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RESEARCH ARTICLE

Quantifying quantum correlation via quantum coherence

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Resource theory is applied to quantify the quantum correlation of a bipartite state and a computable measure is proposed. Since this measure is based on quantum coherence, we present another possible physical meaning for quantum correlation, i.e., the minimum quantum coherence achieved under local unitary transformations. This measure satisfies the basic requirements for quantifying quantum correlation and coincides with concurrence for pure states. Since no optimization is involved in the final definition, this measure is easy to compute irrespective of the Hilbert space dimension of the bipartite state.

Keywords resource theory, quantum correlation, quantum coherence

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1 Introduction

A quantum system may exhibit many interesting properties, such as nonlocality, which has no analogue in classical systems. In the last two decades, quantum correlation of a bipartite or multipartite system has attracted much attention owing to its potential applications in quantum information processing such as the model of deterministic quantum computation with one qubit for quantum simulations [1–3]. In 2001, Ollivier and Zurek proposed a measure for the quantum correlation of a quantum bipartite state through two classically equivalent expressions of mutual information and called it quantum discord. They found that a separable state may have nonzero quantum correlation [4]. From the viewpoint of resource theory [5–7], the quantum correlation measured by quantum discord could be understood in the following manner: all the quantum states with zero quantum discord are free and all the rest are the resource. In accordance with this, all bipartite local operations, where one subsystem undergoes a unitary transformation and the other is allowed to undergo a unitary transformation together with an ancilla, are considered as free operations.

As a fundamental feather of quantum mechanics, quantum coherence is an easily achievable resource in an experiment, and has become a topic of considerable interest in the recent times. [8–16]. According to the resource theory, two basic requirements have to be satisfied by a good measure of a quantum resource: (i) The measure should present a zero result for a free state and a positive value for a resource; (ii) Any free operation should not create or increase the resource. By imposing two other constraints, monotonicity and convexity, on the measure of quantum coherence, Baumgratz *et al.* proposed a framework for quantifying coherence [17, 18]. In the following years, a number of coherence measures were proposed, such as the relative entropy of coherence, the l_1 norm of coherence, and the coherence of formation [19, 20]. Various properties of quantum coherence, such as the relationship between quantum coherence and quantum correlations [21–24], the freezing phenomenon of coherence [25, 26], and the distillation of coherence [20, 27] were investigated. In these works, a basis is fixed for the quantum states and the operations under consideration. In a recent work, a measure of quantum coherence, called *basis-free* quantum coherence, was proposed that used the quantum relative entropy as the distance measure between the resource state and the set of free states. It was found that this basis-free quantum coherence is exactly equivalent to quantum discord, even in multipartite systems [21]. In this work, we investigate the quantification of quantum correlation from the viewpoint of resource theory and a computable measure is proposed irrespective of the dimension of the bipartite state. It is found that quantum correlation could be interpreted as the minimum quantum coherence achieved under local unitary transformations. This measure satisfies the basic requirements for quantifying quantum correlation and coincides with concurrence of pure state. Furthermore, it is easy to compute because the optimization involved in the initial definition is finally removed.

2 Concept and definition

From the viewpoint of resource theory, the local operations and classical communications classify the quantum states into two categories, separable states as free states and entangled states as resource. Just as we mentioned above, if the free operations are composed of all bipartite local operations with one subsystem undergoing a unitary transformation and the other undergoing a unitary transformation together with an ancilla optionally, the bipartite states are divided into classical states with zero quantum discord and quantum states with positive quantum discord. Both, quantum entanglement and quantum discord, are used to quantify the quantum correlation of bipartite states. They are based on different physical contexts and the former is stronger than the latter. Although these definitions of quantum entanglement and quantum discord have clear physical meaning, they are not easy to compute, especially in case of mixed quantum states in high dimensional systems; an optimization is usually required in their definitions.

In this work, we study the quantum correlation of bipartite states with respect to the local operations in the form of $U_A \otimes U_B$, where U_A and U_B are unitary transformations imposed on the two subsystems A and B, respectively. This consideration is based on the basic fact that if two states, ρ_A and ρ_B, in the two subsystems A and B, are completely independent of each other, then the whole state of the composite system could be described by the tensor product of the two states, i.e., $\rho_{AB} = \rho_A \otimes \rho_B$, and any local unitary operation imposed on the subsystems A and B cannot change the independence of the two states. On the contrary, if the two subsystems are correlated to each other, the whole state of the composite system cannot be described by a tensor product. $\rho_{AB} \neq \rho_A \otimes \rho_B$, irrespective of the basis chosen for the subsystem A or B. In other words, by defining the local operations in the simple form of $U_A \otimes U_B$ as the free operations, all bipartite tensor product states are free states, and the rest are resource states.

We start the discussion on the measure of quantum correlation with the distance between the state ρ_{AB} under consideration and the set of tensor product states, denoted as $\mathcal{T}$ hereafter,

$$d(\rho_{AB}) \equiv \min_{\delta \subset \mathcal{T}} \|\rho_{AB} - \delta\|_F, \tag{1}$$

here we choose Frobenius norm [28], $\|X\|_F \equiv \sqrt{\sum_{ij} |X_{ij}|^2}$, to measure the distance between the two density matrices ρ and δ.

In the following, we assume that the tensor product state $\sigma_{AB} = \sigma_A \otimes \sigma_B$ is the state closest to the state ρ_{AB}, so the distance mentioned above is simplified to

$$d(\rho_{AB}) = \|\rho_{AB} - \sigma_{AB}\|_F. \tag{2}$$

Now we consider a free operation $U_{AB} = U_A \otimes U_B$, which turns the above free state σ_{AB} to a diagonal density matrix, $U_{AB}\sigma_{AB}U_{AB}^\dagger = \Lambda_{AB}$. Since the Frobenius norm is unitarily invariant, we can rewrite the above equation as

$$d(\rho_{AB}) = \|\rho'_{AB} - \Lambda_{AB}\|_F, \tag{3}$$

with $\rho'_{AB} = U_{AB}\rho_{AB}U_{AB}^\dagger$. Since the Frobenius norm is given by the squareroot of the sum of the squares of the absolute values of all elements, every diagonal element of the matrix Λ_{AB} has to be equal to the corresponding diagonal element of the density matrix ρ'_{AB}, so that the above result is minimized for the measure of the quantum correlation of ρ_{AB}. In other words, the quantity in the above Eq. (3) can in fact be regarded as a measure of the quantum coherence of the density matrix ρ'_{AB} because it is determined by the off-diagonal elements only and is independent of the diagonal ones.

In order to work out the explicit expression of the local operation $U_{AB} = U_A \otimes U_B$, which minimizes the above distance of ρ_{AB}, we rewrite the above equation as

$$\begin{aligned} d(\rho_{AB}) &= \sqrt{\|\rho'_{AB}\|_F^2 - \sum_i |(\Lambda_{AB})_{ii}|^2} \\ &= \min_{U_{AB}=U_A\otimes U_B} \sqrt{\|\rho_{AB}\|_F^2 - \sum_i |(\rho'_{AB})_{ii}|^2}. \end{aligned} \tag{4}$$

Here, the Frobenius norm $\|\rho'_{AB}\|_F = \|\rho_{AB}\|_F$ is not affected by any unitary operation. Since local unitary operations imposed on the subsystem A (or B) do not change the reduced density matrix of the other subsystem B (or A), the squared sum of the diagonal elements of ρ'_{AB} reaches its maximum value when both the reduced density matrices $\rho'_{A(B)} = \mathrm{Tr}_{B(A)}\{\rho'_{AB}\}$ become diagonal matrices after the local operation $U_{AB} = U_A \otimes U_B$, because

$$\sum_i |(\rho'_{AB})_{ii}|^2 \leq \|\rho'_A\|_F^2 \cdot \|\rho'_B\|_F^2. \tag{5}$$

Since the Frobenius norm of the two reduced density matrices, ρ'_j ($j = A, B$), remains invariant under the above local operation, $\|\rho'_j\|_F = \|\rho_j\|_F$, the distance of the state ρ_{AB} from the set of tensor product states $\mathcal{T}$ could be finally expressed as

$$d(\rho_{AB}) = \sqrt{\|\rho_{AB}\|_F^2 - \|\rho_A\|_F^2 \cdot \|\rho_B\|_F^2}. \tag{6}$$

Now, we introduce a monotonically increasing function of the distance $d(\rho_{AB})$ to measure the quantum correlation of a given bipartite state ρ_{AB}, which is

$$\mathcal{C}(\rho_{AB}) = \sqrt{2 - 2\sqrt{1 - d^2(\rho_{AB})}}. \tag{7}$$

We have two reasons to choose such a function. First, as a monotonically increasing function of the distance $d(\rho_{AB})$, it is confined to the range $[0, 1]$, and has a one-to-one correspondence with distance. The above process for optimizing the distance between the state ρ_{AB} and the set of tensor product states is also valid for optimizing the current measure of quantum correlation. It will be shown that the quantum correlation defined in this way is equivalent to the *basis-free* quantum coherence, i.e., the minimum quantum coherence achieved under local unitary transformations. Second, the quantum correlation defined in this way satisfies all the requirements for a good measure of quantum correlation and coincides with quantum entanglement for pure states. More details are presented in the next section.

As an alternative, we can use the eigenvalues of the density matrix of ρ_{AB}, referred as λ_i, and the eigenvalues of the two reduced density matrices of ρ_A and ρ_B, referred as μ_m and ν_n, respectively, to express the above quantum correlation, which is represented as below.

$$\mathcal{C}(\rho_{AB}) = \sqrt{2 - 2\sqrt{1 - \sum_i \lambda_i^2 + \sum_m \mu_m^2 \sum_n \nu_n^2}}. \tag{8}$$

3 Properties of the current measure of quantum correlation

It is obvious that the quantum correlation defined above could be easily computed based on Eq. (7) or (8), where we only need to work out the Frobenius norms of the quantum state ρ_{AB} and its two reduced density matrices, or their eigenvalues; besides, no optimization is involved in the calculation. In this section, we will discuss some properties of the current quantum correlation.

3.1 As a measure of quantum correlation

In general, three basic conditions have to be satisfied for A good measure of bipartite quantum correlation [29]: (A1) It should generate a zero result for tensor product states, and a positive result for entangled states; (A2) The measuring result should be invariant under local unitary transformations on each one of the two subsystems; (A3) Quantum correlation should coincide with quantum entanglement for pure states. It is obvious that the first condition, (A1), is satisfied by the current measure, because all the tensor product states are free in this case, and the entangled states belong to the resource. Since the current measure of quantum correlation could be considered as the *basis-free* quantum coherence, the second basic condition, (A2), is also satisfied. We prove the third condition, (A3), in the following part of this subsection.

For a given arbitrary bipartite pure state $|\psi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i_A j_B\rangle$, we first work out its Schmidt decomposition $|\psi\rangle_{AB} = \sum_m \sqrt{\mu_m} |m_A m_B\rangle$, with μ_m As the eigenvalues of the two reduced density matrices of $|\psi\rangle_{AB}$. According to the measure in Eq. (8), the quantum correlation of the pure state $|\psi\rangle_{AB}$ is

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2 - 2\sum_m \mu_m^2}. \tag{9}$$

At the same time, the quantum entanglement of the above pure state, measured by Wootters concurrence [30] and its generalized version, I-concurrence [31], is

$$\mathcal{E}(|\psi\rangle_{AB}) = \sqrt{2 - 2\sum_m \mu_m^2}. \tag{10}$$

We observe that the quantum correlation defined here is in accord with the quantum entanglement for the bipartite pure states, measured by I-concurrence, and hence the third condition, (A3), is satisfied.

3.2 As a measure of quantum coherence

For the basis where both the reduced density matrices of ρ_{AB} are diagonal matrices, the distance defined in Eq. (1), as well as the quantum correlation defined in Eq. (7), could both be considered as a measure of quantum coherence in this quantum state because they are determined by the off-diagonal elements of the density matrix in this particular basis, and are independent of the diagonal ones. It provides another possible physical meaning to quantum correlation, i.e., the minimum quantum coherence achieved under local unitary transformations. In fact, the first condition, (A1), mentioned above for the measure of quantum correlation is also required to be fulfilled by the measure of quantum coherence. As the second requirement for the measure of quantum coherence, free operation can neither create resource from a free state, nor increase the resource contained in a quantum state. In the current case, the free operations are composed of local operations formulated as the tensor product of the two local unitary transformations implying that this requirement is satisfied directly. The convexity, i.e., the property of nonincreasing under mixing of quantum states, is satisfied by the measures of quantum coherence proposed in the recent work [17]. However, it is violated by the measure proposed here because the set of free states is not convex and hence the mixture of the two free states might contain a resource. Furthermore, the current *basis-free* measure is equal to the minimum quantum coherence achieved under local unitary transformations, which is different from the measures proposed in some of the previous studies [17, 19, 20], where a fixed basis is required. We affirm that it is a reasonable result because the operation of mixing is not free according to the current definition of free operation where it can be interpreted as "rotation" of the basis, rather than the mixture of two or more states.

3.3 Relationship between quantum coherence, quantum discord, and quantum entanglement

Although quantum entanglement, quantum discord, and quantum coherence could be considered as different types of quantum resources, some connections are found between them. For example, Chitambar and Hsieh unified the resource theories of entanglement and coherence by studying their combined behavior in the framework of local incoherent operations and classical communication (LIOCC) and concluded that entanglement and coherence are indeed closely linked to each other [32]. The quantum coherence contained in a system could be converted to (distillable) entanglement between the system and an initially incoherent ancilla by means of incoherent operations, which makes it possible to quantify coherence in terms of entanglement [33]. In recent past, it was found that the quantum discord created under multipartite incoherent operations is bound by the amount of quantum coherence consumed in the process [34]. By definition, quantum entanglement is more robust than quantum discord. However, many separable bipartite states with zero quantum entanglement could still not be regarded as "classical", because they have nonvanishing quantum discord [35, 36]. At the same time, the density matrix of a "classical" state with vanishing quantum discord is a block diagonal matrix in a particular basis. However, for a bipartite state with zero quantum coherence, based on the current measure discussed above, its whole density matrix has to be a diagonal matrix in a particular basis, which could also be considered as a special type of block diagonal matrix. If we call the states with vanishing quantum discord "type-I classical", and the states with vanishing quantum coherence, based on the current measure, "type-II classical", we have the following conclusion: a "type-I classical" state does not need to be "type-II classical"; however, a "type-II classical" state has to be "type-I classical". For example, a two-qubit mixed state $\rho_{\mathcal{AS}} = \frac{1}{2}|0\rangle_{\mathcal{A}}\langle 0| \otimes |0\rangle_{\mathcal{S}}\langle 0| + \frac{1}{2}|1\rangle_{\mathcal{A}}\langle 1| \otimes |+\rangle_{\mathcal{S}}\langle +|$, with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, is a "type-I classical" state, but not "type-II classical", with $\mathcal{C}(\rho_{\mathcal{AS}}) = \frac{1}{2}$, measured in the aforementioned method. All the bipartite tensor product states are both "type-I classical" and "type-II classical". In other words, the quantum correlation defined here is weaker than quantum discord and quantum entanglement is the strongest among these three.

3.4 Other properties

According to the definition proposed by Ollivier and Zurek [4], which one plays the role of the "apparatus" and which one plays the role of the "system", between the two subsystems A and B, has to be determined before evaluating the quantum discord of a bipartite state. In other words, the evaluation of the quantum discord of a bipartite state is asymmetric with respect to the two subsystems. We can even find many examples where a "type-I classical" quantum state with subsystem A chosen as the "apparatus" might contain nonzero quantum discord on choosing subsystem B as the "apparatus". On the contrary, the evaluation of quantum entanglement is symmetric with respect to the permutation among the subsystems and this property is also satisfied by the measure proposed here. In other words, the permutation between the two parties of a bipartite system would not change its quantum correlation or quantum coherence, if the current measure is applied. Another merit of the current measure is that it is easily computable. Since no optimization is involved in the current measure of quantum correlation [Refer to Eqs. (7) and (8)] and the quantum correlation of a given bipartite quantum state relies only on the Frobenius norm of its density matrix and the two reduced density matrices, it can be easily calculated irrespective of the Hilbert space dimension of the quantum state.

4 conclusions

In a bipartite system, if we regard local unitary operations as free, then all tensor produced states are free states and all the rest are resource. From this viewpoint, we connected the two concepts, quantum correlation and quantum coherence. It is found that quantum correlation could be interpreted as the minimum quantum coherence achieved under local unitary transformations. Based on this idea, we propose a measure of quantum bipartite correlation via quantum coherence. We confirmed that the quantum correlation contained in a bipartite state is equal to the quantum coherence of this state in the basis where both of its reduced density matrices are diagonal. Mathematically, the current measure of quantum correlation depends only on the Frobenius norms of the density matrix of the state and its two reduced density matrices, or alternatively, their eigenvalues. This measure satisfies the basic requirements for quantifying quantum correlation and quantum coherence, and coincides with concurrence for pure states. Since no optimization is involved in the definition, this measure is easy to compute, irrespective of the Hilbert space dimension of the bipartite state. As is well known, the concepts of quantum entanglement, quantum correlation, and quantum coherence have been generalized from the bipartite systems to multipartite systems [21, 37–41], and it will be an interesting topic to quantify quantum correlation via quantum coherence in multipartite quantum systems.

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