between states and resulting order, could be due to mutual interactions of the valence nucleons [13]. She carried out simple calculations for $j = 3/2$ and $j = 5/2$ and for $j = 7/2$ for $n = 3$. The interaction she chose "for simplicity" was the attractive zero range δ-potential. Her results were in agreement with her coupling rules. Moreover, she found that the interaction energy in the case of *odd* n in the $J = j$ ground state is equal to the one in the $J = 0$ ground state of the j^{n-1} configuration. This feature of odd-even variation in binding energies is in agreement with the experimental observation of the pairing energy. The choice of the δ-potential was very fortunate. A long range interaction, which may be considered to be constant over the nucleus, had been used by researchers of earlier publications. Its usage led to results contrary to those of the experiment (Racah [14]).

Mayer was aware of the schematic nature of the δ-potential and suggested that exceptions to her second rule may be due to the non-zero range of the interaction. This point and others were taken up by several rather young theorists [15, 16]. Calculations of interaction energies in nuclear ground and excited states were carried out for a variety of interactions. The popular wave functions used in the calculations were those of the harmonic oscillator, especially after a method was found to facilitate their use [17]. The choice of a two-body interaction between nucleons has been more difficult. Interactions between free protons and neutrons were determined through scattering experiments. Early results suggested simple potentials which could be used in shell model calculations. More detailed and accurate results were expected for better shell model calculations. More accurate experiments, however, led to difficulties of the shell model. The interaction between free nucleons turned out to be too singular at short distances ("hard core") to be accessible to the shell model.

The shell model was then widely accepted and useful and was not going to be discarded. A way to keep it, in view of the short range strong interaction, was shown by Keith Brueckner. Shell model wave functions, with definite independent orbits of nucleons, may be used with a modified or effective version of the interaction. Matrix elements obtained in this manner, should be equal to matrix elements of the interaction between free nucleons calculated with the real nuclear wave functions. The latter contain short range correlations imposed by the free nucleon interaction.

Over the decades many reports were published attempting to calculate the effective interaction using the one between free nucleons. In spite of the great theoretical efforts, no clear results that could be used in the prediction of energies and other observables of states were obtained. It is not surprising that physicists tried to determine the effective interaction from energies of actual nuclei.

2 Effective interactions

In 1954, I had been looking for simple nuclear states whose structure could be identified. Levels of ^{40}K seemed to be due to coupling of $(1d_{3/2})^3$ protons with $J = 3/2$, and a valence $1f_{7/2}$ neutron. Experimentally, four states with $J = 4, 3, 2, 5$ were the only ones measured up to 1 MeV. I, along with a student, used the extracted matrix elements (actually differences) to calculate the spectrum of ^{38}Cl, with the proton $1d_{3/2}$–neutron $1f_{7/2}$ configuration. Our results did not agree with the published data. We were disappointed but not shocked. There was the standard excuse of configuration mixing and it was not clear that matrix elements determined from one nucleus could be safely used in another. Naturally, we did not publish our results.

In 1955, Ford and Levinson [18] published shell model calculations of calcium nuclei. They looked at the four lowest states of ^{42}Ca, $J = 0$ (ground state), $J = 2$ (1.53 MeV) and two following states which they assumed to have $J = 4$ and $J = 6$. They took these states as due to the $(1f_{7/2})^2$ configuration and took from their energies, the matrix elements, actually differences, of the effective interaction between valence neutrons. They tried to calculate with this interaction, energies of $(1f_{7/2})^3$ states, which should be the lowest states of ^{43}Ca. There was no agreement between calculated and experimental energies. They then embarked on complicated mixing of configurations and as a result, obtained very good agreement with the experiment values. This work, hailed as great success of the shell model, started from wrong spin assignments of ^{42}Ca levels. The two levels above the lowest $J = 2$ state do not have spins $J = 4$ and $J = 6$ but they are "intruder states" with spins $J = 0$ and $J = 2$. Spectra of $(1f_{7/2})^n$ configurations are considered in the following.

The relation between ^{40}K and ^{38}Cl turned out to be more acceptable. Energy levels of ^{38}Cl were measured accurately afterwards. They turned out to be in very good agreement with our calculated values [19]. This was the first case where no assumptions about the nature of the interaction were needed for a quantitatively successful shell model calculation. It opened a new era in which it was not necessary to overcome the difficulties in calculating the effective interaction. In many cases, some of its matrix elements could be obtained by analysis of measured energies. Very soon afterwards, Pandya published the same results [20], which he used to demonstrate an analytic formula that he discovered, connecting energies of nucleon-hole states to those of nucleon-nucleon states.

Arthur Kerman told me then, that when he was in Caltech, in the early fifties, Feynman was attempting to apply this approach to light nuclei. In the spring of 1962, I gave a colloquium talk in Caltech on this approach. Feynman was in the audience and seemed interested. After the talk he told me that he was trying to apply this approach to light nuclei. One of his results was that in a certain nucleus, the four lowest levels should have almost the same energy. This seemed to him so unlikely that it made him drop the whole approach (probably ^{16}N where the lowest measured levels lie within 0.4 MeV. They are due to couplings of a $1p_{1/2}$ proton with a $1d_{5/2}$ neutron ($J = 2$ and $J = 3$) and with a $2s_{1/2}$ neutron ($J = 0$ and $J = 1$) [21]). It was very interesting and I enjoyed the encouragement of Feynman very much.

The problem with the $1f_{7/2}$ shell was resolved soon. In the fall of 1956, during a short visit to Berkely, I gave a talk about extracting matrix elements of the effective interaction from the measured energies. Bob Lawson and Jack Uretsky, who attended my talk applied this method to protons in the $1f_{7/2}$ orbit, in nuclei with a closed neutron orbit, $N = 28$ [22]. In some nuclei, levels were found with spins $J = 2, 4, 6$ above the $J = 0$ ground state. They used energy differences of these levels to calculate spectra of odd-even nuclei. The agreement with experiment was good. In particular, their calculated positions of $J = 5/2$ levels, rather close to the $J = 7/2$ ground states, showed how different was the effective interaction from some popular ones. In the case of the δ-potential, the $J \neq 7/2$ levels, lie rather close compared to their excitation energies. In the case of the pairing interaction, they all have the same excitation energy. The origin of the difference was a puzzle until it was traced to the level spacing in even-even nuclei.

Looking at their paper, I noted the similarity between the spectra of the $N = 28$ nuclei and nuclei of Ca isotopes. In particular, $J = 4$ and $J = 6$ levels in ^{42}Ca, lying above the two levels taken by Ford and Levinson to be the $J = 4, 6$ levels (later found to have spins $J = 0$ and $J = 2$), had excitation energies rather close to those of corresponding levels of proton configurations. I looked at binding energies of Ca nuclei and obtained very good agreement between calculated and experimental energies. Such agreement was also obtained between the calculated and experimental binding energies of nuclei with $N = 28$ and $20 \leq Z \leq 28$ [23].

It is important to understand the quantitative significance of the results of the calculations mentioned above. A number of matrix elements could be determined to reproduce a much larger set of measured energies using these calculations. This approach of extracting information on the effective interaction from measured energies became rather standard. In the case of nuclei where several configurations are mixed, diagonal as well as non-diagonal matrix elements were extracted from measured energies.

In light nuclei, full major shells were considered. The

$1p$ shell, including the $1p_{3/2}$ and $1p_{1/2}$ orbits, was considered by Amit and Katz [24] and by Cohen and Kurath [25]. Then followed the full $2s$, $1d$ shell considered by Wildenthal [26] and by Brown and Wildenthal [28]. Very large scale calculations in higher shells, involving millions of states of numerous configurations, have been carried out by the Strasbourg–Madrid group [27] and the Tokyo group [28]. In the complicated cases, there were not enough measured energies to determine all matrix elements, but the most important ones were taken from experiment.

The need to determine matrix elements dictated the adoption of a *two-body interaction*. In all the calculations mentioned above, there was no need to add explicitly three-body forces between valence nucleons. On the other hand, in all theoretical attempts to calculate nuclear energies, inclusion of three-nucleon interactions is essential. This point is further considered in the next section.

Even in successful calculations, the information gained on the effective interaction was limited to several matrix elements. Still, some general properties were soon identified. The $T = 1$ interaction acts between identical nucleons, and between protons and neutrons in antisymmetric states. The $T = 1$ interaction in $j^2 J = 0$ states is strong and attractive. The centroid of the $j^2 J > 0$ *even* states is rather weak, either attractive or repulsive. The centroid of $T = 1$ levels of jj' states is weak and repulsive. The $T = 0$ interaction is attractive and leads to the potential well of the shell model. It is not important for the present paper.

3 States and energies in the seniority scheme

The expectation value in the ground state of a two-nucleon interaction energy of n identical j-nucleons, which is diagonal in seniority, has a rather simple expression that will be derived bellow. The seniority scheme was introduced by Racah [29] for specifying states of atomic electrons. It turned out to be more useful for nuclear structure than for atomic structure physics. Its generalization to jj-coupling (and to states of protons and neutrons) was carried out by Flowers [30] and Racah [31]. It is the scheme of eigenstates of the j^n-configuration in which the *pairing interaction*

$$\langle j^2 JM|v_{12}|j^2 J'M'\rangle = (2j+1)\delta_{J0}\delta_{J'0}\delta_{M0}\delta_{M'0} \tag{1}$$

is diagonal.

The seniority of a state of the j^n configuration of identical nucleons is determined by an integer v. It measures in some way the number of unpaired nucleons in a given state. More precisely, a state of the j^v configuration in which there are no paired nucleons, coupled to $J = 0$ states, has seniority v. More states with seniority v, may be obtained by adding a coupled pair and antisymmetrizing. This may be repeated until such a v state is reached in the $j^{(2j+1-v)}$ configuration. The $j^2 J = 0$ state has $v = 0$ whereas the $j^2 J > 0$ *even* states have seniority $v = 2$. There is a $v = 0$, $J = 0$ state in all j^n configurations with even values of n. The only state with $J = j$ in the $n = 1$ case, has $v = 1$. There is such a state in all j^n configurations with odd values of n. Clearly, the lower the v, the higher is the pairing.

A detailed description of the seniority scheme and derivation of its properties may be found in several books [32, 33]. In the following, the only derivations are presented that deal directly with the subjects of the present paper. To define the notation, the Wigner–Eckart theorem [34] is expressed as

$$\langle JM|T_\kappa^{(k)}|J'M'\rangle = (-1)^{J-M}(J\|T^{(k)}\|J')\begin{pmatrix} J & k & J' \\ -M & \kappa & M' \end{pmatrix}. \tag{2}$$

In (2), the matrix element of the κ component of any irreducible tensor operator of rank k, is expressed as a product of its essential part (the double barred element) with its geometrical part which depends also on M, M' and κ.

Thus, non-vanishing corresponding matrix elements of components of irreducible tensor operators are proportional. It is therefore convenient to define unit tensor operators

$$(j\|\boldsymbol{u}^{(k)}\|j) = 1, \quad \boldsymbol{U}^{(k)} = \sum_i \boldsymbol{u}_i^{(k)}. \tag{3}$$

In jj-coupling, any two-body interaction can be expanded as a linear combination of scalar products of irreducible tensor operatos. The latter products are defined by

$$T_1^{(k)} \cdot T_2^{(k)} = \sum_\kappa (-1)^\kappa T_{1\kappa}^{(k)} T_{2,-\kappa}^{(k)}. \tag{4}$$

The summation in (4) is on κ between $-k$ and k.

In a two-nucleon state, the interaction energies may be expressed as

$$\begin{aligned} V_J &= \langle j^2 JM|V_{12}|j^2 JM\rangle \\ &= \sum_k F_k \langle j^2 JM|\boldsymbol{u}_1^{(k)} \cdot \boldsymbol{u}_2^{(k)}|j^2 JM\rangle \\ &= \sum_k F_k (-1)^{J+1} \begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} (j\|\boldsymbol{u}^{(k)}\|j)^2 \\ &= \sum_k F_k (-1)^{J+1} \begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix}. \end{aligned} \tag{5}$$

The summation over k extends from 0 to $2j$, which is the extension of $j^2 J$ states. The symbols on the last row of (5) are Racah coefficients [35] or Wigner $6j$-symbols [36]. Due to their orthogonality properties, (5) may be transformed into an expression of the F_k in terms of the V_J as follows:

$$F_k = (2k+1)\sum_J (-1)^{J+1}(2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_J. \quad (6)$$

In the expansion (5), the energies V_J are uniquely determined by the F_k coefficients and the latter are uniquely determined in (6) by the set of V_J.

In the case of two j-protons or two j-neutrons, the antisymmetric states allowed by the Pauli principle have *even* values of J. Such states have isospin $T = 1$ and so are proton-neutron states with even values of J. Proton-neutron states with $T = 0$ are symmetric in space and spin variables and have odd values of J. Only if states of identical nucleons are considered, the expansion in terms of F_k is no longer unique as discussed bellow.

The orthogonality of the $6j$-symbols used here, is

$$\sum_k (2k+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix}\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix} = \delta_{JJ'}/(2j+1). \quad (7)$$

Another useful relation is

$$\sum_k (-1)^{k+J+J'}(2k+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix}\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix} = \begin{Bmatrix} j & j & J \\ j & j & J' \end{Bmatrix}. \quad (8)$$

In (7) and (8), J and J' may be either odd or even integers. Since we consider identical j-nucleons, we take J and J' to be even. The summations are over odd and even values of k. Hence, substracting (8) from (7) yields

$$2\sum_{k \text{ odd}} (2k+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix}\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix} = \frac{\delta_{JJ'}}{2J'+1} - \begin{Bmatrix} j & j & J \\ j & j & J' \end{Bmatrix}. \quad (9)$$

Using (6), (9) may be expressed by matrix elements of two-body interactions,

$$-2\sum_{k \text{ odd}} (2k+1)\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix} \times \langle j^2 JM|\boldsymbol{u}_1^{(k)} \cdot \boldsymbol{u}_2^{(k)}|j^2 JM\rangle = \frac{\delta_{JJ'}}{2J'+1} + \langle j^2 JM|\boldsymbol{u}_1^{(J')} \cdot \boldsymbol{u}_2^{(J')}|j^2 JM\rangle. \quad (10)$$

In (10), the expansion includes scalar products of odd tensor operators (odd k values) and ones with even ranks J'. Hence, in this general case, the two-body interaction does not have any special properties. There is, however, a special interesting case of (10) which is considered below.

The simple special case of (10) is the pairing interaction, defined by (1). In this case, the only non-vanishing interaction between two j-nucleons is V_0. In (1) it is set to be $2j+1$. Thus, putting $J' = 0$, the expansion of δ_{J0} contains odd tensors and the constant term with $J' = 0$. Inserting the values $\begin{Bmatrix} j & j & 0 \\ j & j & k \end{Bmatrix} = (-1)^{2j+k}/(2j+1)$, the following expansion of the pairing interaction is obtained,

$$\begin{aligned}(2j+1)\delta_{J0} &= -(2j+1)\langle j^2 JM|\boldsymbol{u}_1^{(0)} \cdot \boldsymbol{u}_2^{(0)}|j^2 JM\rangle \\ &\quad -2\sum_{k \text{ odd}} (2k+1) \times \langle j^2 JM|\boldsymbol{u}_1^{(k)} \cdot \boldsymbol{u}_2^{(k)}|j^2 JM\rangle. \end{aligned} \quad (11)$$

The first term on the r.h.s. of (11) is a constant term, independent of J (and M). The rest is an odd tensor interaction. Odd tensors are closely related to the seniority scheme. Any odd tensor interaction in the j^n configurations of identical nucleons is diagonal in the seniority scheme. This is not changed if a constant interaction is added, like in (11).

A two-body interaction between n identical j-nucleons may be expressed, in view of (5), as

$$\sum_k F_k \sum_{i<p} \boldsymbol{u}_i^{(k)} \cdot \boldsymbol{u}_p^{(k)} = \frac{1}{2}\sum_k F_k \boldsymbol{u}^{(k)} \cdot \boldsymbol{u}^{(k)} - \frac{1}{2}\sum_k F_k \sum_i \boldsymbol{u}_i^{(k)} \cdot \boldsymbol{u}_i^{(k)}. \quad (12)$$

The second sum on the r.h.s. of (12) eliminates the "self interactions" in the first sum. It is independent of J (and M) and its contribution to any eigenstate of the j^n configuration is equal to

$$-\frac{n}{2}\sum_k F_k/(2j+1). \quad (13)$$

The relation between odd tensor operators and the seniority scheme is due to the following relation:

$$\{\boldsymbol{u}_1^{(k)} + \boldsymbol{u}_2^{(k)}\}|j^2 J=0\rangle = 0, \quad k \text{ odd}. \quad (14)$$

According to the Wigner–Eckart Theorem, the l.h.s. of (14) is a state with $J = k$ which is symmetric in the two nucleons. On the other hand, construction of the state on the l.h.s. of (14) leads to an antisymmetric state and hence it must vanish. In view of (14), it is not surprising that the following can be proved,

$$\begin{aligned}&\langle j^n v\alpha JM|\boldsymbol{U}^{(k)}|j^n v'\alpha' J'M'\rangle \\ &= \langle j^v v\alpha JM|\boldsymbol{U}^{(k)}|j^v v'\alpha' J'M'\rangle\delta_{vv'}, \quad k \text{ odd}. \end{aligned} \quad (15)$$

In (15), α and α' are additional quantum numbers which may be needed to distinguish between states with the same seniority and J, M. They are not necessary in $(7/2)^n$ configurations and do not appear in other formulae. Scalar products of $k = 0$ tensor operators are

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diagonal in any scheme. They depend only on n due to the matrix elements

$$\langle j^n JM|\boldsymbol{U}^{(0)}|j^n J'M'\rangle = n\langle jm|\boldsymbol{u}^{(0)}|jm\rangle\delta_{JJ'}\delta_{MM'} = n(2j+1)^{-\frac{1}{2}}\delta_{JJ'}\delta_{MM'}. \quad (16)$$

The pairing interaction (11) is a special case of a more general Hamiltonian including *any* odd tensor interaction and a constant, $k = 0$ term, which is diagonal in the seniority scheme

$$\frac{1}{2}\sum_{k\text{ odd}} F_k\boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)} + \frac{1}{2}F_0\boldsymbol{U}^{(0)}\cdot\boldsymbol{U}^{(0)} - \binom{n}{2}\sum_{k\text{ odd}} F_k/(2j+1) - \binom{n}{2}F_0/(2j+1). \quad (17)$$

From (15) follows that matrix elements of the first term in (17) is independent of n. All other terms depend only on n and are independent of v, J, and M. This proves that *if the interaction is diagonal in seniority, spacings between energy levels are independent of* n. More specific results follow from (17) for states with lowest seniorities $v = 0$ and $v = 1$. They will be considered in the next section. Before proceeding, a different expansion of the interaction will be presented.

Instead of subtracting, (7) and (8) may be added yielding

$$2\sum_{k\text{ even}}(2k+1)\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix}\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} = \frac{\delta_{JJ'}}{2J'+1} + \begin{Bmatrix} j & j & J' \\ j & j & J \end{Bmatrix}. \quad (18)$$

As above, (18) may be written as

$$\delta_{JJ'} = -2\sum_{k\text{ even}}(2k+1)(2J'+1) \times\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix}\langle j^2JM|\boldsymbol{u}_1^{(k)}\cdot\boldsymbol{u}_2^{(k)}|j^2JM\rangle - (2J'+1)\langle j^2JM|\boldsymbol{u}_1^{(J')}\cdot\boldsymbol{u}_2^{(J')}|j^2JM\rangle. \quad (19)$$

The expansion in (19) contains scalar products of only even ranks, k and J'. To make from it a two-body interaction, (19) could be multiplied by $V_{J'}$ and summed over all values of $J'(0, 2, \ldots, 2j-1)$. The resulting interaction is given by

$$\sum_{J'}V_{J'}\delta_{JJ'} = -2\sum_{k}\sum_{J'\text{even}}(2k+1)(2J'+1) \times V_{J'}\begin{Bmatrix} j & j & J' \\ j & j & k \end{Bmatrix}\langle j^2JM|\boldsymbol{u}_1^{(k)}\cdot\boldsymbol{u}_2^{(k)}|j^2JM\rangle - \sum_{J'\text{even}}(2J'+1)V_{J'} \times\langle j^2JM|\boldsymbol{u}_1^{(J')}\cdot\boldsymbol{u}_2^{(J')}|j^2JM\rangle. \quad (20)$$

Eq. (20) demonstrates that given the eigenvalues V_J of the interaction in the j^2 configuration of identical nucleons, a two-body interaction can be constructed with scalar products of only *even* rank tensors. If the dependence (20) of the coefficients on the V_J is not required, even rank interactions may be used with arbitrary coefficients F_k, k even. The Hamiltonian is then given by an expression analogous to (17) as

$$\frac{1}{2}\sum_{k>0\text{ even}} F_k\boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)} + \frac{1}{2}F_0\boldsymbol{U}^{(0)}\cdot\boldsymbol{U}^{(0)} - \binom{n}{2}\sum_{k>0\text{ even}} F_k/(2j+1) - \binom{n}{2}F_0/(2j+1). \quad (21)$$

To evaluate states and energies of (21) we may use the seniority scheme. Matrix elements of even tensor operators, $k > 0$ even, have a rather simple dependence on n in the seniority scheme

$$(j^nvJ\|\boldsymbol{U}^{(k)}\|j^nvJ') = \left\{\frac{2j+1-2n}{2j+1-2v}\right\}(j^vvJ\|\boldsymbol{U}^{(k)}\|j^vvJ'). \quad (22)$$

Unlike odd rank tensors, even ones are not diagonal in the seniority scheme. Thus,

$$(j^nvJ\|\boldsymbol{U}^{(k)}\|j^nv-2, J') = \left[\frac{(n-v+2)(2j+3-n-v)}{2(2j+3-2v)}\right]^{\frac{1}{2}} \times(j^vvJ\|\boldsymbol{U}^{(k)}\|j^vv-2, J'), \quad k > 0\text{ even}. \quad (23)$$

To calculate matrix elements of (21), Eqs. (22) and (23) may be used. First, however, note that the terms with F_0 are, due to (17), diagonal in any scheme and depend only on n. The contribution of the first term of (21) is different and is considered in the Appendix. It is interesting that in (21) every term with $k > 0$ even, breaks seniority. Still, this expansion is valid for all interactions which are diagonal in the seniority scheme. In such cases, the coefficients of the various terms add up to make the linear combination (21) diagonal in seniority.

4 Identical nucleons in the $1f_{7/2}$ orbit

Valence neutrons in calcium nuclei between ^{40}Ca ($n = 0$) and ^{48}Ca ($n = 8$) occupy the $1f_{7/2}$ orbit. Valence protons in this orbit may be conveniently studied in nuclei with a closed proton shell of $N = 28$.

Spin $j = 7/2$ is small enough and hence, *any two-body interaction is diagonal in the seniority scheme of* $(7/2)^n$ *configurations.* A more precise condition will be given below. Thus, any deviation from predictions of the seniority scheme is due to deviation of wave functions from pure $(7/2)^n$ ones.

Looking at the spectra of the relevant nuclei, energy levels which should have spacings which are independent of n, do not follow this basic result of the seniority scheme. They seem to be perturbed by other configurations. This situation will be discussed bellow, but an exception should be noted. The excitation energies of $J = 5/2$ in nuclei with $n = 3, 5$ are fairly equal. Perhaps the $J = 7/2$ ground states and $J = 5/2$ states, which are farther from possible perturbing states, are less perturbed than other states. To check this possibility, ground states energies (binding energies) are now considered.

The shell model expression of binding energies (mass formula) of semi magic nuclei, with closed shells and n identical nucleons in the j-orbit, is

$$BE(j^n) = BE(n{=}0)+nC_j+\langle j^n\text{g.s.}|\sum_{i<l} V(il)|j^n\text{g.s.}\rangle. \tag{24}$$

On the r.h.s. of (24) are listed the binding energy of the closed shells nucleus and n times the single nucleon energy of a nucleon in the j-orbit. It is equal to the sum of the kinetic energy of a j-nucleon and its potential energy in the shell model potential. The latter is due to the effective interaction energy between a j-nucleon and the nucleons in the closed shells. The last term is the expectation value of the (two-body) interaction between the n j-nucleons in the ground state.

The ground states of all even n nuclei have $J = 0$ and are interpreted as the $v = 0$ ones. To calculate the interaction energy in these states, the expression (17) of the interaction is used. The following derivations hold for any value of j and for any interaction diagonal in seniority.

From (15) and (16) follows that the first sum in (17) vanishes in the expectation values (which are eigenvalues) of states with $J = 0$ and $v = 0$ in j^n configurations (n even). The contribution of the odd tensor interaction is

$$-\binom{n}{2}\sum_{k\text{ odd}} F_k/(2j+1). \tag{25}$$

The two terms proportional to F_0 contribute terms proportional to n and n^2.

As a result, the eigenvalue of (17) of the $v = 0$, $J = 0$ state in the j^n configuration of identical nucleons, is equal to

$$n(n-1)/2\{F_0/(2j+1)\} - (n/2)\left\{\sum_{k\text{ odd}} F_k/(2j+1)\right\}. \tag{26}$$

Eq. (26) is a very simple function of the number n of valence identical nucleons. The coefficients of n and n^2 are functions of the V_J, J even. There is a simple way to calculate these functions.

The value of (26) for n=2 is the eigenvalue V_0 of the $j^2 v = 0$, $J = 0$ state

$$F_0/(2j+1) - \sum_{k\text{ odd}} F_k/(2j+1) = V_0. \tag{27}$$

There is another state where the eigenvalue of the interaction energy is easy to calculate. It is the $J = 0$ state of the full or closed orbit, with $n = 2j+1$, which is equal to

$$\sum_{J\text{ even}} (2J+1)V_J = V_0 + (j+1)(2j-1)\bar{V}_2. \tag{28}$$

The centroid of the $v = 2$ levels, $\bar{V}_2$, is defined by (28). Equating (28) to (26) where we put $n = 2j + 1$, another equation, in addition to (27) for the coefficients in (26) is obtained,

$$2j(2j+1)/2\{F_0/(2j{+}1)\}(2j{+}1)\left\{\sum_{k\text{ odd}} F_k/(2j+1)\right\}/2$$
$$= V_0 + (j+1)(2j-1)\overline{V_2}. \tag{29}$$

Multiplying (27) by $(2j + 1)/2$ and subtracting it from (29), the value of F_0 is obtained from

$$\left\{\frac{2j-1}{2}\right\} F_0 = -\left\{\frac{2j-1}{2}\right\} V_0 + (j{+}1)(2j{-}1)\overline{V_2}. \tag{30}$$

The coefficient of the quadratic term in (26) is equal to

$$F_0/(2j+1) = \{(2j+2)\overline{V_2} - V_0\}/(2j+1). \tag{31}$$

From (27), the coefficient of $n/2$ is obtained as

$$-\frac{\sum_{k\text{ odd}} F_k}{2j+1} = \frac{(2j+2)(V_0-\overline{V_2})}{2j+1}. \tag{32}$$

In the case of the pairing interaction (1), $V_0 = 2j + 1$ and $\overline{V_2} = 0$. The eigenvalues of the $v = 0$, $J = 0$ states in j^n configurations are given by

$$-\frac{n(n-1)}{2} + \left(\frac{n}{2}\right)(2j+2) = n(2j+3-n)/2. \tag{33}$$

The states with maximum pairing in configurations with odd n are the $J = j$, $v = 1$ states. Their eigenvalues of a Hamiltonian with a two-body interaction which is diagonal in seniority can be obtained from (17). In the present case, the first term does not vanish. Using (16), its contribution can be evaluated as follows:

$$\frac{1}{2}\sum_{k\text{ odd}} F_k \times$$
$$\langle j^n v{=}1, J{=}j, M{=}m|\boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)}|j^n v{=}1, J{=}j, M{=}m\rangle$$
$$= \frac{1}{2}\sum_{k\text{ odd}} F_k\langle jm|\boldsymbol{u}_1^{(k)}\cdot\boldsymbol{u}_1^{(k)}|jm\rangle$$
$$= \frac{1}{2}\sum_{k\text{ odd}} F_k/(2j+1). \tag{34}$$

It cancels a portion of the $n/2$ term in (18), reducing it to

$$(n-1)/2\left\{\sum_{k\text{ odd}} F_k/(2j+1)\right\}. \tag{35}$$

The other term, proportional to $n(n-1)/2$, is the same as in (17). The coefficient of the quadratic term may be denoted by α and the one of the linear term, by β. The eigenvalues of states with $v=0$, $J=0$ and $v=1$, $J=j$ in j^n configurations may be combined to

$$\frac{\alpha n(n-1)}{2}+\frac{\beta(n-v)}{2}=\alpha n(n-1)/2+\beta[n/2]. \tag{36}$$

In (36), $[n/2]$ is defined as the largest integer not exceeding $n/2$. It is the *pairing term* which gives rise to the odd-even variation of binding energies of nuclei. From the derivation follows that in the case of an odd tensor interaction, $F_0=0$ in (26), the quadratic term vanishes, $\alpha=0$. This applies to the δ-interaction. Eigenvalues of the pairing interaction (1), are obtained from (36) by putting $\alpha=-1$, $\beta=2j+2$. These eigenvalues are special cases, for $v=0$ and $v=1$, of the expression of all eigenvalues of (1)

$$E(n,v)=(n-v)(2j+3-n-v)/2. \tag{37}$$

This formula will not be proved here

To calculate energies of ground states, with $J=0$ in n even nuclei and $J=7/2$ in odd n ones, we use (24) with the expression (36) for the interaction energies,

$$\begin{aligned} BE(j^n)–BE(n=0) &= BE(^{40+n}\text{Ca})-BE(^{40}\text{Ca}) \\ &= nC+\alpha n(n-1)/2+\beta[n/2]. \end{aligned} \tag{38}$$

As seen in (38), we start with calcium isotopes. We look for values of C, α, and β which reproduce the measured binding energies on the l.h.s. of (38). A least squares fit yields the following values (binding energies are taken as *positive* quantities),

$$C=8.4227,\quad \alpha=-0.2264,\quad \beta=3.2328\text{ MeV}.$$

The binding energy differences, calculated from (38) with these values agree very well with measured values. The r.m.s. deviation between calculated and experimental values is 0.091 MeV which is 0.12% of the 75 MeV range of δ energies. The root mean square deviation is defined by

$$\left[\sum_n (BE(n)_{\text{exp.}}-BE(n)_{\text{cal.}})^2/(N-K)\right]^{\frac{1}{2}}. \tag{39}$$

In (39), N is the number of data points (8 here) and K is the number of adjustable parameters (3 here).

To display the good agreement with the data, it is convenient to plot experimental and calculated *separation energies*, $BE(j^n)-BE(j^{n-1})$. From (38) follows that they should lie on two parallel straight lines, one for odd and the other for even values of n. These are shown in Fig. 1.

An interesting feature which is displayed in Fig. 1 is due to the values of the coefficients of (38). A nucleus with closed shells (^{48}Ca) is no more "tightly bound" than its even-even neighbors. The erroneous statement, that is even mentioned today, about the extra binding energies of magic nuclei is due to an incorrect interpretation of atomic ionization energies [37]. The extra stability of magic nuclei is due to the weaker binding of nuclei *beyond* them as shown in Fig. 1 for ^{49}Ca and ^{50}Ca.

Another point that should be mentioned, is concerned with possible effects of *polarization* of the closed shells nucleus by the valence nucleons. In the following sections clear effects of such core polarization will be considered. They may contribute to binding energy terms proportional to n and n^2. There is no way to disentangle such effects from matrix elements extracted from experimental energies.

Looking at Fig. 1, there seems to be no indication for 3-nucleon interactions between valence neutrons. There was an attempt to attribute the perturbed positions of higher levels to 3-body interactions [38]. A simpler explanation is that these perturbations are due to mixing of configurations as argued in Ref. [39]. The parameters of (38) could certainly contain contributions from 3-body interactions with core (^{40}Ca) nucleons. A possible interaction between *two* core nucleons and a valence nucleon could contribute to C. Such interaction between a core nucleon and *two* valence nucleons could contribute to the

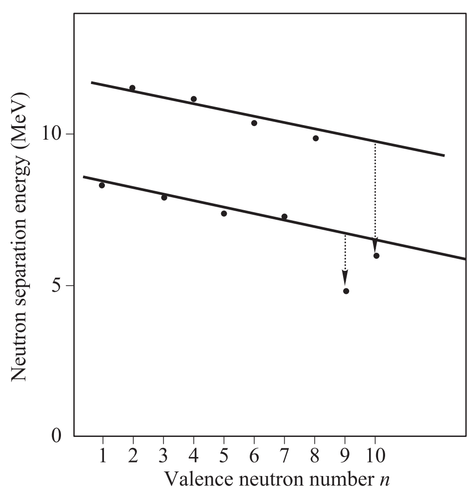

Fig. 1 Neutron separation energies from calcium isotopes. The experimental data, dots, follow closely the theoretical predictions, on the straight parallel lines.

interaction between the latter. Even if these interactions are weak, the number of core nucleons is rather large.

It is interesting to see whether the value of β obtained above from analysis of binding energies, with its expression (33). As explained above, energy levels with $J > 0$ are perturbed and hence, it is not easy to find a reliable value of $V_0 - \bar{V}_2$. If levels of ^{42}Ca are taken, we find from (33), $j = 7/2$,

$$\bar{V}_2 – V_0 = \{5 \times 1.53 + 9 \times 2.76 + 13 \times 3.19\} = 2.739 \text{ MeV},$$

and hence, the value of β should be

$$9 \times 2.739/8 = 3.0817 \text{ MeV},$$

which is in fair agreement with the value 3.2328 MeV obtained above.

The situation of proton $1f_{7/2}$ configurations in $N = 28$ nuclei is very similar. The number n of valence protons in this orbit, goes from $n = 1$ in ^{49}Sc to $n = 8$ in ^{56}Ni. The effective interaction between protons includes the contribution of their mutual Coulomb interaction. As explained above, if $j = 7/2$ (of course, if $j < 7/2$), any two-body interaction is diagonal in the seniority scheme. Thus, the formulae derived above may be safely used. Like in the case of (38), we look for values of C, α and β which would reproduce the measured binding energies on the l.h.s. of the following equation:

$$BE(Z = 20 + n, N = 28) - BE(^{48}\text{Ca}) \\ = nC + \alpha n(n-1)/2 + \beta\,[n/2]\,. \tag{40}$$

This is a result of the shell model and the assignment of the configuration. A least squares fit yields the following values of the coefficients of (40) in MeV,

$$C = 9.7064, \quad \alpha = -0.791, \quad \beta = 3.1384.$$

Binding energies, taken as positive quantities, calculated from (40) with these values agree rather well with measured energies. The r.m.s. deviation, defined by (39), is 0.05 MeV which is 0.086% of the 58 MeV range of binding energies. Like the case in neutron configurations, to display the agreement between measured energies and those calculated in the shell model, separation energies are plotted in Fig. 2. The measured points are rather close to the theoretical values which lie on two parallel straight lines.

To look for agreement between the coefficients thus determined, and two-nucleon energy levels, the centroid of $J > 0$ even levels in ^{50}Ti is calculated to be at

$$\bar{V}_2 - \bar{V}_0 = \frac{\{5 \times 1.553 + 9 \times 2.675 + 13 \times 3.199\}}{27} \\ = 2.71955 \text{ MeV}.$$

above the ground state which leads to $\beta = 2.7195 \times 9/8 = 3.059$ MeV. This is in fair agreement with the value

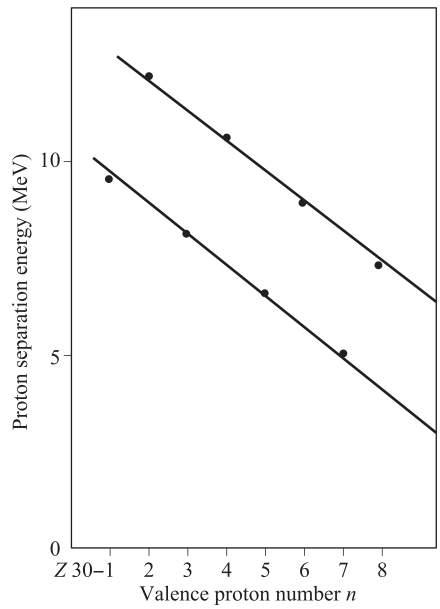

Fig. 2 Proton separation energies from $N = 28$ nuclei. The experimental data, dots, follow closely the theoretical predictions, on the straight parallel lines.

obtained above, in view of the perturbations of $J > 0$ states. It is interesting to look at the differences between the parameters of the neutron and the proton configurations. The value of β of the protons is less attractive by close to 0.1 MeV which could be due to the contribution of the repulsive Coulomb interaction to the pairing interaction. The value of α of the protons is more repulsive by 0.56 MeV. This difference seems bigger than the possible contribution of the Coulomb energy. An obvious reason for it could be the difference between the cores with the valence protons and neutrons, ^{48}Ca and ^{40}Ca. This could lead to differences in the effective interaction.

To see the effects due to the Coulomb interaction, it may be instructive to look at proton $(1f_{7/2})^n$ configurations outside the ^{40}Ca core. Such nuclei are difficult to measure and binding energies are published only for 4 nuclei. Still, it is interesting to carry out the analysis carried above to find the values of the parameters in this case. The least square fit to the measured binding energies yields the values $C = 1.075$, $\alpha = 0.488$ and $\beta = 3.191$ MeV. Binding energies calculated with these coefficients are in good agreement with measured ones. The r.m.s. deviation in this agreement is 0.0167 MeV, 0.25% of the energy range. It is meaningful since there are 4 experimental energies and only 3 parameters. The value of β is reduced compared to the result of the neutrons, although less than in the case of the protons. It is significant that the value of α is more repulsive by 0.26 MeV than in the case of the neutrons but definitely not as much as in the case of protons above.

5 Core polarization: Neutron effective charge

Neutrons do not have electric charges. Still, there are processes in which neutrons participate and seem to carry an *effective charge*. In the shell model, this is attributed to polarization of the core, the nucleus with only closed shells, by the valence neutrons. A striking demonstration of this phenomenon is offered by the quadrupole moments of calcium isotopes with valence neutron numbers $n = 1$, 3, 5, and 7 shown in Fig. 3 [40]. In this section, a possible mechanism is considered which shows how shell model wave functions could be modified to mimic $1f_{7/2}$ neutrons with electric charges.

Shell model wave functions considered here, are those of n neutrons in the $j = 7/2$ orbit outside the closed shells of ^{40}Ca whose spin is $J_c = 0$, $|j^n(vJM)J_c = 0, JM\rangle$. This state may have non-vanishing matrix elements with excited states of the core with $J_c = k$ and positive parity, like

$$\begin{aligned} V_{\gamma k} &= F_{\gamma k}\langle j^n(vJM)J_c = 0, JM|\boldsymbol{U}_j^{(k)} \cdot \boldsymbol{U}_c^{(k)}|j^n(v'J'M')\gamma J_c = k, JM\rangle \\ &= (-1)^k F_{\gamma k}[(2k+1)(2J+1)]^{-1/2}(j^n vJ\|\boldsymbol{U}_j^{(k)}\|j^n v'J')(J_c = 0\|\boldsymbol{U}_c^{(k)}\|\gamma J_c = k). \end{aligned} \tag{41}$$

The equality in (41) is due to a simple formula of tensor algebra. In (41), γ is a quantum number which distinguishes between orthogonal states with the same value of k. To lowest order in perturbation theory, the modified ground state wave function is

$$\begin{aligned} &\Big\{|j^n(vJM)J_c = 0, JM\rangle \\ &\quad + \sum_{\gamma k} V_{\gamma k}|j^n(v'J'M')\gamma J_c = k, JM\rangle/\Delta_{\gamma k}\Big\} \\ &\quad \times\Big\{1 + \sum_{\gamma k}(V_{\gamma k}/\Delta_{\gamma k})^2\Big\}^{-1/2}. \end{aligned} \tag{42}$$

In (42), $\Delta_{\gamma k}$ is the energy difference between the $J_c = 0$ and $\gamma J_c = k$ state. The energy difference between the $j^n(vJ)$ and $j^n(v'J')$ states is neglected here. The excited core states may well include excitations of protons and such states with $k = 2$, contribute to ground states quadrupole moments. The operator of the quadrupole moment in j^n configurations is proportional to $\boldsymbol{U}^{(2)}$ so we calculate its expectation value in the ground state (42).

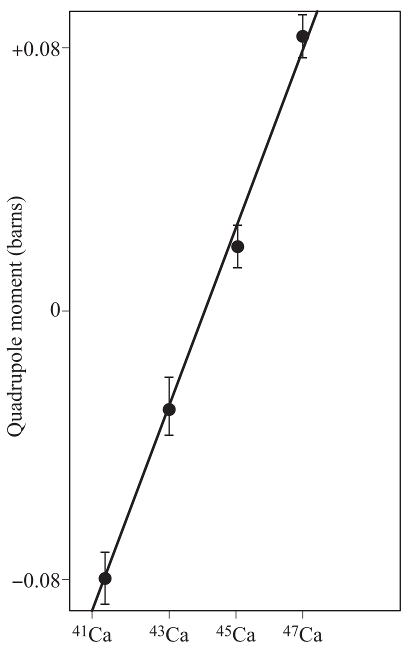

Fig. 3 Quadrupole moments of odd calcium isotopes. The measured values follow closely the theoretical straight line.

Its expectation value in the $(7/2)^n$ neutron configuration is equal to zero. The important contributions come from the cross terms in (42). These terms are, for $v = v'$ and $J = J'$,

$$\begin{aligned} &2\sum_{\gamma} V_{\gamma 2}(j^n(vJ)J_c = 0, J\|U_c^{(2)}\|j^n(vJ)\gamma J_c 2, J)/\Delta_{\gamma 2} \\ &= 2\sum_{\gamma} V_{\gamma 2}[(2J+1)/5]^{1/2}(J_c{=}0\|\boldsymbol{U}_c^{(2)}\|\gamma J_c{=}2)/\Delta_{\gamma 2}. \end{aligned} \tag{43}$$

Substituting the expression of $V_{\gamma 2}$ from (41) we obtain

$$(j^n vJ\|\boldsymbol{U}^{(2)}\|j^n vJ)\sum_{\gamma}\Big\{\frac{2}{5}F_{\gamma 2}(J_c{=}0\|\boldsymbol{U}_c^{(2)}\|\gamma J_c{=}2)^2/\Delta_{\gamma 2}\Big\}. \tag{44}$$

Thus, the modified ground state wave function has a quadrupole moment, which looks as being due to the j-neutrons carrying an *effective electric charge*. If the coefficient in (44) is fairly independent of n, the dependence of the effective quadrupole on n is given by $(j^n vJ\|\boldsymbol{U}^{(2)}\|j^n vJ)$. This dependence is given by (22) as

$$(2j + 1 - 2n)/(2j + 1 - 2v). \tag{45}$$

The single nucleon quadrupole moment is *negative* for a *positive charge* and thus (45) agrees well with measured values as clearly seen in Fig. 3.

The contribution of matrix elements between the additional states in (42) are of higher order of perturbation theory and are negligible. So are the corrections to the normalization, presented in (42) and not shown in subsequent equations. The contribution to the quadrupole moment is proportional to the *amplitudes* of the added states. They may be fairly large and still their square may give a small percentage to the wave function. A question may be asked whether such corrections to the wave functions are consistent with the good agreement obtained above for binding energies. The use of effective interaction in j^n configurations indicates that the wave functions are not pure. Configuration mixing of some kind must be present. In the next section, the contribution of some of these corrections to the effective interaction is calculated.

Possible effects of core polarization in even-even nuclei may be observed in electromagnetic transitions. Such quadrupole transitions take place between the lowest $J = 2$ excited states and the $J = 0$ ground states. Core polarization due to the $J = 0$ ground state of the j^n configuration is given in (42) for $v = 0$, $J = 0$. Core polarization of the $J = 2$ state adds to the shell model state

$$|j^n(v = 2, J = 2)J_c = 0, J = 2M\rangle$$

states which are

$$|j^n(v = 0, J = 0)\gamma J_c = 2, J = 2, M\rangle,$$

so that the state becomes equal to

$$|j^n(v=2, J=2, M)J_c=0, J=2, M\rangle + \frac{\sum_\gamma V_{\gamma 2}|j^n(v=0, J=0, M=0)\gamma J_c = 2, J = 2, M\rangle}{\Delta_{\gamma 2}}\left\{1+\sum_\gamma\left(\frac{V_{\gamma 2}}{\Delta_{\gamma 2}}\right)^2\right\}^{-1/2}. \quad (46)$$

The core excited states in (46) include proton excitations and hence, have non-vanishing matrix elements of the quadrupole operator, proportional to $\boldsymbol{U}_c^{(2)}$, with the shell model ground state wave function. As above, the normalization denominator is ignored, and the matrix element is

$$\begin{aligned}&\sum_\gamma V_{\gamma 2}(j^n(v = 0, J = 0)\gamma J_c = J = 2\\ &\|\boldsymbol{U}_c^{(2)}\|j^n(v = 0, J = 0)J_c = J = 0)/\Delta_{\gamma 2}\\ &= \sum_\gamma V_{\gamma 2}(\gamma J_c = 2\|\boldsymbol{U}_c^{(2)}\|J_c = 0)/\Delta_{\gamma 2}. \qquad (47)\end{aligned}$$

Substituting the values of $V_{\gamma 2}$ from (41), with some necessary changes, the matrix element becomes equal to

$$\begin{aligned}&(j^n v = 2J = 2\|\boldsymbol{U}^{(2)}\|j^n v = 0, J = 0)\\ &\quad\times\sum_\gamma F'_{\gamma 2}(J_c = 0\|\boldsymbol{U}_c^{(2)}\|\gamma J_c = 2)^2/(5\Delta_{\gamma 2}). \qquad (48)\end{aligned}$$

There is another contribution, of the same order, to the matrix element. It is due to the one between the shell model component in (45) and the core excited state in (42),

$$\begin{aligned}&\sum_\gamma V''_{\gamma 2}(j^n(v = 2J = 2)J_c = 0, J = 2\\ &\|\boldsymbol{U}_c^{(2)}\|j^n(v'' = 2J'' = 2)\gamma J_c = 2, J = 0)/(5\Delta''_{\gamma 2})\\ &= (j^n v = 0, J = 0\|\boldsymbol{U}^{(k)}\|(j^n v = 2, J = 2)\\ &\quad\times\sum_\gamma F''_{\gamma 2}((J_c = 0\|\boldsymbol{U}_c^{(k)}\|\gamma J_c = 2)^2/(5\Delta''_{\gamma 2}). \qquad (49)\end{aligned}$$

The sum of (47) and (48) is equal to the matrix element

$$(j^n v = 0, J = 0\|U^{(2)}\|j^n v = 2, J = 2) \qquad (50)$$

multiplied by a function of core excited states. This function includes the matrix element of the quadrupole operator between the $\gamma J_c = 2$ excited core states and its $J_c = 0$ ground state. With present day knowledge of the nuclear many body system, it is not possible to calculate this function. If, however, the energy difference Δ is fairly independent of n, this dependence is determined in (50) by the matrix element between states of the j^n configuration of neutrons as if they carry an effective electric charge. The dependence on n of the matrix element (50) is given by (23). The reduced transition probability $B(E2)$ is proportional to the square of the reduced matrix element. Hence, the expected dependence of $BE(2)$ on n is, in the case considered,

$$(n - v + 2)(2j + 3 - n - v) = n(2j + 1–n). \qquad (51)$$

It is possible to see whether the dependence in (51) is obeyed in calcium nuclei with $n = 2$, 4, and 6. In these nuclei the $B(E2)$ values should be proportional to 12, 16 and 12. The reported measured $B(E2)$ values in these nuclei are, in units of $(\mathrm{eb})^2$, In ^{42}Ca 0.037, in ^{44}Ca 0.047 or 0.049 and in ^{46}Ca 0.017.

Whereas the first two values agree well with the predicted n-dependence, the third is considerably lower. The core polarization in ^{46}Ca seems to be smaller than in the others. This effect is evident also in its radius which is considered in the next section.

6 Core polarization: Nuclear radii

Another effect which is beyond the simple shell model, is the change of nuclear radii when valence neutrons are added to them. Such changes occur in calcium nuclei when $1f_{7/2}$ neutrons are added although they hardly contribute to the charge distribution. If the ^{40}Ca core does not change, there will be no increase in nuclear radii, contrary to experiment. Here, core polarization will be considered which gives rise to changes in nuclear radii in agreement with experiment.

Polarization of the ^{40}Ca by valence nucleons was considered in the last section. In (41), the ground state wave function, perturbed by core polarization was presented, somewhat more generally, it is

$$\left\{|j^n(vJ)J_c=0,JM\rangle-\sum_{v'J'\gamma k}V_{v'J'\gamma k}|j^n(v'J')\gamma J_c=k,JM\rangle/\Delta_{v'J'\gamma k}\right\}\times\left\{1+\sum_{v'J'\gamma k}(V_{v'J'\gamma k}/\Delta_{v'J'\gamma k})^2\right\}^{-1/2}. \quad (52)$$

In (52), $\Delta_{v'J'\gamma k}$ is the energy difference between the $|j^n(v'J')\gamma J_c=k,J'\rangle$ and the the $|j^n(vJ)J_c=0,J\rangle$ states. The aim is to calculate the expectation value of the sum over protons of $\sum_i r_i^2$ which is a scalar. The expectation value in the first, simple shell model wave function, apart from the normalization denominator, is the mean square radius of the core, ^{40}Ca. The next term is due to the matrix element between the core state and excited states. Non-vanishing contributions are obtained only from states with $J_c=k=0$. The result is

$$\begin{aligned}&2\left\{\sum_\gamma V_{\gamma 0}\left(j^n(vJ)J_c=0,J\|\sum_{i\text{ protons}}r_i^2\|j^n(vJ)\gamma J_c=0,J\right)\Big/\Delta_{vJ\gamma_0}\right\}\times\left\{1+\sum_{v'J'\gamma'k'}(V_{v'J'\gamma'k'}/\Delta_{v'J'\gamma'k'})^2\right\}^{-1}\\&=(j^nvJ\|\boldsymbol{U}^{(0)}\|j^nvJ)2\sum_\gamma F_{\gamma 0}(J_c=0\|\boldsymbol{U}_c^{(0)}\|\gamma J_c=0)\left(J_c=0\|\sum_{i\text{ protons}}r_i^2\|\gamma J_c=0\right)\Big/\Delta_{vJ\gamma_0}\\&\quad\times\left[1+\sum_{v'J'\gamma'k'}(V_{v'J'\gamma'k'}/\Delta_{v'J'\gamma'k'})^2\right]^{-1}. \end{aligned}\quad (53)$$

Thus, the result is a product of an expectation value in a state of the j^n configuration by a function of matrix elements of excited states of the core. The former is according to (16), simply equal to $n[(2J+1)/(2j+1)]^{1/2}$.

The last term is due to the next order of perturbation theory. In it, contributions to the expectation value are taken between the modifications of the simple shell model wave functions. These contributions are given by

$$\begin{aligned}&\sum_{v'J'\gamma'\gamma''k}V_{v'J'\gamma'k}V_{v'J'\gamma''k}\times\left(j^n(v'J')\gamma'J_c'=k\|\sum_{i\text{ protons}}r_i^2\|j^n(v'J')\gamma''J_c=k,J\right)\\&\times[\Delta_{v'J'\gamma'k}\Delta_{v'J'\gamma''k}]^{-1}\left[1+\sum_{v'J'\gamma'k}(V_{v'J'\gamma'k}/\Delta_{v'J'\gamma'k})^2\right]^{-1}\\&=\sum_{v'J'\gamma'\gamma''k}V_{v'J'\gamma'k}V_{v'J'\gamma''k}\left(\gamma'J'c=k\|\sum_{i\text{ protons}}r_i^2\|\gamma''J_c''=k\right)\\&\quad\times\left[1+\sum_{v'J'\gamma'k}(V_{v'J'\gamma'k}/\Delta_{v'J'\gamma'k})^2\right]^{-1}[\Delta_{v'J'\gamma'k}\Delta_{v'J'\gamma''k}]^{-1}.\end{aligned}\quad (54)$$

The expression (54) is a perturbation of order $(V/\Delta)^2$. Keeping the normalization denominators is approximately equal to

$$[1+(V/\Delta)^2]^{-1}\sim 1-(V/\Delta)^2. \quad (55)$$

Hence, leaving the normalization denominators in (53), adds higher orders terms of perturbation which are not considered here. These corrections are relevant only when applied to the first term, the expectation value in the ^{40}Ca ground state and will be added later. Replacing here (55) by 1 and making use of (41), the following form of (53) is obtained,

$$\sum_{v'J'\gamma'\gamma''k} F_{\gamma'k}F_{\gamma''k}[(2k+1)(2J+1)]^{-1}(j^n vJ\|\boldsymbol{U}^{(k)}\|j^n v'J')^2 \times (J_c=0\|\boldsymbol{U}_c^{(k)}\|\gamma' J_c'=k)(J_c=0\|\boldsymbol{U}_c^{(k)}\|\gamma'' J_c'=k)$$
$$\times\left(\gamma' J_c'=k\|\sum_{i\text{ protons}} r_i^2\|\gamma'' J_c''=k\right)\times[\Delta_{v'J'\gamma'k}\Delta_{v'J'\gamma''k}]^{-1}\left[1+\sum_{v'J'\gamma'k}(V_{v'J'\gamma'k}/\Delta_{v'J'\gamma'k})^2\right]^{-1}. \tag{56}$$

Taking care of the normalization according to (54), (55) is replaced by

$$\sum_{v'J'J''\gamma'\gamma''k} F_{\gamma'k}F_{\gamma''k}[(2k+1)(2J+1)]^{-1}\times(j^n vJ\|\boldsymbol{U}^{(k)}\|j^n v'J')^2$$
$$\times\{J_c=0\|\boldsymbol{U}_c^{(k)}\|\gamma' J_c'=k)\times(J_c=0\|\boldsymbol{U}_c^{(k)}\|\gamma'' J_c'=k)$$
$$\times\left(\gamma' J_c'=k\|\sum_{i\text{ protons}} r_i^2\|\gamma'' J_c''=k\right)-\delta_{\gamma'\gamma''}\left(J_c=0\|\sum_{i\text{ protons}} r_i^2\|J_c 0\right)\}[\Delta_{v'J'\gamma'k}\Delta_{v'J'\gamma''k}]^{-1}. \tag{57}$$

The expression (57) can be expressed as the product of a term due to the j^n configuration

$$\sum_{v'J'}(j^n vJ\|\boldsymbol{U}^{(k)}\|j^n v'J')^2/(2J+1)$$
$$=(j^n vJ\|\boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)}\|j^n vJ) \tag{58}$$

and a function of core excited states. This is possible if we make the assumption that the energy differences in (57) may be replaced by values which are the same for all values of v', J' and n. A linear combination of scalar products like that on the r.h.s. of (58), with various values of k, is equal, according to (12), to a two-body interaction plus a term proportional to n. Hence, the deviations of mean square radii may be expressed by

$$\langle r^2(^{40+n}\text{Ca})\rangle-\langle r^2(^{40}\text{Ca})\rangle$$
$$=nC+\alpha n(n-1)/2+\beta[n/2]. \tag{59}$$

This was realized by Zamick [41] and Talmi [42] and the formula (59) gave a fair description of the odd-even variation of the data. Results of new measurements are shown in Fig. 4. They are fairly reproduced by (59) with the following values of the parameters,

$$C=-0.0083\alpha=-0.034\beta=0.256\ \text{fm}^2.$$

A notable exception is for ^{46}Ca where it seems that the core polarization is considerably weaker than in other isotopes. This fact was already evident by the smaller transition probability shown in last section. The near vanishing of the linear term is rather unique. The odd-even variation is prevalent also in other cases. The fair agreement in Fig. 4 is certainly better than the one, attributed to (59), in Fig. 7, Ref. [43].

The effective interaction is due to mixing of configurations into the shell model wave functions. In that approach described here, it is determined from experimental data and not by theory. In the last two sections, mixing of such configurations is presented. Their contribution to the effective interaction can be derived. In second order of perturbation theory, the contribution to the perturbed state is equal to $\sum_i V_i^2/\Delta_i$ where V_i is the matrix element of the interaction between the perturbed state and the i-th perturbing state. The energy difference between them is Δ_i.

The square of the matrix element between the shell model state $|j^n(vJ)J_c=0,JM\rangle$ and the excited state $|j^n(v'J')\gamma J_c=k,JM\rangle$ is, according to (56), equal to

$$\sum_{\gamma k}F_{\gamma k}^2[(2k+1)(2J+1)]^{-1}$$
$$\times(j^n vJ\|\boldsymbol{U}^{(k)}\|j^n v'J')^2(J_c\!=\!0\|U_c^{(k)}\|\gamma J_c'\!=\!k)^2\Delta_{v'J'\gamma'k}^{-1}. \tag{60}$$

If the dependence of the energy denominator in (58) on $v'J'$ can be ignored, the expression (58) becomes equal to a product of a term which is an expectation value in

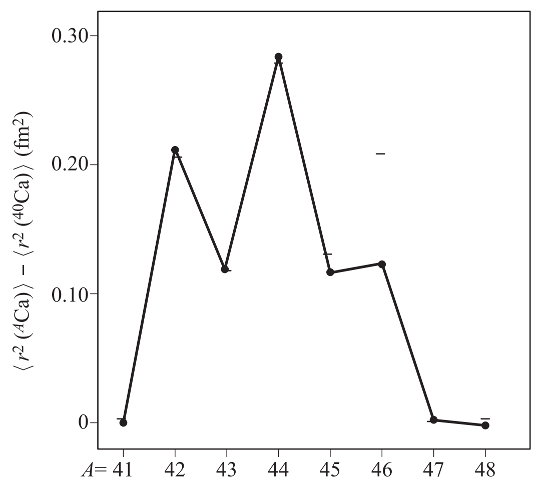

Fig. 4 Radii of calcium isotopes beyond ^{40}Ca. Squared charge radii minus that of ^{40}Ca are plotted. Measured values are connected by straight lines. They are fairly reproduced by theoretical ones, short segments. A clear exception is ^{46}Ca (see text).

the shell model given by (57) as

$$\sum_{v'J'}(j^n vJ\|\boldsymbol{U}^{(k)}\|j^n v'J')^2/(2J+1)$$
$$= (j^n vJ\|\boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)}\|j^n vJ). \qquad (61)$$

and a function of core excited states. The r.h.s. of (61) is, according to (12), equal to a two-body interaction plus a linear term in n.

7 Simplicity from complexity?

The title of this section was coined by Franco Iachello. The shell model is a very simple system describing nuclei, which are very complex systems. Franco referred this expression to an even simpler model – the interacting boson model, introduced by Arima and Iachello [44]. In the present review, only the simplicity of the shell model is considered. In heavier nuclei with valence protons and neutrons, the simplicity of the shell model, independent nucleon motion, seems to continue into strongly deformed nuclei.

Physicists were aware of the successful shell model of the electrons in atoms. There, the massive, positively charged nucleus gives rise to a strong central potential. The Coulomb interaction between the light electrons is not short ranged and hence, yields a rather smooth central field. However, there are deviations from the simple picture. In their classical book [45], Condon and Shortley consider several p^2 and p^3 electron configurations. They show that the two-level spacing in each configuration (between ^{1}D ^{3}P ^{1}S levels in p^2 and ^{2}D ^{2}P ^{4}S levels in p^3 configurations) cannot be due to $1/r$ interaction.

Motivated by the successful nuclear calculations, Racah [46] calculated energies of p^3 states using (effective) two-body matrix elements determined from levels of p^2 configuration in the same orbit. The agreement between the calculated and measured energies was very good. It is clear that in this case, configuration mixing yields the changes in the two-body effective interaction. Still, the results are very simple. States of p^n configuration may be used, provided effective interactions and perhaps other effective operators, are adopted. The real wave functions are certainly more complicated than the model p^n states.

In nuclei, the effective interaction is a very tame and strongly renormalized version of the interaction between free nucleons. There is no strong central potential well as in atoms. The interaction between free nucleons is very strong at short distances. Still, it seems that the shell model is a very good approximation in certain regions, of the nuclear many body system. It would be very instructive to show this fact. It would then be possible to know in which regions of mass and isospin, the simple shell model may be useful.

The shell model is also used in another very different approach. In that approach, for the effective interaction, some empirical data are used but most matrix elements are extracted using some theoretical approximation based on the bare interaction. The matrix of the interaction plus single nucleon energies, usually taken from experiment, if possible, is a very large value. Its order may reach many millions if all states with a given value of J in a major shell are included in the calculation. Ingenious methods for diagonalization of giant matrices have been developed by the Strasbourg–Madrid [28] and Tokyo [29] groups. These calculations exhibit the complexity of the nuclear many body problem. Their results, however, yield wave functions with millions of all possible states with a given J in the major shell.

Even more complicated are the wave functions, obtained from the *ab initio* approach. In them, no central potential is assumed and some version of the interaction between free nucleons is adopted. In some cases, the authors use harmonic oscillator wave functions as the basis and include in the calculation states with higher values of $n\hbar\omega$. The higher the n, the more accurate are the results. The highest value of n included is limited by the computing capacity. The results of these calculations may be of great interest. They yield, however, what may be described as complexity from complexity.

It is important to realize that the simplicity discussed in this article, is demonstrated for observables and not for wave functions. In the case of configuration mixing, wave functions may be far from simple. In the last two sections there are examples of mixing of simple shell model states with core-exited states. The states of several nucleons could be rather complex but other configuration mixings could be more complicated. In the wave functions described above, states of the j^n configuration are present in all components of the state. The perturbing states, however, may include states in which j-nucleons are promoted to higher orbits, specially by the short range part of the interaction. Still, the effect of such mixings could also contribute to the effective interaction.

The observables calculated with the wave functions in the last two sections can be well reproduced by simple shell model j^n wave functions with effective operators. These are contributions to a two-body effective interaction, effective single nucleon (quadrupole) operator (effective charge) for electromagnetic moments and transitions. For squares of nuclear radii, a combination of a single nucleon and a two-body effective operator is necessary.

The situation is similar to the case of $v = 0$, $J = 0$ states of identical nucleons in the seniority scheme. It was shown above that, for an odd tensor interaction, the eigenvalues of these states are equal to nV_0 where V_0 is the eigenvalue of the $j^2J = 0$ state. This seems as if the states $j^nv = 0$, $J = 0$ are "condenstates of $J = 0$ pairs". In a boson model these states correspond to states of n s-bosons. Due to the Pauli principle, the wave functions of nucleons are very different. In the closed shell, $n = 2j+1$, the weight of each j^2J state is proportional to $2J+1$. Only one pair is coupled to $J = 0$. If eigenvalues of the Hamiltonian are calculated by matrix diagonalization, specially in the m-scheme, eigenstates with many components are obtained. For higher values of j it may not be easy to identify the states with $v = 0$.

Diagonalizations of the complete $1f$-$2p$ have been carried out. For the calcium isotopes, the complex wave functions do not include excitations of the ^{40}Ca core. They do not lead to effective charges of the $f_{7/2}$ neutrons, described in the last two sections. To include core excitations, it is necessary to adopt a much bigger shell model space. To establish the connection to the simple shell model, the complex wave functions should be copaired with antisymmetrized wave functions of "dressed" $1f_{7/2}$ nucleons. Each wave function of the latter is like those considered in the last two sections. It includes the $1f_{7/2}$ state and the states of mixed configurations.

It seems that it is a complicated task to make the connection between the very complex wave functions obtained using nuclear many-body calculations and the simple shell model. Good agreement with experiment is obtained by using the simple shell model, with effective operators, in many cases, many more than described above. This indicates that the connection exists in those cases. It is a challenge to demonstrate it and thus prove how the simplicity of the shell model emerges from the complexity of the nuclear many-body calculations.

Appendix A

It was shown above that in j^n configurations with only identical nucleons, *any* two-body interaction can be constructed as a linear combination of scalar products of only *even* tensors with arbitrary coefficients F_k, k even. The Hamiltonian is then given by an expression analogous to (17) as

$$\begin{aligned}&\frac{1}{2}\sum_{k>0 \text{ even}} F_k \boldsymbol{U}^{(k)}\cdot\boldsymbol{U}^{(k)} + \frac{1}{2}F_0\boldsymbol{U}^{(0)}\cdot\boldsymbol{U}^{(0)}\\ &\quad -\frac{n}{2}\sum_{k>0 \text{ even}} F_k/(2j+1) - (n/2)F_0/(2j+1). \qquad \text{(A1)}\end{aligned}$$

To evaluate states and energies of (A1) we may use the seniority scheme. Matrix elements of even tensor operators have a rather simple dependence on n in the seniority scheme

$$\begin{aligned}(j^nvJ\|\boldsymbol{U}^{(k)}\|j^nvJ') &= \{(2j+1-2n)/(2j+1-2v)\}\\ &\quad\times(j^vvJ\|\boldsymbol{U}^{(k)}\|j^vvJ'). \qquad \text{(A2)}\end{aligned}$$

Unlike odd rank tensors, even ones are not diagonal in the seniority scheme. Thus,

$$\begin{aligned}&(j^nvJ\|\boldsymbol{U}^{(k)}\|j^nv-2,J')\\ &= [(n-v+2)\times(2j+3-n-v)/2(2j+3-2v)]^{1/2}\\ &\quad\times(j^vvJ\|\boldsymbol{U}^{(k)}\|j^vv-2,J'),\quad k>0 \text{ even}. \qquad \text{(A3)}\end{aligned}$$

To calculate matrix elements of (A1), Eqs. (A2) and (A3) may be used. First, however, note that the terms with F_0 contribute, like in the case of (17), the result (20) which is diagonal in any scheme. The contribution of the first term of (A1) is different. Non-vanishing matrix elements with the $j^nv = 0$, $J = 0$ state are

$$\frac{1}{2}\sum_{k>0 \text{ even}} F_k\langle j^nv=0, J=0|\boldsymbol{U}(k)\cdot\boldsymbol{U}(k)|j^nv', J'=0\rangle. \qquad \text{(A4)}$$

The matrix element (A4) is equal to

$$\begin{aligned}&\sum_{v''J''}(j^nv=0,J=0\|\boldsymbol{U}^{(k)}\|j^nv''=2,J''=k)\\ &\quad\times(j^nv'=0,J'=0\|\boldsymbol{U}^{(k)}\|j^nv''=2,J''=k)\\ &=[n(2j+1-n)/2(2j-1)]\\ &\quad\times\sum_{v''J''}(j^vv=0,J=0\|\boldsymbol{U}^{(k)}\|j^vv''=2,J''=k)\\ &\quad\times(j^vv'=0,J'=0\|\boldsymbol{U}^{(k)}\|j^vv''=2,J''=k). \qquad \text{(A5)}\end{aligned}$$

The expectation value, diagonal matrix element, of the $v = 0$, $J = 0$ state is given by (A5) for $v' = v$. If the interaction is diagonal in the seniority scheme, (A5) is the eigenvalue of this state. The condition for this situation is that the matrix element (A4) vanishes for any state with $J' = 0$ and $v' = 4$ (there is no $J = 0$ state with $v' = 2$). This is always the case for any $j \leq 7/2$, since the *only* $J = 0$ state in such j^n configurations has $v = 0$. It was shown that if the j^4 state with $v = 0$, $J = 0$ state is an eigenstate of the interaction, the latter is diagonal in *all* states of the seniority scheme [47].

If the interaction is not diagonal in seniority, the expectation value (A4) with $v = v'$ is its contribution to the first order in perturbation theory. This is a linear and quadratic function of the number n of valence identical nucleons, like in (30) for $v = 0$.

If the interaction can be treated as a perturbation, matrix elements which are non-diagonal in the seniority scheme may be ignored. In first order, only the expectation values of the interaction in eigenstates of the seniority scheme are taken into account.

In this case, the energies of states, due to the perturbation, are equal to those due to an interaction diagonal in seniority. Thus, the expectation values of the interaction in states with $v = 0$, $J = 0$ and with $v = 1$, $J = j$, are equal to (36), i.e.,

$$\alpha n(n-1)/2 + \beta[n/2]. \tag{A6}$$

The coefficients in (A6) are given by (25) and (26) as

$$\alpha = \{(2j+2)\overline{V_2} - V_0\}/(2j+1),$$
$$\beta = (2j+2)(V_0 - \overline{V_2})/(2j+1). \tag{A7}$$

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