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REVIEW ARTICLE

Regularity of atomic nuclei with random interactions: *sd* bosons

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Atomic nuclei are complex systems with gigantic configuration spaces, therefore truncations of model spaces are indispensable. Due to the short-range nature of the nuclear interactions, one may resort to a truncation by using coherent nucleon-pairs which are conveniently further simplified as bosons, such as *sd* bosons. The discovery of the spin-zero ground state dominance with random two-body interactions led to a series of studies on regular structure for *sd* bosons in the presence of random interactions, and this review article summarizes studies along this line in last two decades. We concentrate on various patterns exhibited in *sd* boson systems, and demonstrate that many random samples which were thought to be noisy exhibit very regular patterns, some of which are interpreted in terms of the U(5), O(6), $\overline{\mathrm{O(6)}}$, SU(3), and $\overline{\mathrm{SU(3)}}$ dynamical symmetries of the *sd* interacting boson model.

Keywords regularity, random interactions, *sd* bosons

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Contents

1 Introduction

Although atomic nuclei are very complicated systems, their low-lying states are found to reflect extremely simple patterns, such as vibrational and rotational spectrum, seniority patterns, phase transitions, and so on. Such simplicity provides us with important clues to study the nuclear structure. Dynamical symmetries, found in nuclear low-lying states, are not obviously implicit in the underlying interactions. Thus one might speculate that, given an emsemble of Hamiltonians with such symmetries, some properties might arise dominantly while the others occur at very small probabilities, and this feature may not depend on details of the interactions.

In 1997, Johnson, Bertsch and Dean took a two-body random ensemble, and diagonalized the shell model Hamiltonian of a few even-even nuclei. To their surprise, they found that spin-zero ground state dominance, despite that the number of spin-zero states is very small in comparison with the full model space, which is consisted of basis states with various spins [1, 2]. There were many studies of the microscopic explanation on this phenomenon, for instance, there have been a number of groups studied this problem in the case of single-j orbit [3–9] and for bosons with spin l [6, 10–12]; yet the origin of this famous puzzle has remained to be open [13, 14]. A simple surrogate of this complicated question was suggested in Refs. [15–17], where it was pointed out that positive parity is dominant for even-even nuclei with random two-body interactions. Various patterns related to atomic nuclei with random interactions were investigated in Refs. [15, 17–27].

The same idea as in Refs. [1, 2] was extensively studied within the *sd* interacting boson model (the *sd* IBM). This model was suggested by Arima and Iachello in the seventies [28–30]. These *sd* bosons are approximations or

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surrogates of coherent SD nucleon pairs which are dominantly favored in low-lying states, due to the short-range nature of interactions between nucleons inside nucleus. The IBM, together with the Mayer–Jensen shell model [31–33], and the Bohr–Mottelson collective model [34], are fundamental frameworks in theoretical nuclear structure. The triad of these models are often referred to as the shell, geometric, and the algebraic models, each of which generated a family of offshoots. The most important virtue of the IBM is its elegance and simplicity in theoretical formulation and applications to explain the nuclear data. See Ref. [35] for a review.

The first study of *sd* bosons with random two-body interactions after Ref. [1] was performed by Bijker and Frank [36], who pointed out a strong evidence for the occurrence of both vibrational and rotational band structures, as well as the spin-zero ground state dominance of *sd* boson systems. This means that such regular structures represent a much more robust property of the collective model space than is generally thought. Similar results were also pointed out [37] in another algebraic model, the Fermion Dynamical Symmetry Model [38–42]. Bijker and Frank further pointed out [43] that for *sd* bosons, once the number of bosons is sufficiently large in comparison with the rank of interactions, the resulted spectral properties are characterized by a dominance of spin-zero ground states and the occurrence of both vibrational and rotational band structures. In Ref. [44], the calculations of Refs. [36, 43] were generalized by considering the degree of freedom of the F-spin [45], where it was shown that the dominance of both spin-zero ground states and the occurrence of vibrational and rotational motions survives for neutron-proton *sd* boson systems, for both F-spin conserving interactions and F-spin breaking interactions, while these dominances are more prominent for the former, i.e., F-spin conserving interactions; for both types of interactions, ground state favors F-spin maximum. Table 1 presents the frequency of spin and F-spin, (I, F), of the ground states by 1000 runs of F-spin conserving interactions, with $(N_\pi, N_\nu) = (6, 6)$ bosons.

Table 1 Frequency of spin and F-spin (I, F) of the ground state in 1000 runs of random F-spin conserving interactions, with $(N_\pi, N_\nu) = (6, 6)$ bosons. Taken from Ref. [44].

$I \setminus F$	0	1	2	3	4	5	6
0	73	1	19	7	16	1	571
1	0	2	0	1	0	1	0
2	0	8	5	2	0	3	14
3	0	0	0	1	0	0	0
4	2	0	0	0	1	0	2
5	0	0	0	0	0	0	0
6	1	0	0	1	0	0	2
7	0	0	0	0	0	0	0
8	0	1	1	0	0	0	4
9	0	0	0	0	0	0	0
10	0	2	0	0	0	0	2
11	0	0	0	0	0	0	0
12	4	0	0	0	0	0	6
13	0	0	0	0	0	1	0
14	0	2	0	0	0	0	11
15	1	1	1	0	0	0	0
16	0	0	3	1	0	0	4
17	0	1	0	0	0	1	0
18	5	0	0	2	1	0	12
19	0	9	0	0	0	0	0
20	0	0	10	0	3	0	13
21	0	0	0	4	0	0	0
22	0	0	0	0	3	5	4
23	0	0	0	0	0	3	0
24	0	0	0	0	0	0	145

Despite of these regularities, numerical calculations of the shell model with random interactions showed [46] that rotational behavior does not generically arise in fermion systems, and that some special feature of the ensemble might be necessary to obtain a generic rotation in fermion systems. Since the *sd* bosons in the IBM are mapped from coherent SD nucleon pairs, a quick question of the generic rotation with random interactions is whether or not this rotational behavior arises in the SD nucleon-pair truncated subspace. In order to answer this question, in Ref. [47] the shell model Hamiltonian were diagonalized with random two-body interactions within the SD nucleon-pair truncated subspace, and the results are very close to those in the full shell model space, namely, no rotational peaks are observed. However, with appropriate restrictions on the form the Hamiltonian, in particular, inclusion of quadrupole-quadrupole interaction, the statistical rotational peak arises. This means that the IBM should not be thought of as simply a truncation of the configuration space, but rather as a truncation of the model space that arises from the dominance of quadrupole correlations. In other words, the IBM, whether or not modeled by random interaction, is a consequence of quadrupole and pairing correlations.

In this paper we summarize regular patterns arises in *sd* boson systems with random interactions, in particular, relevant results in the last ten years. We begin with the dominance of the spin-zero ground states and the collective motions with random interactions. Then we come to Malman plot [48] of excited states which demonstrate regular patterns of random samples which are neither close to the vibrational peak nor to the rotational peak. Such unexpected regular patterns are found to be independent of the ground state spin values. A comparison between wave functions of given dynamical symmetry and those of the two-body random ensemble showed that the three limit cases, i.e., the U(5), the SO(6) and the SU(3), are actually much more realizable than expected in the *sd* boson systems. This review is organized in chronological sequence of publications, by which we hope this review is more friendly to readers, as it could be instructive in understanding how this topic evolves.

2 Two-body random Hamiltonian of *sd* bosons

In the *sd* IBM the building blocks of the Hamiltonian and the model space are *sd* bosons which are respectively interpreted as correlated S nucleon pairs with spin zero and D nucleon pairs with spin two. Namely, the IBM is a phenomenological model which first truncates the full shell model space to the S and D pair subspace and next maps the S and D pair subspace to the s and d boson subspace. The total boson number is conserved in the Hamiltonian, and is equal to half of the valence nucleon number for an even-even nucleus. The parity of s and d bosons is positive. The relationships between the IBM and the geometric description were discussed in Refs. [35, 49–51], and the relations between the IBM and the microscopic nuclear shell model were discussed in Refs. [45, 52, 53].

The IBM in which the distinction between protons and neutrons is left out is called the IBM-1. The IBM-1 Hamiltonian [35] that we used in this paper is as follows:

$$\begin{aligned} H = {} & E_0 + e_d \sum_m d^\dagger_m d_m + V^{(0)}_{ssss} s^\dagger s^\dagger ss \\ & + \sum_{L=0,2,4} \frac{1}{2} V^{(L)}_{dddd} \sum_M (d^\dagger d^\dagger)^{(L)}_M (dd)^{(L)}_M \\ & + \frac{1}{\sqrt{2}} V^{(2)}_{ddds} \sum_M \left[(d^\dagger d^\dagger)^{(2)}_M d_M s + h.c. \right] \\ & + \frac{1}{2} V^{(0)}_{ddss} \left[(d^\dagger d^\dagger)^{(0)} (ss)^{(0)} + h.c. \right] \\ & + V^{(2)}_{dsds} \sum_M (d^\dagger s^\dagger)^{(2)}_M (ds)^{(2)}_M , \end{aligned} \qquad (1)$$

where

$$\begin{aligned} (d^\dagger d^\dagger)^{(L)}_M &= \sum_{m_1 m_2} (2m_1 2m_2|LM) d^\dagger_{m_1} d^\dagger_{m_2}, \\ (dd)^{(L)}_M &= ((d^\dagger d^\dagger)^{(L)}_M)^\dagger, \end{aligned}$$

and we take this notation for bosons hereafter. In Eq. (1) there are one independent single-particle energy, e_d, and seven two-body parameters, $V^{(0)}_{ssss}$, $V^{(L)}_{dddd}$ ($L = 0, 2, 4$), $V^{(2)}_{ddds}$, $V^{(0)}_{ddss}$, and $V^{(2)}_{dsds}$. In this paper we are interested in two-body random ensemble, we set $E_0 = 0$, $e_d = 0$ in Eq. (1). For short, we denote two-body matrix elements by using $V_{\alpha\beta} = \langle\alpha|V|\beta\rangle$; e.g., $|\alpha\rangle = |dd, 0\rangle$ and $|\beta\rangle = |ss, 0\rangle$ for $V^{(0)}_{ddss}$.

We assume that $V_{\alpha\beta}$ of two-body matrix elements for *sd* bosons are random and follow Gaussian distribution with an average being zero. The distribution width is set to be 1 for diagonal two-body matrix elements and $1/\sqrt{2}$ for off-diagonal two-body matrix elements. More specifically, if $V_{\alpha\beta}$ and $V_{\alpha'\beta'}$ are two arbitrary (different) two-body matrix elements for *sd* bosons, one has

$$\begin{aligned} \rho(V_{\alpha\beta}) &= \frac{1}{\sqrt{2\pi x}} \exp\left(- \frac{[V_{\alpha\beta}]^2}{2x} \right), \\ \rho(V_{\alpha'\beta'}) &= \frac{1}{\sqrt{2\pi x'}} \exp\left(- \frac{[V_{\alpha'\beta'}]^2}{2x'} \right), \end{aligned} \qquad (2)$$

where

$$x = \begin{cases} 1, & \text{if } |\alpha\rangle = |\beta\rangle, \\ \dfrac{1}{2}, & \text{otherwise,} \end{cases} \qquad x' = \begin{cases} 1, & \text{if } |\alpha'\rangle = |\beta'\rangle, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

One sees for the ensemble average that

$$\begin{aligned} &\langle [V_{\alpha'\beta'}]^2 \rangle = x, \\ &\langle V_{\alpha\beta} V_{\alpha'\beta'} \rangle = 0. \end{aligned}$$

Such random ensemble is called two-body random ensemble, or the TBRE for short.

3 Vibrational and rotational peaks

In this section we summarize results obtained by Bijker and Frank [36, 43]. This discussion is exemplified by *sd* bosons with boson number $N_B = 10$. In Fig. 1 we show the $P(I)$, the probability of spin I to be the ground state, versus I, with a TBRE Hamiltonian of ten *sd* bosons. One sees indeed the spin-zero ground state dominance with random two-body interactions.

We first concentrate on random samples which give spin zero ground states, and investigate the distribution

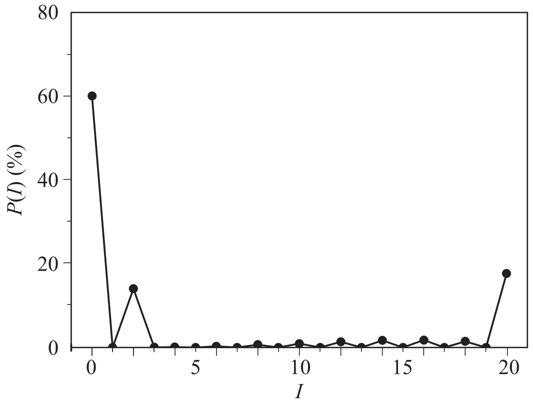

Fig. 1 Probability of spin-I ground state of ten *sd* bosons with two-body random ensemble.

of R_4, defined as below:

$$R_4 = \frac{E_{4_1} - E_{0_1}}{E_{2_1} - E_{0_1}}.$$

In nuclear structure studies, the value of R_4 is often used as a tag for different collective motions: $R_4 \simeq 1$ for seniority type, $R_4 \simeq 2$ for vibration, and $R_4 \simeq 3.33$ for rotation. Below we also use the notation of R_I,

$$R_I = \frac{E_{I_1} - E_{0_1}}{E_{2_1} - E_{0_1}}.$$

In Fig. 2 we plot the distribution of R_4 for random samples with spin zero ground states. One sees that two prominent peaks, one around $R_4 \sim 1.8$ and the other at $R_4 \sim 3.3$. These two statistical peaks were believed to represent generic vibrational and rotational motions under random two-body interactions.

In Fig. 3 we plot a relative B(E2) value, $r = B(E2, 4_1^+ \to 2_1^+)/B(E2, 2_1^+ \to 0_1^+)$ of *sd* bosons with same set of random interactions, versus R_4, in the case of $N_B = 10$. One sees (R_4, r) converges around $(R_4, r) = \left(2, \frac{2(N_B-1)}{N_B}\right) = (2, 1.8)$ and $\left(10/3, \frac{10(N_B-1)(2N_B+5)}{7N_B(2N_B+3)}\right) \simeq (3.33, 1.40)$, corresponding to the U(5) and SU(3) limit of the IBM.

The results of Refs. [36, 43] are very interesting, because these regularities suggest that not only the spin-zero ground state dominance but also typical collective motions, vibrational and rotational, are generically favored by *sd* bosons in the presence of random interactions. Bijker and Frank [54] also studied the origin of the above patterns, by using a Hartree–Bose mean field analysis [49, 50]. They further generalized this approach to studies of *sp* bosons (called vibron model [55, 56]) in Ref. [57]. As the group structure of *sp* bosons is relatively simple, the mean field analysis of *sp* bosons provides us with a more accurate description and more transparent interpretation of $P(I)$ values for *sp* bosons than *sd* bosons.

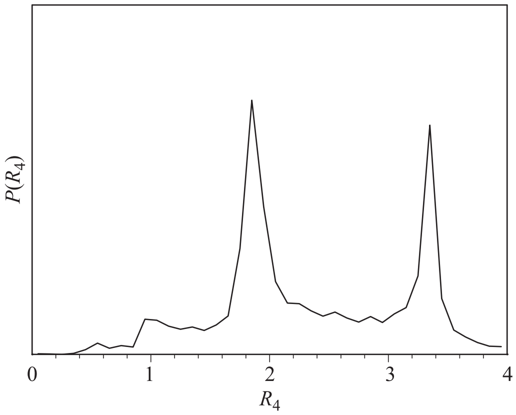

Fig. 2 Distribution of R_4 for ten *sd* bosons with two-body random ensemble.

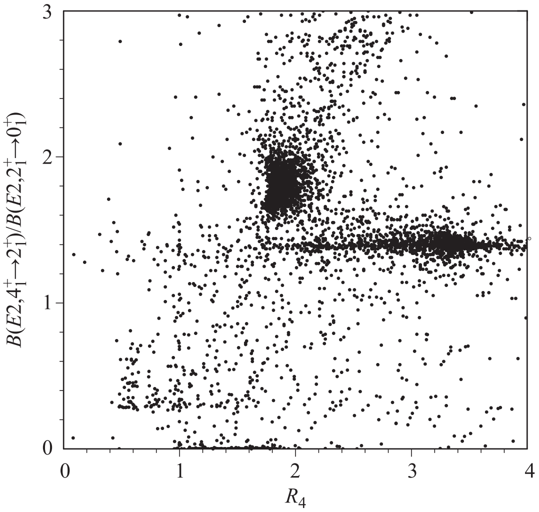

Fig. 3 Distribution of $r = B(E2, 4_1^+ \to 2_1^+)/B(E2, 2_1^+ \to 0_1^+)$ for ten *sd* bosons with two-body random ensemble.

4 "Noisy" samples of *sd* bosons

As random samples of $R_4 \simeq 2$ and 3.33 are interpreted to be vibrational and rotational, one easily conjectures that random samples of R_4 relatively far from these two values can be treated to be noisy samples, which are actually random or chaotic parts of the whole ensemble for *sd* boson systems. However, it was realized in Refs. [58, 59] that this hand-waving comment is not the case.

In Fig. 4 we present a Mallmann plot [48], distributions of calculated R_6 and R_8 versus R_4 for *sd* bosons with boson number $N_B = 6$–11. In consistent with results of Refs. [36, 43], two very sharp peaks at $R_4 \simeq 2.0, 3.3$ (the solid black dots) are reproduced. In addition, one easily observes a few other linear correlations between R_I and R_4, denoted by (α), (β) and (γ), and a relatively smaller "peak" at $R_I \simeq R_4 \simeq 1.0$ which was first pointed out in Ref. [20]. We shall leave the discussion of correlation α in Section 4. Here we discuss correlations β and γ.

The β correlation is best seen when $N_B = 3k$ (k is an integer), and there are much less statistic samples if $N_B \neq 3k$. We find that the low-lying states for random samples of the correlation β are well represented by d-boson condensation. In order to study this correlation, let us look at the eigenvalues of d bosons. According to Eqs. (2.79, 2.82) of Ref. [35] and Eq. (2.8) of Ref. [10], yrast state energies of d bosons are given by

$$E_{\tau I} = \mathcal{C}_1 \tau(\tau+3) + \mathcal{C}_2 I(I+1), \qquad (3)$$

where τ is the "seniority" number of d bosons, and I is the spin of the given state, and

$$\mathcal{C}_1 = \frac{-7V_{dddd}^{(0)} + 10V_{dddd}^{(2)} - 3V_{dddd}^{(4)}}{70},$$

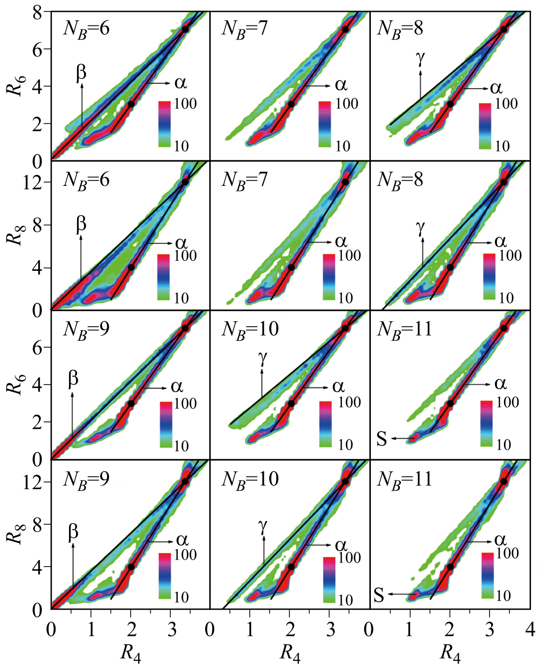

Fig. 4 Distribution of R_I and R_4 for boson numbers $N_B = 6 \sim 11$. For each N_B we perform the calculations by 60000 sets of the TBRE. In addition to two very sharp vibrational and rotational peaks (denoted by solid black circles in each panel, i.e., $(R_4, R_6) = (2.0, 3.0)$ and $(3.3, 7.0)$, $(R_4, R_8) = (2.0, 4.0)$ and $(3.3, 12.0)$, three linear correlations between R_I ($I = 6, 8$) and R_4 are easily noticed. We denote these new correlations by the (α), (β) and (γ). The slopes of solid lines corresponding to the (α), (β) and (γ) correlations are 3, $\frac{21}{10}$ and $\frac{9}{5}$ in the (R_4, R_6) plots; and they are 6, $\frac{18}{5}$ and $\frac{138}{35}$ in the (R_4, R_8) plots. See the text for details. Another relatively smaller "peak" at $R_I \simeq 1.0$ ($I = 4, 6, 8$) is denoted by "S" (called the seniority-type correlation in Ref. [20], see the text) for $N_B = 11$.

$$\mathcal{C}_2 = \frac{-V^{(2)}_{dddd} + V^{(4)}_{dddd}}{14}. \tag{4}$$

The value of τ is given in the reduction rule of the U(5) limit, which has been discussed in details in Eqs. (2.28–2.35) in Ref. [35]. For convenience we repeat it as follows. Here the number of d bosons is $n_d = N_B$ (i.e., the number of s boson is zero), as we focus on d boson condensation in low-lying states. For given boson number N_B, we decompose n by $N_B = 2v + \tau$, thus τ takes

$$N_B, N_B - 2, \cdots, 1 \text{ or } 0 \quad (N_B = \text{odd or even}),$$

corresponding to $v = 0, 1, 2, \cdots, \left[\frac{n}{2}\right]$. For each τ we obtain possible values of λ, with requirement of $\tau = 3n_\Delta + \lambda$. In other words, for given τ,

$$\lambda = \tau, \tau - 3, \ldots, 0 \text{ or } 1, 2 \quad (\tau \bmod 3),$$

corresponding to $n_\Delta = 0, 1, 2, \cdots, \left[\frac{\tau}{3}\right]$. The value of spin of the systems for each λ (independent of the value of τ) is given by

$$I = \lambda, \lambda + 1, \cdots, 2\lambda - 2, 2\lambda.$$

If the two-body matrix element $V^{(2)}_{2222}$ is negative with a large amplitude, the probability to have both $\mathcal{C}_1 < 0$ and $\mathcal{C}_2 > 0$ is very large. In this case the low-lying levels are those with the maximal value of $\tau_{\max} = n_d$ and the minimal value of I. If $n_d = N_B \neq 3k$, $I_{\min}$ with $\tau = \tau_{\max}$ is two, and such samples are excluded here (only 0-g.s. cases are considered). Among states with $\tau = \tau_{\max}$ and $n_d = N_B = 3k$, $I_{\min} = 0$ while the spin-two state is missing. The value of τ in the calculated 2_1^+ state is not equal to N_B. This leads to $E_{I_1^+} - E_{0_1^+} \propto I(I+1)$ except for $I = 2$. On the one hand these random samples fades in the plot of Fig. 2; on the other hand, they exhibit a regular pattern in Mallmann plot. In the R_I–R_4 plot, $R_I/R_4 = I(I+1)/(4 \times 5) = \frac{I(I+1)}{20}$, and $E_{2_1^+}$ is canceled out here. In Fig. 4, one sees that the correlation (β) is indeed described by the simple relation $R_I/R_4 = \frac{I(I+1)}{20}$, that is, $\frac{R_6}{R_4} = \frac{21}{10}$ and $\frac{R_8}{R_4} = \frac{18}{5}$.

Our analysis is also supported by the distributions of $V^{(2)}_{sdsd}$, $V^{(2)}_{dddd}$, n_d and τ in the random samples of the correlation (β). In Fig. 5, we present the distributions of $V^{(2)}_{sdsd}$ and $V^{(2)}_{dddd}$ of the correlation (β) for $N_B = 6$. One sees that the value of $V^{(2)}_{sdsd}$ in the random samples of the correlation (β) is repulsive. This leads to a large probability for low-lying states to have either s^{N_B} or d^{N_B} configurations. On the other hand, the value of $V^{(2)}_{dddd}$ in these random samples are mostly negative with a large amplitude, which favors d-boson condensation in low-lying states, with $\mathcal{C}_1 < 0$ and $\mathcal{C}_2 > 0$ in Eq. (3). In Fig. 6, we further present the distribution of expectation value of n_d and τ for the yrast $I = 0$, 2, 4, 6 and 8 states, for

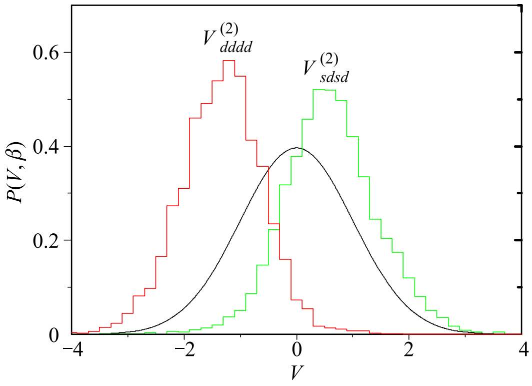

Fig. 5 Distribution of two-body interactions $V^{(2)}_{sdsd}$ and $V^{(2)}_{dddd}$ for random samples of the correlation (β) for $N_B = 6$. The histogram plot in red corresponds to the distribution of $V^{(2)}_{dddd}$, that in green corresponds to the distribution of $V^{(2)}_{sdsd}$, and the plot in black the TBRE. All distributions are normalized by using $\int P(V, \beta)\mathrm{d}V = 1$. One sees the repulsive $V^{(2)}_{sdsd}$ and attractive $V^{(2)}_{dddd}$ in the correlation (β).

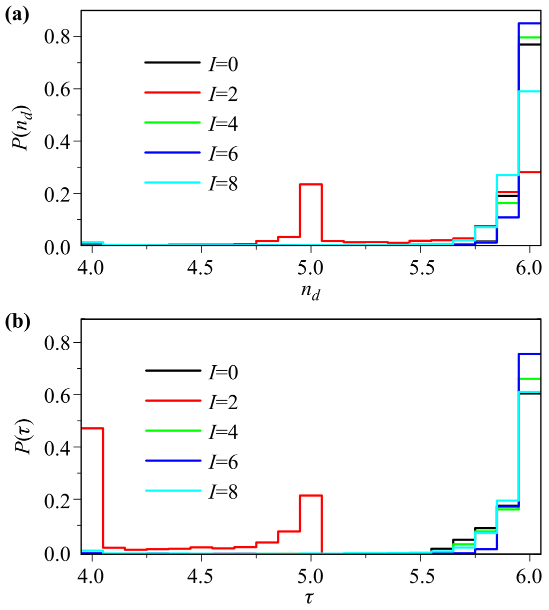

Fig. 6 Distribution of d-boson number (n_d) and that of d-boson seniority (τ) for the first $I = 0$, 2, 4, 6 and 8 states satisfying the correlation β, for $N_B = 6$. The probabilities, $P(n_d)$ and $P(\tau)$, are normalized to 1. One sees that $n_d \simeq N_B = 6$ and that the value of τ is dominated by $\tau = n_d = 6$ except $I = 2$ (plot in red).

$N_B = 6$. One sees the yrast states are all dominated by $\tau = n_d = N_B$ except for $I = 2$. All these are consistent with our above analysis of the correlation (β).

For $N_B = 2k$, another correlation (γ) arises (at ∼ 1% among the TBRE ensemble). Similar to the correlation (β), the correlation (γ) is also originated from d-boson condensation. The only difference is that $\mathcal{C}_1$ in Eq. (3) for the correlation (γ) should be positive. In this case, the yrast states have the smallest τ among in the U(5) irreducible representations. Therefore, for $N_B = 2k$, $\tau_{\min} = 0$, 2, 2, 4 and 4 for $I = 0$, 2, 4, 6 and 8. This is indeed the case, as shown in Fig. 7.

The mechanism of the correlation (γ) for $N_B = 2k$ is very similar to that in the correlation (β). As shown in Fig. 8, the interaction between s and d boson, $V^{(2)}_{sdsd}$, is dominantly repulsive, and the "pairing interaction" between d bosons, $V^{(0)}_{dddd}$, is attractive with a large amplitude in the random samples of the correlation (γ). As in the correlation (β) a repulsive $V^{(2)}_{sdsd}$ pushes states with s bosons upwards, the low-lying states are dominated either by d bosons or s bosons; An attractive $V^{(0)}_{dddd}$ favors d bosons in low-lying states. Therefore the low-lying states exhibit a seniority-type structure: with a positive value of $\mathcal{C}_1$ in Eq. (2), the 0^+_1, 2^+_1, 4^+_1, 6^+_1, 8^+_1 states dominated by $\tau = 0$, 2, 2, 4, 4 configurations, respectively. Assuming this pattern in the correlation (γ), we have $\frac{R_6-7}{R_4-\frac{10}{3}} = \frac{9}{5}$ and $\frac{R_8-12}{R_4-\frac{10}{3}} = \frac{138}{35}$. These two relations nicely describes the correlation (γ).

For $N_B = 6k \pm 1$, there is another correlation which looks similar to the correlation (γ). This correlation fades when N_B becomes large, with its probability being $\simeq 1\%$ for $N_B = 7$ and 0.6% for $N_B = 11$ among the TBRE ensemble. We are not yet able to relate this correlation to any features in the two-body interactions of the Hamiltonian or simple configurations of sd bosons.

There exists a small but sharp statistical peak labeled "S" at $(R_I, R_4) \simeq (1, 1)$ in Fig. 4. This peak is very robust

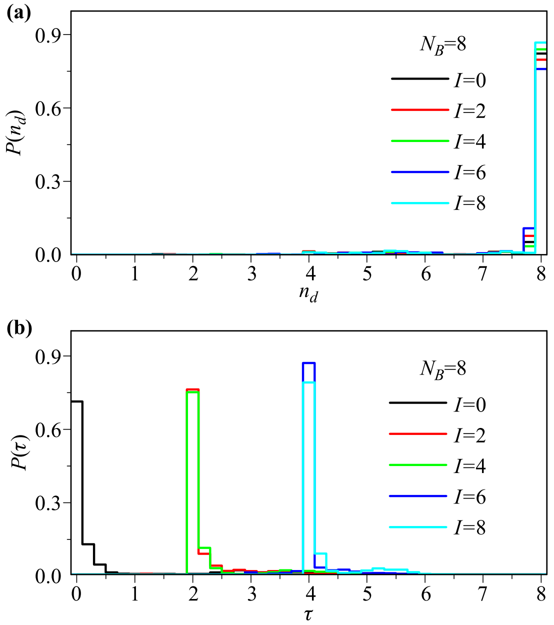

Fig. 7 Similar to Fig. 5, except for the correlation (γ) with $N_B = 8$. One sees that these yrast states are dominated by d bosons, with $\tau = 0$, 2, 2, 4 and 4 for $I = 0$, 2, 4, 6 and 8. The probabilities $P(n_d)$ and $P(\tau)$ are normalized to 1.

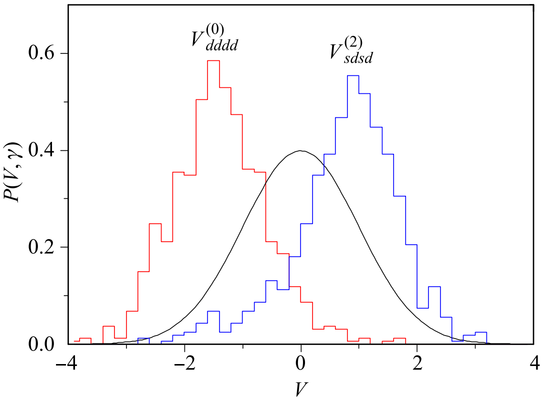

Fig. 8 Distribution of two-body interactions $V^{(2)}_{sdsd}$ and $V^{(0)}_{dddd}$ for random samples of the correlation (γ) with $N_B = 8$. The histogram plot in red corresponds to the distribution of $V^{(0)}_{dddd}$, that in blue corresponds to the distribution of $V^{(2)}_{sdsd}$, and the plot in black the TBRE. All distributions are normalized for $\int P(V, \beta)\mathrm{d}V = 1$. One sees the repulsive $V^{(2)}_{sdsd}$ and attractive $V^{(0)}_{dddd}$ in the correlation (γ).

even for larger N_B ($\sim 0.6\%$ of the TBRE ensemble). It was called seniority-like spectra in Ref. [20]. According to Fig. 9, we find that the ground states of these random samples are dominated by s-boson condensation, and the other low-lying yrast states are dominated by d-boson condensation; the $I \neq 0$ yrast states have either the maximal or minimal τ values.

We similarly find that the correlation $(R_I, R_4) \simeq (1, 1)$ is originated from the special features of two-body interactions among the TBRE Hamiltonian. A repulsive interaction between s and d bosons ($V^{(2)}_{sdsd}$) (see Fig. 10) favors s or d boson condensation in low-lying states; and a strong and attractive $V^{(0)}_{ssss}$ interaction favors s^{N_B} configuration. The energy gap between the two configurations is approximately $|V^{(0)}_{ssss}|N_B(N_B-1)$. Such interactions lead to s-boson condensation in the ground states and d-boson "condensation" above the ground state. If $\mathcal{C}_1 < 0$, $\tau = \tau_{\max} = N_B$ for all the $I \neq 0$ yrast state; if $\mathcal{C}_1 > 0$, $\tau = \tau_{\min} \simeq 1$, 3, 3, and 5 in the 2^+_1, 4^+_1, 6^+_1 and 8^+_1 states, respectively.

It is of interest to investigate in more detail why $R_I = R_4 \simeq 1$ in this case. Let us consider two yrast states consisting of d bosons, with spin I, I' and seniority τ and τ'. The difference between these two states is given by $|\mathcal{C}_1\delta_1 + \mathcal{C}_2\delta_2|$ [see Eq. (3)], where $\delta_1 = \tau(\tau+3) - \tau'(\tau'+3)$ and $\delta_2 = I(I+1) - I'(I'+1)$. For $0 < I, I' \leq 8$, the maximum δ_1 equals $5\times 8 - 1\times 4 = 36$, and the maximum of δ_2 equals $8\times 9 - 2\times 3 = 66$. These two values are much smaller than $N_B(N_B-1)$ when $N_B > 10$. Furthermore, the width of $\mathcal{C}_1$ and $\mathcal{C}_2$ approximately equals 0.18 and 0.1, respectively, which is much smaller than the width of $V^{(0)}_{ssss}$ (~ 1). Thus $|\mathcal{C}_1\delta_1 + \mathcal{C}_2\delta_2| \ll |V^{(0)}_{ssss}|N_B(N_B-1)$, leading to $R_I \sim R_4 \simeq 1$. This also explains why the peak at $R_I \sim R_4 \simeq 1$ becomes sharper for larger N_B.

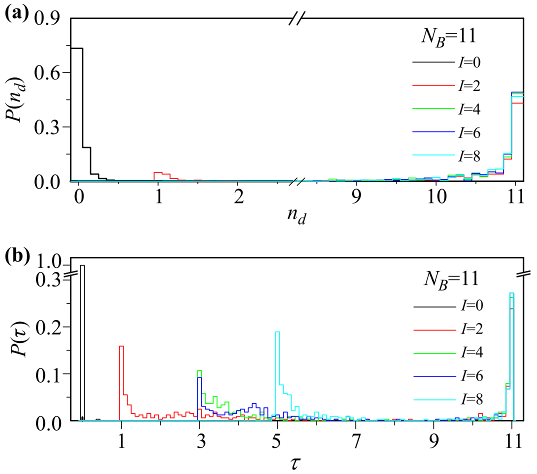

Fig. 9 Similar to Fig. 5 except for the restriction of $R_I \simeq R_4 \simeq 1$ with $N_B = 11$. One sees that the ground states are dominated by s bosons and the yrast states with $I = 2$, 4, 6 and 8 are dominated by d bosons. The dominant value of $\tau \simeq 1$, 3, 3, and 5 in the 2^+_1, 4^+_1, 6^+_1 and 8^+_1 states, respectively. The probabilities $P(n_d)$ and $P(\tau)$ are normalized to 1.

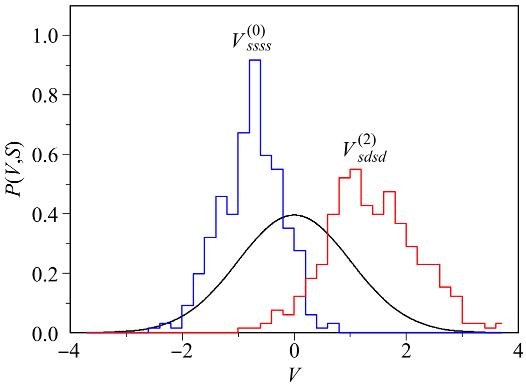

Fig. 10 Distribution of two-body interactions $V^{(2)}_{sdsd}$ and $V^{(0)}_{ssss}$ for random samples of $R_I \simeq R_4 \simeq 1$ with $N_B = 11$. The histogram plot in blue corresponds to the distribution of $V^{(0)}_{ssss}$, that in red corresponds to the distribution of $V^{(2)}_{sdsd}$, and the plot in black the TBRE. All distributions are normalized for $\int P(V,\beta)\mathrm{d}V = 1$. One sees the repulsive $V^{(2)}_{sdsd}$ and attractive $V^{(0)}_{ssss}$ in the correlation (γ).

We note without details that, in the Fermion Dynamical Symmetry Model, similar correlations are also pointed out [60] by using the Mallmann plot, and each correlation can be connected with certain symmetry [61], while these correlations are not obvious in the distribution of R_4 [37].

5 Spin-nonzero random samples

In the above, the discussions are based on random samples which have spin zero ground states. In this section we show that yrast states of sd bosons are also *highly correlated*, regardless the ground spin is zero or not. For these correlations, we show that the low-lying states of random samples with spin non-zero ground states are dominantly represented by d boson condensation, and this is done from two perspectives; one of which is to derive formulas representing these linear correlations, and the other of which is to further investigate the wave functions for yrast states of the random samples.

We present Mallmann plot of R_6 and R_8 versus R_4, for random samples of ground state spin *non-zero* cases, with boson number n from 6 to 11, in Figs. 11 and 12. For each boson number n, we take 60 000 runs of the random Hamiltonians. One sees that *very strong linear correlation* between R_6 (and R_8) versus R_4 arises in these random *spin-non-zero* samplings. We denote these correlations by $\alpha, \beta, \gamma, \delta, \xi$ (α, β, γ correlations are already discussed in the above for cases with spin-zero ground states). For convenience, we summarize these linear correlations in Table 2. All these correlations, except correlation α, can be "derived" from the U(5) limit of the IBM.

Now we derive the formulas corresponding to linear correlations $\beta, \gamma, \delta, \xi$, from d boson condensation of low-lying states. By using the above reduction rule of d bosons, we are now able to discuss yrast states for different signs of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Eq. (4). Suppose that $n = 2k + 1$ (odd n), and that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are positive. According to Eq. (3), the yrast states have minimal value of τ (i.e., $v = 0$). Easily one obtains that the $|\tau, \lambda\rangle = |3, 0\rangle, |1, 1\rangle, |3, 3\rangle, |3, 3\rangle$, and $|5, 5\rangle$, respectively for yrast states of spin 0, 2, 4, 6, and 8. The eigenvalues of these states are respectively $18\mathcal{C}_1$, $4\mathcal{C}_1 + 6\mathcal{C}_2$, $18\mathcal{C}_1 + 20\mathcal{C}_2$, $18\mathcal{C}_1 + 42\mathcal{C}_2$, and $40\mathcal{C}_1 + 72\mathcal{C}_2$. These yrast state energies lead to

$$R_6 = \frac{21}{10} R_4, \quad R_8 = \frac{57}{14} R_4 - \frac{11}{7}.$$

The above R_6–R_4 relation corresponds to β correlation, which coincides with the β correlation arising in the case of $n = 3k$ with both $\mathcal{C}_1$ and $\mathcal{C}_2$ negative (with maximal value of τ). The R_8–R_4 relation corresponds to ξ correlation. The above discussion are repeated for other cases of $\mathcal{C}_1$, $\mathcal{C}_2$ and n, and the results of correlations are summarized in Table 3. It is worthy to note that the g.s. probabilities of d boson systems with different signs of $\mathcal{C}_1$ and $\mathcal{C}_2$ can be derived analytically [6, 12].

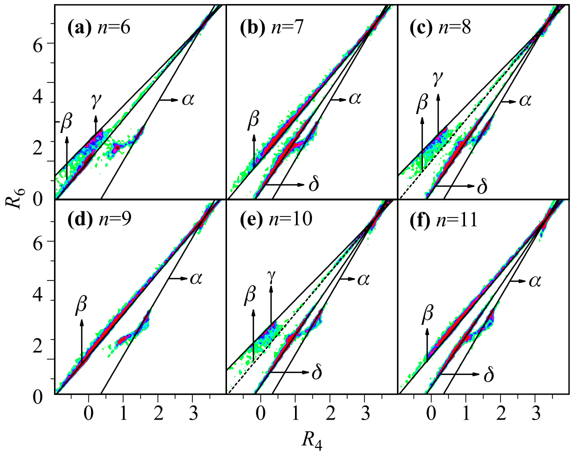

Fig. 11 Correlation of R_6 versus R_4 of *sd* bosons with two-body random interactions. n is boson number. Only random samples with spin *non-zero* ground states are considered here. In addition to the peaks at $(R_4, R_6) = (3.3, 7.0)$ which corresponds to rotational motion, and $(R_4, R_6) = (1.0, 1.0)$ which corresponds to the seniority-type correlation, strong correlations (denoted by α, β, γ, δ and ξ) between R_4 and R_6 are seen. Correlations β and ξ are described by the same formula in the R_6–R_4 plot, and thus are "mixed" in the case of odd n when $n = 3k$. See the text for more details.

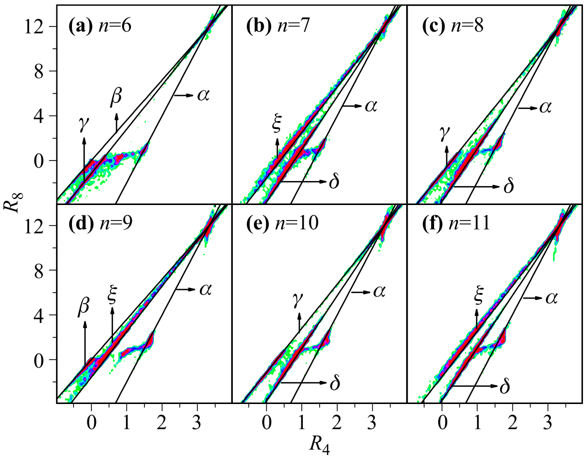

Fig. 12 Same as in Fig. 1 but R_6 is replaced by R_8. In the R_8–R_4 plot, correlations β and ξ are different.

Table 2 Description of linear correlations (denoted by $\alpha, \beta, \gamma, \delta, \xi$) between R_I and R_4. Let us define $R_6 = k_1 R_4 + b_1$, and $R_8 = k_2 R_4 + b_2$. For each correlation we present its k_1, b_1, k_2, b_2 as well as corresponding requirement of boson number n. One sees that correlations β and ξ are described by the same formula in terms of R_6 and R_4.

Correlation	k_1	b_1	k_2	b_2	n
α	3	-3	6	-8	both even and odd n
β	$\frac{21}{10}$	0	$\frac{15}{8}$	0	$n = 3k$ or odd n
γ	$\frac{9}{5}$	1	$\frac{138}{35}$	$-\frac{8}{7}$	even n
δ	$\frac{18}{7}$	$-\frac{11}{7}$	$\frac{33}{7}$	$-\frac{26}{7}$	$n \neq 3k$
ξ	$\frac{21}{10}$	0	$\frac{57}{14}$	$-\frac{11}{7}$	odd n

Now we further examine wave functions of yrast states for random samples with $\beta, \gamma, \delta, \xi$ correlations. We show that they are well represented by the d boson condensation, as assumed in the above. Let us first come to correlation β (arising for *sd* boson systems with $n = 3k$), for which $R_6 = \frac{21}{10} R_4$, and $R_8 = \frac{18}{5} R_4$. This correlation arises for $\mathcal{C}_1 < 0$ and $n = 3k$ (k is an integer), in this case ground state spin is zero or $2n$ (maximum spin), and d boson seniority number τ takes the maximal value. Here we focus on random samples with spin non-zero ground states, the β correlation is given by d boson condensate states with both $\mathcal{C}_1$ and $\mathcal{C}_2$ negative, and ground state spin $2n$. In Fig. 13(a) we plot the distribution of expected value of n_d (d boson number) of random samples with spin-$2n$ ground states for β correlation of the *sd*-boson system (boson number n is six), where one sees that n_d is dominantly equal to $n = 6$. In Fig. 13(a′) for the same system we plot the distribution of seniority τ, which is also consistent with the values in Table 3: the value of τ equals boson number n for yrast states with spin 0, 4, 6, 8, and equals $n - 2$ for the yrast 2^+ state. In Fig. 14(a), we present the distribution of $\mathcal{C}_1$ and $\mathcal{C}_2$ for random samples with spin-$2n$ ground states

Table 3 Summary of quantum numbers for yrast spin I states under different n, different signs of $\mathcal{C}_1$ and $\mathcal{C}_2$. n is the total number of *sd* bosons, and “g.s.” is the abbreviation for “ground state”. For each case one has either one or two possible spin(s) in the g.s. for random samples of d bosons. For cases with two possible spins in the g.s., we use ▲I to label spin I which has a larger probability to be the g.s. spin and △ I to label spin I which has a relatively smaller probability to be the g.s. spin of given random sample. For example, for $\mathcal{C}_1 < 0$ and $\mathcal{C}_2 > 0$ and $n = 3k + 1$, the ground state spin I of d bosons may be $I = 2$ with a larger probability or $I = 0$ with a relatively probability, and these two possibilities sum up to 100%.

$\mathcal{C}_1, \mathcal{C}_2$	$\mathcal{C}_1 < 0, \mathcal{C}_2 > 0$						$\mathcal{C}_1 > 0, \mathcal{C}_2 > 0$			
n	$3k$		$3k+1$		$3k+2$		$2k$		$2k+1$	
τ, λ	τ	λ	τ	λ	τ	λ	τ	λ	τ	λ
$\vert 0_1\rangle$	n	0	$n-4$	0	$n-2$	0	0	0	3	0
$\vert 2_1\rangle$	$n-2$	1	n	1	n	2	2	2	1	1
$\vert 4_1\rangle$	n	3	n	4	n	2	2	2	3	3
$\vert 6_1\rangle$	n	3	n	4	n	5	4	4	3	3
$\vert 8_1\rangle$	n	6	n	4	n	5	4	4	5	5
g.s. spin	0		▲2, △ 0		▲2, △ 0		0		▲2, △ 0	
Correlation	β		δ		δ		γ		ξ	
$\mathcal{C}_1, \mathcal{C}_2$	$\mathcal{C}_1 < 0, \mathcal{C}_2 < 0$						$\mathcal{C}_1 > 0, \mathcal{C}_2 < 0$			
n	$3k$		$3k+1$		$3k+2$		$2k$		$2k+1$	
τ, λ	τ	λ	τ	λ	τ	λ	τ	λ	τ	λ
$\vert 0_1\rangle$	n	0	$n-4$	0	$n-2$	0	0	0	3	0
$\vert 2_1\rangle$	$n-2$	1	n	1	n	2	2	2	1	1
$\vert 4_1\rangle$	n	3	n	4	n	2	2	2	3	3
$\vert 6_1\rangle$	n	3	n	4	n	5	4	4	3	3
$\vert 8_1\rangle$	n	6	n	4	n	5	4	4	5	5
g.s. spin	$2n$		$2n$		$2n$		▲$2n$, △ 0		▲$2n$, △ 2	
Correlation	β		δ		δ		γ		ξ	

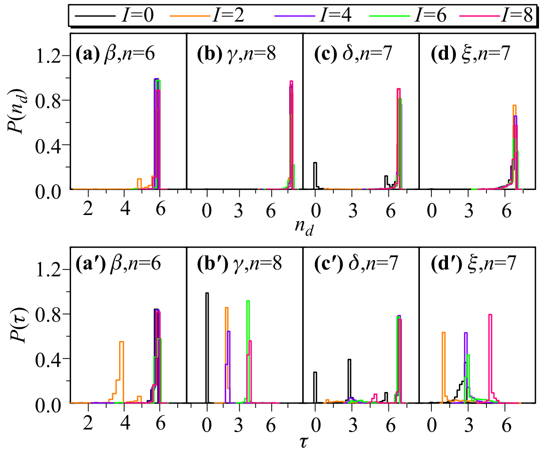

Fig. 13 Histogram plots of expected values of d boson number d-boson seniority (τ) in yrast states of spin $0, 2, 4, 6, 8$, for random samples with spin-$2n$ ground states corresponding to correlations β [panels **(a, a′)**, with boson number $n = 6$], γ [panels **(b, b′)**, with $n = 8$], δ [panels **(c, c′)**, with $n = 7$], and ξ [panels **(d, d′)**, with $n = 7$]. Both $P(n_d)$ and $P(\tau)$ are normalized to 1. One sees that the distributions of n_d and τ are consistent with those predicted by assuming d boson condensation for these random samples.

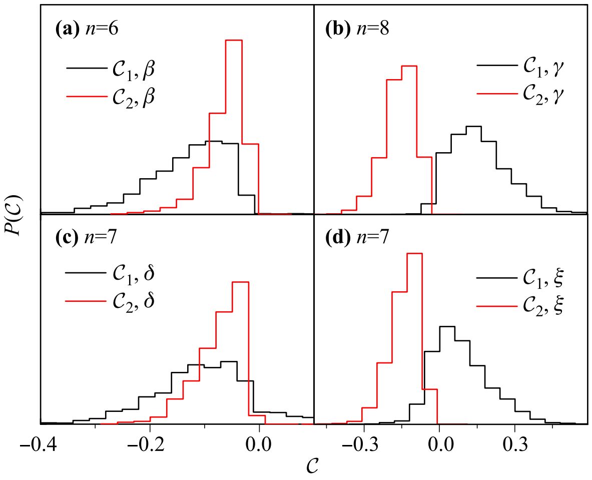

Fig. 14 Distribution of coefficients $\mathcal{C}_1, \mathcal{C}_2$ of Eq. (3) for random samples with spin-$2n$ ground states, corresponding to correlations β [panel **(a)**: $n = 6$], γ [panel **(b)**: $n = 8$], δ [panel **(c)**: $n = 7$] and ξ [panel **(d)**: $n = 7$]. One sees here that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are dominantly negative for correlations β and δ, and that $\mathcal{C}_1 < 0$ and $\mathcal{C}_2 > 0$ for correlations γ and ξ. Such $\mathcal{C}_1$ and $\mathcal{C}_2$ values are consistent with predictions in Table 3 by assuming d boson condensation for these correlations.

corresponding to the β correlation for $n = 6$. One sees that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are dominantly negative. All are consistent with our conjecture that correlation β is originated from d boson condensation.

The case of correlation γ is very similar to correlation β (arising for sd boson systems with even number of n). As shown in Fig. 13(b, b′) the expected values of n_d and τ equals those predicted in Table 3 ($n_d = n = 8$, τ takes the smallest value, and here the value of τ equals 0, 2, 2, 4, 4 for yrast state with spin 0, 2, 4, 6, 8, respectively), namely, for ground states of d boson condensation under the requirement that $\mathcal{C}_1 > 0$ and $\mathcal{C}_2 < 0$, well consistent with the results presented in Fig. 14(b).

The correlation δ arises for $n \neq 3k$, with a more complex pattern than the above correlations β and γ. *Only* for $n = 3k+2$ and random samples with spin two ground states, the statistics of n_d and τ are consistent with those tabulated for $\mathcal{C}_1 < 0$ and $\mathcal{C}_2 > 0$ in Table 3. In this case the expected values of both n_d and τ in yrast states equals to n, [except that τ equals $n - 2$ for the yrast state of spin 0]. For other random samples of correlation δ, the number of n_d in the yrast state of spin 0 does not exhibit an apparent regularity, while other yrast states that we investigate well follow the results tabulated in Table 3. Therefore it is interesting to investigate why the "irregular" spin-zero yrast state of the random samples does not break the correlation δ. According to Table 3, for correlation δ the value of $\tau = n$ for yrast states of spin $I = 2$, 4, 6, 8. Therefore the yrast energy of these four states are given by $\mathcal{C}_1 n(n+3) + \mathcal{C}_2 I(I+1)$. Let us denote the energy of the lowest spin-I state by using E_I, we obtain

$$R_4 = \frac{14\mathcal{C}_2}{E_2 - E_0} + 1, \quad R_6 = \frac{36\mathcal{C}_2}{E_2 - E_0} + 1,$$
$$R_8 = \frac{66\mathcal{C}_2}{E_2 - E_0} + 1.$$

This gives

$$R_6 = \frac{18}{7}R_4 - \frac{11}{7}, \quad R_8 = \frac{33}{7}R_4 - \frac{26}{7},$$

regardless of the value of E_0. Therefore, the correlation δ is more general than cases given in Table 3. Correlation δ requires that the lowest states of spin I ($I \neq 0$) are given by d boson condensation while it does not have additional requirement of the lowest spin-zero state for random samples.

We finally come to the correlation ξ which arises when $\mathcal{C}_1 > 0$ and n is odd. In this case the ground state spin can be 0 and 2 (if $\mathcal{C}_2 > 0$), and $2n$ or 2 (if $\mathcal{C}_2 < 0$), depending on the relative magnitudes of $\mathcal{C}_1$ and $\mathcal{C}_2$, as discussed in Table 3. In Fig. 13(d, d′) we plot expected value of n_d and τ in yrast states of spin 0, 2, 4, 6, 8 for random samples with spin-$2n$ ground states which exhibit the correlation ξ for $n = 7$. One sees that both n_d and τ in these states of such random samples follow Table 3. We also further investigate the random samples with spin $2n$ ground states, and obtain the distributions of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Fig. 14(d), where one sees dominantly positive $\mathcal{C}_1$ and negative $\mathcal{C}_2$, consistent with predictions in Table 3. We should note that the wave functions, quantum number τ and sign of $\mathcal{C}_2$ for correlation ξ are fully different from those for correlation β, although the formula of the R_6–R_4 relation for ξ is interestingly and accidentally the same as that for correlation β.

6 Wave functions

As we discussed in the above sections, random samples with R_4 far from the vibrational and rotational peaks follow a number of correlations which can be interpreted in terms of d boson condensations, seniority type and so on. In this part we examine the wave functions of random sd-boson systems.

Let us begin our discussion with random samplings close to the vibrational peak, $R_4 \approx 2$. More precisely, we find the vibrational peak for the systems with $n = 6, 8, 10$ at $R_4 = 1.73, 1.82, 1.86$, respectively. For these vibrational samplings we calculate the overlap between the wave function obtained under the TBRE, $|r\rangle$, and the U(5)-limit states, $|\mathrm{U}(5), i\rangle$ ($i = 1, 2, \ldots, d$, where d is the dimension of the sd-boson space). The largest overlap is denoted by $x^2_{\mathrm{U}(5)}$, i.e.,

$$x^2_{\mathrm{U}(5)} = \max_i \langle r|\mathrm{U}(5), i\rangle^2. \tag{5}$$

The distribution of $x^2_{\mathrm{U}(5)}$ for the $0^+_1, 2^+_1, 4^+_1$ states are presented in Fig. 15, and one sees a statistical peak at $x^2_{\mathrm{U}(5)} = 1$ [61]. The probability of $x^2_{\mathrm{U}(5)} > 0.8$ is around 83%, 75%, and 63% for the 0^+_1, 2^+_1, and 4^+_1 states, respectively, and the rest of the vibrational samplings does not follow any of the above dynamical symmetry limits.

For random samplings close to the rotational peak, $R_4 = 3.33$, we calculate the overlap between the wave function obtained under TBRE, $|r\rangle$, and the SU(3)-limit states, $|\mathrm{SU}(3), i\rangle$ ($i = 1, 2, \ldots, d$), and the largest overlap is denoted by $x^2_{\mathrm{SU}(3)}$:

$$x^2_{\mathrm{SU}(3)} = \max_i \langle r|\mathrm{SU}(3), i\rangle^2. \tag{6}$$

The distribution of $x^2_{\mathrm{SU}(3)}$ for the 0^+_1 state are presented (dashed curves in red) in Fig. 16, and one sees two statistical peaks at small and large overlaps, $x^2_{\mathrm{SU}(3)} \approx 0.3$ and $x^2_{\mathrm{SU}(3)} = 1$, respectively. Our numerical results show that the samplings with $x^2_{\mathrm{SU}(3)} \approx 0.3$ corresponds to the conjugate $\overline{\mathrm{SU}(3)}$ limit. Similarly, we present the distribution of $x^2_{\overline{\mathrm{SU}(3)}}$ (dotted curves in green) in Fig. 16, and one also sees two statistical peaks at small and large overlaps,

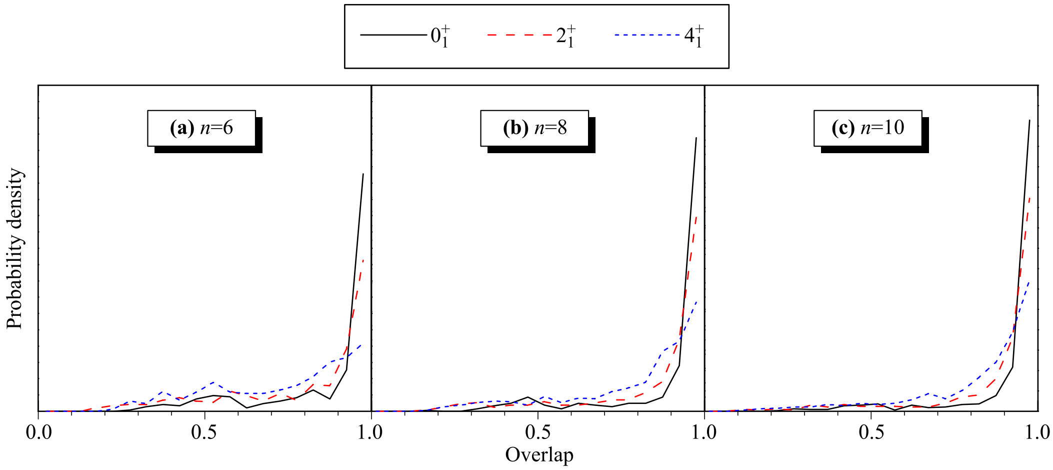

Fig. 15 Distribution of the largest overlap between the wave function obtained under the TBRE and U(5)-limit states for the random samplings of the vibrational peak.

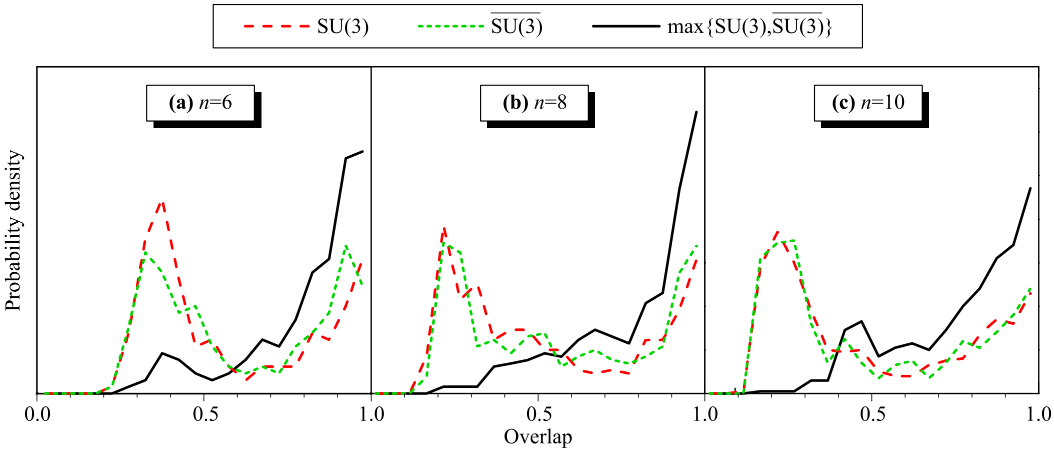

Fig. 16 Distribution of the largest overlap between the wave function obtained under the TBRE and SU(3)/$\overline{\mathrm{SU(3)}}$-limit states for the 0_1^+ state of the random samplings with $R_4 \approx 3.33$.

respectively. For the distribution of $\max(x^2_{\mathrm{SU(3)}}, x^2_{\overline{\mathrm{SU(3)}}})$, there is one pronounced peak at a large value of overlaps.

Now we come to the R_4–R_6 Mallmann plot, exemplified with $n = 6$ and 8, both of which have strong α and γ correlations, and the seniority type $(R_4, R_6) = (1, 1)$. The case of $n = 6$ has a strong β correlation, while the $n = 8$ system has weak β and δ correlations. The largest overlaps between the wave function obtained under TBRE and the dynamical-symmetry-limit states [i.e., U(5), SU(3), $\overline{\mathrm{SU(3)}}$, O(6), and $\overline{\mathrm{O(6)}}$] for the 0_1^+ state are presented in Fig. 17. One sees that most of the random samplings with $R_4 \le 2$ are well explained by the U(5) limit, especially those following the R_6–R_4 linear correlations. Our calculated overlaps for random samples of the strong β, γ, δ correlations demonstrate that they are indeed well represented by the d-boson condensation, as already pointed out in Refs. [58, 59]. In Figs. 17(a) and (b) one sees that the $n = 6$ samplings following the β correlation are well represented by both the U(5) limit and the O(6)/$\overline{\mathrm{O(6)}}$ limit. This phenomenon is easy to understand: Each random sample represented by the $n_d = \tau = 6$ state of the U(5) limit is precisely given by unique $\tau = 6$ state of the O(6) limit (with $\sigma = 6$). Here n_d is the quantum number of the U(5) symmetry, which equals the number of d bosons; σ is the quantum number of the O(6) symmetry, and τ is the quantum number of the O(5) symmetry.

Random samplings following the seniority-like correlation, i.e., $R_6 \approx R_4$, are well explained by the U(5) limit, whose 4_1^+ and 6_1^+ states are well represented by the d-boson condensation, while the 0_1^+ state is well represented by the s-boson dominance.

Finally, we discuss the α correlation, and our results are as below:

1) Random samplings of $R_4 \le 2$ with the α correlation are well represented by the s-boson dominance in the

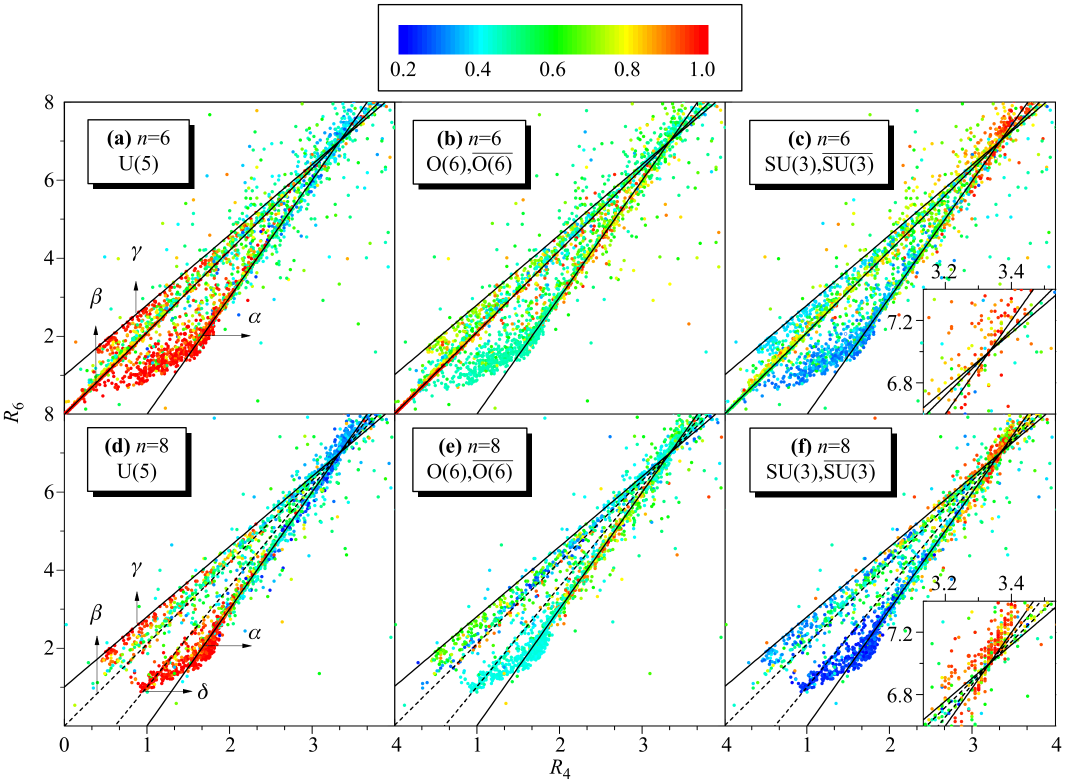

Fig. 17 Mallmann plot (R_6 vs. R_4) for the random ensemble of the *sd* bosons. The color represent the largest overlap between the wave function obtained under the TBRE and dynamical-symmetry-limit states. In the insert of panels (c) and (f) we show the fine structure of the Mallmann plot around the rotational point, $(R_4, R_6) = (3.33, 7)$, by magnifying the scales.

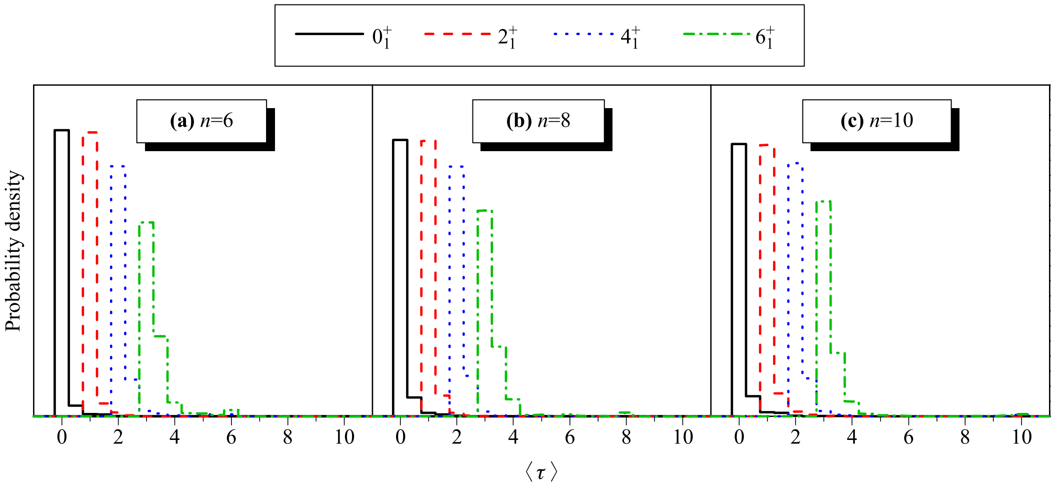

Fig. 18 Expectation value of the O(5)-symmetry quantum number τ for random samplings following the α correlation with $R_4 < 2.9$.

U(5) $\supset$ O(5) limit. In this case the yrast states have $n_d = \tau = \tau_{\min} = 0, 1, 2, 3$ for $I = 0, 2, 4, 6$, respectively. This leads to the correlation $R_6 = 3R_4 - 3$.

2) Random samplings of $2 < R_4 < 2.9$ are no longer good U(5)-symmetric. There exist strong mixing of the U(5), O(6), and $\overline{\text{O(6)}}$ configurations for these random samples. On the other hand, and most random samplings of the α correlation with $2 < R_4 < 2.9$ can be approximately represented by the $\tau = \tau_{\min}$ states of the O(5) symmetry [the U(5), O(6), and $\overline{\text{O(6)}}$ limits share the same O(5) subgroup], as shown in Fig. 18: the expectation value of τ in these yrast states $\approx \tau_{\min}$.

3) For random samplings of $R_4 \approx 3$, the SU(3) and $\overline{\mathrm{SU}(3)}$ symmetry emerges. However, The statistics of these random samplings do not favor dominantly any of the dynamical symmetry limits.

4) Random samplings of $R_4 \approx 3.33$ dominantly follow the SU(3) or $\overline{\mathrm{SU}(3)}$ limit.

It is also interesting to study the α correlation with the consistent-Q Hamiltonian, i.e.,

$$H(\zeta,\chi) = (1-\zeta)\hat{n}_d - \frac{\zeta}{4n}\hat{Q}^{\chi}\cdot\hat{Q}^{\chi}. \tag{7}$$

Both ζ and χ are control parameters: ζ ranges between 0 and 1, and χ ranges between 0 and $-\sqrt{7}/2$, which are well known as the Casten's symmetry triangle [63]. We find that for any given ζ and χ within the symmetry triangle, the excitation energy of the yrast I states ($I = 0, 2, 4, \ldots, 2n$) follows very well the Ejiri formula [64]:

$$E(I_1^+) \propto I(I+X), \quad \text{with } X \geq 1,$$

which yields $R_6 = 3R_4 - 3$.

7 Summary and conclusions

In this paper we present a review of studies of many-body systems with random two-body interactions, with our focus on *sd*-boson systems. In this case not only the spin-zero ground state dominance but also typical collective motions, e.g., seniority-type, vibrational and rotational motion, are generically favored in the presence of random interactions. It is very interesting to see that many random samples of the whole ensemble which were believed to be noisy (namely, $R_4 = E_{4_1^+}/E_{2_1^+}$ far from 1, 2, or 3.33) exhibit very striking and regular patterns. Some of these noisy samples follow the U(5) limit, and this regularity does not depend on whether or not the ground state spin is zero. Random samples at the rotational peak are dominantly approximated by the SU(3) or the $\overline{\mathrm{SU}(3)}$ symmetry in the *sd* IBM. Among various correlations discussed in this review, the β correlation can be interpreted in terms of both the U(5) and O(6) symmetry limit, this duality is easy to understand, as the O(5) group is the canonic subgroup of both U(5) and O(6) group.

The observations of correlations summarized in this paper are interesting, yet a simple and sound understanding of these features is warranted. One would ask whether or not there are other regular correlations which have not been observed hitherto, and furthermore the noisy random samples which are not covered by correlations of this review are to be explained in future.

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