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RESEARCH ARTICLE

Geometric field theory and weak Euler–Lagrange equation for classical relativistic particle-field systems

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A manifestly covariant, or geometric, field theory of relativistic classical particle-field systems is developed. The connection between the space-time symmetry and energy-momentum conservation laws of the system is established geometrically without splitting the space and time coordinates; i.e., space-time is treated as one entity without choosing a coordinate system. To achieve this goal, we need to overcome two difficulties. The first difficulty arises from the fact that the particles and the field reside on different manifolds. As a result, the geometric Lagrangian density of the system is a function of the 4-potential of the electromagnetic fields and also a functional of the particles' world lines. The other difficulty associated with the geometric setting results from the mass-shell constraint. The standard Euler–Lagrange (EL) equation for a particle is generalized into the geometric EL equation when the mass-shell constraint is imposed. For the particle-field system, the geometric EL equation is further generalized into a weak geometric EL equation for particles. With the EL equation for the field and the geometric weak EL equation for particles, the symmetries and conservation laws can be established geometrically. A geometric expression for the particle energy-momentum tensor is derived for the first time, which recovers the non-geometric form in the literature for a chosen coordinate system.

Keywords relativistic particle-field system, different manifolds, mass-shell constraint, geometric weak Euler–Lagrange equation, symmetry, conservation laws

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1 Introduction

Energy-momentum conservation is a fundamental law of physics. It applies to both quantum systems and classical systems. From the field-theoretical viewpoint, energy-momentum conservation is fundamentally due to the space-time symmetry of the Lagrangian (or Lagrangian density) that the system admits [1–3]. For example, the Lagrangian density of a pure electromagnetic field is

$$\mathcal{L}_F = \frac{1}{8\pi}\left[\left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\varphi\right)^2 - (\nabla\times\boldsymbol{A})^2\right]. \quad (1)$$

The energy and momentum conservation laws of the system,

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}\right) + \nabla\cdot\left(\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right) = 0, \quad (2)$$

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c}\right) + \nabla\cdot\left(\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}\boldsymbol{I} - \frac{\boldsymbol{E}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{B}}{4\pi}\right) = 0, \quad (3)$$

can be derived from the Euler–Lagrange (EL) equations,

$$\frac{\partial \mathcal{L}_F}{\partial \varphi} - \frac{D}{D\boldsymbol{x}}\cdot\frac{\partial \mathcal{L}_F}{\partial(\nabla\varphi)} = 0, \quad (4)$$

$$\frac{\partial \mathcal{L}_F}{\partial \boldsymbol{A}} - \frac{D}{D\boldsymbol{x}}\cdot\frac{\partial \mathcal{L}_F}{\partial \nabla\boldsymbol{A}} - \frac{D}{Dt}\frac{\partial \mathcal{L}_F}{\partial \boldsymbol{A}_{,t}} = 0, \quad (5)$$

and the symmetry conditions,

$$\frac{\partial \mathcal{L}_F}{\partial t} = 0, \quad \frac{\partial \mathcal{L}_F}{\partial \boldsymbol{x}} = 0. \quad (6)$$

Here $D/D\boldsymbol{x}$ and D/Dt denote the partial derivatives

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with respect to $\boldsymbol{x}$ and t, respectively, when they operate on a field defined on the space-time.

Geometrically, the spatial and temporal symmetries of the Lagrangian density are components of the space-time symmetry,

$$\frac{\partial \mathcal{L}_F}{\partial \chi^\mu}=0, \quad (\mu=0,1,2,3), \tag{7}$$

where χ^μ is an arbitrary world point in four-dimensional Minkowski space-time; i.e., $\chi^0 \equiv t$ and $\chi^i \equiv x^i$. Further, the Lagrangian density in Eq. (7) can be equivalently written as

$$\mathcal{L}_F=-\frac{1}{4\pi}\partial^{[\mu}A^{\nu]}\partial_\mu A_\nu, \tag{8}$$

where the 4-potential $A_\nu=(\varphi,-\boldsymbol{A})$ is defined on the space-time, and the symbol $[\mu\,\nu]$ denotes the antisymmetrization of the indexes of μ and ν [4]. In this paper, we assume that the space-time is endowed with a Lorentzian metric and the signature of the metric is $(+---)$. Equation (7) is a manifestly covariant form of Eq. (6). In this paper, the phrase "manifestly covariant" will be replaced by "geometric" to indicate that covariance is the intrinsic coordinate-independent property of the physical system [5].

Similarly, the EL equations, (4) and (5), or the Maxwell equations can be written in geometric form as

$$\frac{\partial \mathcal{L}_F}{\partial A_\mu}-\frac{D}{D\chi^\nu}\left[\frac{\partial \mathcal{L}_F}{\partial\left(\partial_\nu A_\mu\right)}\right]=0. \tag{9}$$

Here, the operator $D/D\chi^\nu$ denotes the partial derivative with respect to χ^ν for a fixed $\chi^\mu(\mu\neq\nu)$ when it is operated on a field in space-time. The geometric energy-momentum conservation law is

$$\partial_\nu T_F^{\mu\nu}=0, \quad (\mu,\nu=0,1,2,3), \tag{10}$$

where $T_F^{\mu\nu}$ is the energy-momentum tensor of the electromagnetic fields and can be written in an explicit form as

$$T_F^{\mu\nu}=\frac{1}{4\pi}\left(-F^{\mu\sigma}F_\sigma^\nu+\frac{1}{4\pi}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right), \tag{11}$$

where F is the electromagnetic tensor, and η is the Lorentzian metric. To briefly summarize, Eq. (8) is the geometric form of Eq. (1), and Eq. (10) is that of Eqs. (2) and (3). Similarly, Eq. (9) is the geometric form of Eqs. (4) and (5), and Eq. (7) is that of Eq. (6). These, of course, are well-known [6].

Classical particle-field systems, where many charged particles evolve under the electromagnetic field generated self-consistently by the particles, are often encountered in astrophysics, accelerator physics, and plasma physics [7–12]. For these systems, the relations between symmetries and conservation laws have been established only recently by Qin *et al.* [13]. It was found that the standard EL equation for particles no longer holds, because the dynamics of the particles and fields are defined on different manifolds having different dimensions. The electromagnetic fields are defined on the space-time domain, whereas the particle trajectories as fields are defined only on the time axis. For particles, a weak EL equation is established to replace the standard EL equation. It was discovered that the weak EL equation can also link symmetries with conservation laws as in standard field theory. This field theory is non-relativistic, but it can easily be extended to relativistic cases, which will be shown in Section 2. However, this approach is based on the split form of space and time. In other words, it is not geometric.

In this paper, we will geometrically reformulate the field theory of classical particle-field systems established in Ref. [13]. A geometric weak EL equation will be derived, and the energy-momentum conservation will be geometrically derived from the space-time symmetry. To achieve this goal, we need to overcome two difficulties. The first difficulty arises from the fact that the particles and the field reside on different manifolds, as noted in Ref. [13]. In the geometric setting, this difference is more prominent. The particles' dynamics are characterized by world lines on the space-time. They are defined on $\mathbb{R}^1$ as a field valued in the space-time, and the domain of the particle fields can be the proper time or any other parameterization of the world lines. The world line of a particle is uniquely defined, regardless of how it is parameterized. The electromagnetic fields, on the other hand, are defined on the space-time. The geometric Lagrangian density of the system will be a function of the 4-potential of the electromagnetic fields and also a functional of the particles' world lines. This is qualitatively different from the standard field theory, where the Lagrangian density is a local function of the fields. The other difficulty associated with the geometric setting results from the mass-shell constraint, which exists even for the geometric variational principle for a single particle [14]. The standard EL equation will be generalized into a geometric EL equation when the mass-shell constraint is imposed. For the particle-field system, the geometric EL equation is further generalized into a weak geometric EL equation.

We emphasize that it is of significant theoretical and practical value to put the physical laws governing the classical particle-field system into geometric forms. The compact geometric forms, or manifestly Lorentz-invariant forms, are especially suitable for analysis of the statistical mechanics of relativistic plasmas [8, 15–

23]. They also serve as the theoretical foundations for developing Lorentz covariant algorithms [24] for numerical simulations. In quantum field theory, both geometric and non-geometric forms are used. The path integral approach is based on the geometric form of the Lagrangian density, whereas the canonical quantization method is not manifestly covariant because the space and time dimensions are split. The path integral form of quantum electrodynamics has been adopted recently to develop a kinetic theory of magnetized plasmas when both quantum and relativistic effects are important [25].

We note that energy-momentum conservation laws are well-known results and can be found, for example, in Ref. [6]. However, in the literature, these conservation laws are not derived from the underpinning symmetries. Often one can establish a conservation law without knowing the underpinning symmetry. As in the case studied here, establishing the connection between a symmetry and a conservation law can sometimes be a difficult but rewarding task.

This paper is organized as follows. In Section 2, we will discuss how to extend the work of Qin *et al.* to the relativistic case in a non-geometric way. The geometric Lagrangian of a single particle and the corresponding geometric EL equation are discussed in Section 3. In Section 4, the geometric Lagrangian density of particle-field systems and its properties are discussed. The geometric weak EL equation and the link it provides between the conservation laws and space-time symmetries are presented in Section 5.

2 Non-geometric field theory and weak Euler–Lagrange equation of relativistic particle-field systems

The classical relativistic particle-field system in flat space is governed by the following Newton–Maxwell equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_{sp}m_s\dot{\boldsymbol{X}}_{sp}) = q_s\left(\boldsymbol{E}+\frac{1}{c}\dot{\boldsymbol{X}}_{sp}\times\boldsymbol{B}\right), \tag{12}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_{s,p}q_s\delta(\boldsymbol{x}-\boldsymbol{X}_{sp}), \tag{13}$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_{s,p}q_s\dot{\boldsymbol{X}}_{sp}\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{14}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \tag{15}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{16}$$

where $\gamma_{sp}\equiv 1/\sqrt{1-\dot{\boldsymbol{X}}_{sp}^2/c^2}$ is the relativistic factor of the p-th particle of species s, m_s is the mass of any particle of species s, $\boldsymbol{X}_{sp}$ is the trajectory, $\boldsymbol{x}$ is a point in configuration space, and $\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})$ is the Dirac delta function on the 3D configuration space. The basic Newton–Maxwell equations, (12)–(14), can be equivalently written as the Vlasov–Maxwell equations

$$\frac{\partial F_s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial F_s}{\partial\boldsymbol{x}}+q_s\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial F_s}{\partial\boldsymbol{p}} = 0, \tag{17}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s\int F_s\mathrm{d}^3\boldsymbol{p}, \tag{18}$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_s q_s\int F_s\boldsymbol{v}\mathrm{d}^3\boldsymbol{p}+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} \tag{19}$$

by defining the Klimontovich distribution function in phase space as

$$F_s = \sum_p\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})\delta\left(\boldsymbol{p}-\boldsymbol{p}_{sp}\right). \tag{20}$$

Here, $\boldsymbol{p}_{sp}=\gamma_{sp}m_s\dot{\boldsymbol{X}}_{sp}$ is the relativistic momentum of sp-particle. The electric field $\boldsymbol{E}(t,\boldsymbol{x})$ and magnetic field $\boldsymbol{B}(t,\boldsymbol{x})$ are functions of space and time. For this system, the action $\mathcal{A}$ is

$$\mathcal{A}[\varphi,\boldsymbol{A},\boldsymbol{X}_{sp}] = \int\mathcal{L}_{PF}\mathrm{d}t\mathrm{d}^3\boldsymbol{x}, \tag{21}$$

where

$$\begin{aligned}\mathcal{L}_{PF} = &-\sum_{s,p}\gamma_{sp}^{-1}m_sc^2\delta\left(\boldsymbol{x}-\boldsymbol{X}_{sp}\right)\\ &+\frac{q_s}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_{sp}\delta\left(\boldsymbol{x}-\boldsymbol{X}_{sp}\right)-q_s\varphi\delta\left(\boldsymbol{x}-\boldsymbol{X}_{sp}\right)\\ &+\frac{1}{8\pi}\left[\left(-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}-\nabla\varphi\right)^2-(\nabla\times\boldsymbol{A})^2\right]\end{aligned} \tag{22}$$

is the Lagrangian density of this system [6]. The variation of $\mathcal{A}$ induced by $\delta\boldsymbol{X}_{sp}$, $\delta\boldsymbol{A}$, and $\delta\varphi$ is

$$\begin{aligned}\delta\mathcal{A} = &\sum_{s,p}\int\mathrm{d}t\delta\boldsymbol{X}_{sp}\cdot\int E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})\mathrm{d}^3\boldsymbol{x}\\ &+\int E_\varphi(\mathcal{L}_{PF})\delta\varphi\mathrm{d}t\mathrm{d}^3\boldsymbol{x}\\ &+\int E_{\boldsymbol{A}}(\mathcal{L}_{PF})\cdot\delta\boldsymbol{A}\mathrm{d}t\mathrm{d}^3\boldsymbol{x},\end{aligned} \tag{23}$$

where

$$E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF}) \equiv \frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{X}_{sp}}-\frac{D}{Dt}\cdot\frac{\partial\mathcal{L}_{PF}}{\partial\dot{\boldsymbol{X}}_{sp}}, \tag{24}$$

$$E_\varphi(\mathcal{L}_{PF}) \equiv \frac{\partial\mathcal{L}_{PF}}{\partial\varphi}-\frac{D}{D\boldsymbol{x}}\cdot\frac{\partial\mathcal{L}_{PF}}{\partial\nabla\varphi}, \tag{25}$$

$$E_{\boldsymbol{A}}(\mathcal{L}_{PF}) \equiv \frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{A}}-\frac{D}{D\boldsymbol{x}}\cdot\frac{\partial\mathcal{L}_{PF}}{\partial\nabla\boldsymbol{A}}-\frac{D}{Dt}\frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{A}_{,t}}. \tag{26}$$

For $\delta\mathcal{A}=0$, we have

$$\int E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})\mathrm{d}^3\boldsymbol{x}=0, \tag{27}$$

$$E_{\varphi}(\mathcal{L}_{PF})=0, \tag{28}$$

$$E_{\boldsymbol{A}}(\mathcal{L}_{PF})=0 \tag{29}$$

owing to the arbitrariness of $\delta\boldsymbol{X}_{sp}$, $\delta\varphi$, and $\delta\boldsymbol{A}$ in Eq. (23). Equation (27) will be called the submanifold EL equation because it is defined only on the time axis after integration over the spatial variable. Moreover, substituting Eq. (22) into Eqs. (24)–(29), we will recover Eqs. (12)–(16) by defining $\boldsymbol{E}=-\partial\boldsymbol{A}/\partial t-\nabla\varphi$ and $\boldsymbol{B}=\nabla\times\boldsymbol{A}$.

For Eq. (27), in general, we expect that $E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})\neq 0$, although the integral of this term vanishes. We can derive an explicit expression for $E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})$ as

$$E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})\equiv\frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{X}_{sp}}-\frac{D}{Dt}\left(\frac{\partial\mathcal{L}_{PF}}{\partial\dot{\boldsymbol{X}}_{sp}}\right)=\frac{\partial}{\partial\boldsymbol{x}}\cdot(\dot{\boldsymbol{X}}_{sp}\boldsymbol{P}_{sp})+\frac{\partial}{\partial\boldsymbol{x}}(H_{sp}-\boldsymbol{P}_{sp}\cdot\dot{\boldsymbol{X}}_{sp}), \tag{30}$$

where

$$H_{sp}=\left(\gamma_{sp}m_sc^2+q_s\varphi\right)\delta(\boldsymbol{x}-\boldsymbol{X}_{sp}), \tag{31}$$

$$\boldsymbol{P}_{sp}=\left(\gamma_{sp}m_s\dot{\boldsymbol{X}}_{sp}+\frac{q_s}{c}\boldsymbol{A}\right)\delta(\boldsymbol{x}-\boldsymbol{X}_{sp}). \tag{32}$$

Equation (30) is called the weak EL equation [13], where the qualifier "weak" indicates that only the spatial integral of $E_{\boldsymbol{X}_{sp}}(\mathcal{L}_{PF})$ is zero [see Eq. (27)].

Next, we define the symmetry of the action $\mathcal{A}[\varphi,\boldsymbol{A},\boldsymbol{X}_{sp}]$ to be a group of transformation

$$(t,\boldsymbol{x},\varphi,\boldsymbol{A},\boldsymbol{X}_{sp})\longmapsto(\tilde{t},\tilde{\boldsymbol{x}},\tilde{\varphi},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{X}}_{sp}), \tag{33}$$

such that

$$\int\mathcal{L}_{PF}\left(t,\boldsymbol{x},\varphi,\boldsymbol{A},\boldsymbol{X}_{sp}\right)\mathrm{d}t\mathrm{d}^3\boldsymbol{x}=\int\tilde{\mathcal{L}}_{PF}(\tilde{t},\tilde{\boldsymbol{x}},\tilde{\varphi},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{X}}_{sp})\mathrm{d}\tilde{t}\mathrm{d}^3\tilde{\boldsymbol{x}}. \tag{34}$$

For our Lagrangian density [see Eq. (22)], if the group transformation is the time translation

$$(\tilde{t},\tilde{\boldsymbol{x}},\tilde{\varphi},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{X}}_{sp})=(t+\epsilon,\boldsymbol{x},\varphi,\boldsymbol{A},\boldsymbol{X}_{sp}),\quad\epsilon\in\mathbb{R}, \tag{35}$$

condition (34) will be satisfied because

$$\frac{\partial\mathcal{L}_{PF}}{\partial t}=0. \tag{36}$$

Using the weak EL equation in (30) for particles, the EL equations for fields [see Eqs. (25), (26), (28), and (29)], we obtain the energy conservation law

$$\frac{\partial}{\partial t}\left[\sum_{s,p}\gamma_{sp}m_sc^2\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})+\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}\right]+\nabla\cdot\left[\sum_{s,p}\gamma_{sp}m_sc^2\dot{\boldsymbol{X}}_{sp}\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})+\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right]=0. \tag{37}$$

Equation (34) holds for spatial translation as well:

$$\left(\tilde{t},\tilde{\boldsymbol{x}},\tilde{\varphi},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{X}}_{sp}\right)=\left(t,\boldsymbol{x}+\epsilon\boldsymbol{X},\varphi,\boldsymbol{A},\boldsymbol{X}_{sp}+\epsilon\boldsymbol{X}\right), \tag{38}$$

where $\epsilon\in\mathbb{R}$, because $\mathcal{L}_{PF}$ satisfies

$$\frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{x}}+\sum_{s,p}\frac{\partial\mathcal{L}_{PF}}{\partial\boldsymbol{X}_{sp}}=0. \tag{39}$$

Consequently, the momentum conservation law resulting from this symmetry can be written as

$$\frac{\partial}{\partial t}\left[\sum_{s,p}\gamma_{sp}m_s\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})\dot{\boldsymbol{X}}_{sp}+\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c}\right]+\nabla\cdot\left[\sum_{s,p}\gamma_{sp}m_s\delta(\boldsymbol{x}-\boldsymbol{X}_{sp})\dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}+\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}\boldsymbol{I}-\frac{\boldsymbol{E}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{B}}{4\pi}\right]. \tag{40}$$

Details of the derivation can be found in Ref. [13]. However, in relativistic cases, this split form of space and time is not elegant. Space and time should be treated as one entity in the most fundamental and geometric approach. In the following sections, we will explore a geometric way to establish the relations between symmetries and conservation laws.

Now, let us discuss the necessity of the weak EL equation introduced in this study. We note that one can construct a velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ on space-time using the particles' trajectories $\boldsymbol{X}_{sp}(t)$ as $\boldsymbol{v}(\boldsymbol{x},t)=\Sigma_{s,p}\boldsymbol{v}_{sp}(\boldsymbol{x},t)\delta(\boldsymbol{x}-\boldsymbol{X}_{sp}(t))$. However, in the variation procedure, we cannot treat the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ defined this way on space-time as an independent field that can be varied freely by an arbitrary $\delta\boldsymbol{v}(\boldsymbol{x},t)$. The quantities that can be independently varied in the variation procedure are the 4-potential $A(\boldsymbol{x},t)$ and the particles' trajectories $\boldsymbol{X}_{sp}(t)$. Obviously, $A(\boldsymbol{x},t)$ and $\boldsymbol{X}_{sp}(t)$ are defined on different domains. Thus, we need to introduce the weak EL equation to overcome this difficulty.

That said, there is indeed an alternative approach, if we insist on varying the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$, instead of

$\boldsymbol{X}_{sp}(t)$. In this case, the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ cannot be varied arbitrarily. The variation $\delta\boldsymbol{v}(\boldsymbol{x},t)$ needs to satisfy certain constraints. A systematic approach to this type of constrained variation has been developed in the context of Euler–Poincare reduction [26–28]. The dynamic equation resulting from the constrained variation will be very different from the standard EL equation and will complicate the symmetry analysis. In this study, we will not pursue this route. Note that the standard EL equation needs to be amended in either approach.

3 Geometric Lagrangian and geometric Euler–Lagrange equation for a single particle

For a classical particle, the action can be expressed as

$$\mathcal{A}=\int_{t_1}^{t_2} L(\boldsymbol{x},\dot{\boldsymbol{x}},t)\mathrm{d}t, \tag{41}$$

where $L(\boldsymbol{x},\dot{\boldsymbol{x}},t)$ is the Lagrangian. Applying the principle of least action yields

$$\begin{aligned}0=\delta\mathcal{A}&=\int \delta L(\boldsymbol{x},\dot{\boldsymbol{x}},t)\mathrm{d}t\\&=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial \boldsymbol{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right)\right]\cdot\delta\boldsymbol{x}\mathrm{d}t+\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\cdot\delta\boldsymbol{x}\right)_{t_1}^{t_2}.\end{aligned} \tag{42}$$

Because $\delta\boldsymbol{x}(t_1)=\delta\boldsymbol{x}(t_2)=0$ and because of the arbitrariness of $\delta\boldsymbol{x}$, we obtain the EL equation for the dynamics of the particle as

$$\frac{\partial L}{\partial \boldsymbol{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right)=0. \tag{43}$$

For relativistic particles, the variational principle is more complicated. In special relativity, the Lagrangian should be Lorentz-invariant. For this reason, the integral variable of the action should be the proper time τ, and the action is

$$\mathcal{A}=\int_{p_1}^{p_2}\widetilde{L}(\boldsymbol{\chi},\dot{\chi},\tau)\mathrm{d}\tau, \tag{44}$$

where $\boldsymbol{\chi}$ and $\dot{\chi}\equiv d\boldsymbol{\chi}/\mathrm{d}\tau$ are the world line and 4-velocity of the particle, respectively. The Lagrangian $\widetilde{L}$ is manifestly covariant [14] and is called the geometric Lagrangian in this study. The integral in Eq. (44) is along any possible world lines passing p_1 and p_2 in space-time. If we parameterize the world line by the proper time τ, then Eq. (44) can be written as

$$\mathcal{A}=\int_{\tau_1}^{\tau_2}\widetilde{L}(\boldsymbol{\chi},\dot{\chi},\tau)\mathrm{d}\tau, \tag{45}$$

where τ_1 is the proper time at the beginning, and τ_2 is the proper time at the end. We can choose the parameter τ_1 to be the same for any possible world line. However, the parameter τ_2 is generally different because different world lines passing the given space-time point p_2 have different lengths, which means that $\delta(\mathrm{d}\tau)\neq 0$. This is different from the non-relativistic case, which implicitly assumes $\delta(\mathrm{d}t)=0$. The Hamiltonian principle is

$$0=\delta\mathcal{A}=\int_{\tau_1}^{\tau_2}\delta\widetilde{L}(\boldsymbol{\chi},\dot{\chi},\tau)\mathrm{d}\tau+\int_{\tau_1}^{\tau_2}\widetilde{L}(\boldsymbol{\chi},\dot{\chi},\tau)\delta(\mathrm{d}\tau), \tag{46}$$

where

$$\begin{aligned}\delta(\mathrm{d}\tau)&=\frac{1}{c^2}\delta\left(\frac{\mathrm{d}\chi^\mu\mathrm{d}\chi_\mu}{\mathrm{d}\tau}\right)=\frac{1}{c^2}\delta\left[\mathrm{d}\chi^\mu\dot{\chi}_\mu\right]\\&=\frac{1}{c^2}\delta(\mathrm{d}\chi^\mu)\dot{\chi}_\mu+\frac{1}{c^2}\mathrm{d}\chi^\mu\delta(\dot{\chi}_\mu)\\&=\frac{1}{c^2}\delta(\mathrm{d}\chi^\mu)\dot{\chi}_\mu+\frac{1}{c^2}\dot{\chi}^\mu\delta(\dot{\chi}_\mu)\mathrm{d}\tau.\end{aligned} \tag{47}$$

Here, χ^μ and $\dot{\chi}^\mu$ $(\mu=0,1,2,3)$ are the components of the world line and 4-velocity of the particle, respectively. From the mass-shell constraint,

$$P^\mu P_\mu=m_0^2c^2 \text{ or } \dot{\chi}^\mu\dot{\chi}_\mu=c^2, \tag{48}$$

where $P^\mu=m_0\dot{\chi}^\mu$ represents the components of the 4-momentum, and m_0 is the rest mass, we have

$$0=\delta(c^2)=\delta(\dot{\chi}^\mu\dot{\chi}_\mu)=2\dot{\chi}^\mu\delta(\dot{\chi}_\mu), \tag{49}$$

which means that Eq. (47) can be reduced to

$$\delta(\mathrm{d}\tau)=\frac{1}{c^2}\dot{\chi}_\mu\delta(\mathrm{d}\chi^\mu). \tag{50}$$

The second term on the right-hand side of Eq. (46) is

$$\begin{aligned}\frac{1}{c^2}\int_{\tau_1}^{\tau_2}\widetilde{L}\dot{\chi}_\mu\delta\left(\mathrm{d}\chi^\mu\right)&=\frac{1}{c^2}\int_{\tau_1}^{\tau_2}\widetilde{L}\dot{\chi}_\mu\frac{d\left(\delta\chi^\mu\right)}{\mathrm{d}\tau}\mathrm{d}\tau=-\frac{1}{c^2}\int_{\tau_1}^{\tau_2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\widetilde{L}\dot{\chi}_\mu\right]\left(\delta\chi^\mu\right)\mathrm{d}\tau+\left[\widetilde{L}\dot{\chi}_\mu\left(\delta\chi^\mu\right)\right]_{\tau_1}^{\tau_2}\\&=-\frac{1}{c^2}\int_{\tau_1}^{\tau_2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\widetilde{L}\dot{\chi}_\mu\right]\delta\chi^\mu\mathrm{d}\tau.\end{aligned} \tag{51}$$

The first term on the right-hand side of Eq. (46) is

$$\int_{\tau_1}^{\tau_2}\left(\frac{\partial\widetilde{L}}{\partial\chi^\mu}\delta\chi^\mu+\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}\delta\dot{\chi}^\mu\right)\mathrm{d}\tau=\int_{\tau_1}^{\tau_2}\left\{\frac{\partial\widetilde{L}}{\partial\chi^\mu}\delta\chi^\mu+\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}\left[\frac{\mathrm{d}(\delta\chi^\mu)}{\mathrm{d}\tau}-\frac{1}{c^2}\dot{\chi}^\mu\dot{\chi}_\nu\frac{\mathrm{d}(\delta\chi^\nu)}{\mathrm{d}\tau}\right]\right\}\mathrm{d}\tau$$
$$=\int_{\tau_1}^{\tau_2}\frac{\partial\widetilde{L}}{\partial\chi^\mu}\delta\chi^\mu\mathrm{d}\tau+\int_{\tau_1}^{\tau_2}\left[\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}\dot{\chi}^\nu\dot{\chi}_\mu\right]\frac{\mathrm{d}(\delta\chi^\mu)}{\mathrm{d}\tau}\mathrm{d}\tau$$
$$=\int_{\tau_1}^{\tau_2}\frac{\partial\widetilde{L}}{\partial\chi^\mu}\delta\chi^\mu\mathrm{d}\tau-\int_{\tau_1}^{\tau_2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\dot{\chi}^\nu\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}\dot{\chi}_\mu\right](\delta\chi^\mu)\mathrm{d}\tau$$
$$+\left\{\left[\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\dot{\chi}^\nu\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}\dot{\chi}_\mu\right]\delta\chi^\mu\right\}_{\tau_1}^{\tau_2}$$
$$=\int_{\tau_1}^{\tau_2}\left[\frac{\partial\widetilde{L}}{\partial\chi^\mu}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\dot{\chi}^\nu\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}\dot{\chi}_\mu\right)\right]\delta\chi^\mu\mathrm{d}\tau, \tag{52}$$

where we used the following identity:

$$\delta\dot{\chi}^\mu=\frac{\mathrm{d}\left(\delta\chi^\mu\right)\mathrm{d}\tau-\frac{1}{c^2}\dot{\chi}^\nu\mathrm{d}\chi^\mu\mathrm{d}\left(\delta\chi^\nu\right)}{\left(\mathrm{d}\tau\right)^2}=\frac{\mathrm{d}\left(\delta\chi^\mu\right)}{\mathrm{d}\tau}-\frac{1}{c^2}\dot{\chi}^\mu\dot{\chi}_\nu\frac{\mathrm{d}\left(\delta\chi^\nu\right)}{\mathrm{d}\tau}. \tag{53}$$

Substituting Eqs. (51) and (52) into Eq. (46), we obtain

$$0=\delta\mathcal{A}=\int_{\tau_1}^{\tau_2}\left\{\frac{\partial\widetilde{L}}{\partial\chi^\mu}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\left(\dot{\chi}^\nu\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}-\widetilde{L}\right)\dot{\chi}_\mu\right]\right\}\delta\chi^\mu\mathrm{d}\tau=0, \tag{54}$$

which implies

$$\frac{\partial\widetilde{L}}{\partial\chi^\mu}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\mu}-\frac{1}{c^2}\left(\dot{\chi}^\nu\frac{\partial\widetilde{L}}{\partial\dot{\chi}^\nu}-\widetilde{L}\right)\dot{\chi}_\mu\right]=0 \tag{55}$$

owing to the arbitrariness of $\delta\chi^\mu$. Equation (55) will be called the geometric EL equation. We note that it has been derived by Infeld using the Lagrange multiplier method [14].

4 Geometric Lagrangian density

For particle-field systems, we need to find the density of the geometric Lagrangian. We start from Eq. (21), the non-geometric form of the action of our system. It can be written as [6]

$$\mathcal{A}=-\sum_{s,p}\int m_sc^2\mathrm{d}\tau-\sum_{s,p}\int\frac{q_s}{c}A_\mu\mathrm{d}\chi_{sp}^\mu$$
$$-\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\mathrm{d}\Omega, \tag{56}$$

where χ_{sp}^μ $(\mu=0,1,2,3)$ represents the components of the world line of the p-th particle of species s, τ is the proper time, and the 4-potential A_μ and the field-strength tensor $F^{\mu\nu}$ are functions on the Minkowski space. The boundary of the domain is taken to be at infinity.

Equation (56) can be easily translated into another form,

$$\mathcal{A}=\sum_{s,p}\int\left[-m_sc^2-\frac{q_s}{c}A_\mu\dot{\chi}^\mu(\tau)\right]\mathrm{d}\tau$$
$$-\frac{1}{4\pi c}\int\partial^{[\mu}A^{\nu]}\partial_\mu A_\nu\mathrm{d}\Omega, \tag{57}$$

by using the relations $\dot{\chi}^\mu=\mathrm{d}\chi^\mu/\mathrm{d}\tau$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. The action is manifestly covariant or geometric. However, to obtain the Lagrangian density of particle-field systems, we should change it into another form by multiplying the first term in Eq. (57) by the equation

$$\int\delta(\chi-\chi_{sp})\mathrm{d}\Omega=1, \tag{58}$$

where χ is an arbitrary world point in Minkowski space, and $\delta\left(\chi-\chi_{sp}\right)$ is the Dirac delta function. Then Eq. (57) becomes

$$\mathcal{A}=\int\left[\sum_{s,p}\int\left(-m_sc^2-\frac{q_s}{c}A_\mu\dot{\chi}^\mu(\tau)\right)\delta\left(\chi-\chi_{sp}\right)\mathrm{d}\tau\right]\mathrm{d}\Omega-\frac{1}{4\pi c}\int\partial^{[\mu}A^{\nu]}\partial_\mu A_\nu\mathrm{d}\Omega. \tag{59}$$

The geometric Lagrangian density is easy to read from Eq. (59) as

$$\mathcal{L}=\sum_{s,p}\int\left(-m_sc^2-\frac{q_s}{c}A_\mu\dot{\chi}^\mu_{sp}(\tau)\right)\delta(\chi-\chi_{sp})\mathrm{d}\tau-\frac{1}{4\pi c}\partial^{[\mu}A^{\nu]}\partial_\mu A_\nu. \tag{60}$$

The geometric Lagrangian above can be simplified as

$$\mathcal{L}=\mathcal{L}_P+\mathcal{L}_F=\int\hat{\mathcal{L}}_P\mathrm{d}\tau+\mathcal{L}_F \tag{61}$$

by defining

$$\hat{\mathcal{L}}_P=\sum_{s,p}\left(-m_sc^2-\frac{q_s}{c}A_\mu\dot{\chi}^\mu_{sp}(\tau)\right)\delta\left(\chi-\chi_{sp}\right), \tag{62}$$

$$\mathcal{L}_P=\int\hat{\mathcal{L}}_P\left(\chi_{sp},\dot{\chi}_{sp},\chi,A\right)\mathrm{d}\tau, \tag{63}$$

and

$$\mathcal{L}_F=-\frac{1}{4\pi c}\partial^{[\mu}A^{\nu]}\partial_\mu A_\nu. \tag{64}$$

Here, $\hat{\mathcal{L}}_P$ is a function of χ, χ_{sp}, $\dot{\chi}_{sp}$, and A. On the other hand, $\mathcal{L}_P$ can be regarded as a functional of the world lines of the particles and a function of the space-time and the field A. Note that $\mathcal{L}_F$ in Eq. (64) differs from $\mathcal{L}_F$ in Eq. (8) by a constant c. The reason is that in the geometric form of the action given by Eq. (59), the volume form $\mathrm{d}\Omega$ has the dimension of [length]^4.

5 Geometric weak Euler–Lagrange equation and energy-momentum conservation

Now we determine how the action given by Eq. (59) and the geometric Lagrangian density vary in response to the variation of $\delta\chi_{sp}$ and δA. From Eqs. (59)–(64), the variation of the action of the particle-field system can be written as

$$\delta\mathcal{A}=\int\delta\left[\int\hat{\mathcal{L}}_P\left(\chi_{sp},\dot{\chi}_{sp},\chi,A\right)\mathrm{d}\tau\right]_A\mathrm{d}\Omega+\int\delta\mathcal{L}_F\mathrm{d}\Omega+\int\delta\left[\int\hat{\mathcal{L}}_P\left(\chi_{sp},\dot{\chi}_{sp},\chi,A\right)\mathrm{d}\tau\right]_{\chi_{sp},\dot{\chi}_{sp}}\mathrm{d}\Omega, \tag{65}$$

where the notation $\delta\left[\int\hat{\mathcal{L}}_P\left(\chi_{sp},\dot{\chi}_{sp},\chi,A\right)\mathrm{d}\tau\right]_\alpha$ $(\alpha=A,\chi_{sp},\dot{\chi}_{sp})$ indicates that α is fixed.

The first term on the right-hand side of Eq. (65) can be treated by the procedure used in the derivation of Eq. (55):

$$\begin{aligned}\int\delta\left[\int\hat{\mathcal{L}}_P\left(\chi_{sp},\dot{\chi}_{sp},\chi,A\right)\mathrm{d}\tau\right]_A\mathrm{d}\Omega&=\iint\left\{\frac{\partial\hat{\mathcal{L}}_P}{\partial\chi^\mu_{sp}}-\frac{D}{D\tau}\left[\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\mu_{sp}}-\frac{1}{c^2}\left(\chi^\nu_{sp}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\nu_{sp}}-\hat{\mathcal{L}}_P\right)\dot{\chi}_{sp\mu}\right]\right\}\delta\chi^\mu_{sp}\mathrm{d}\tau\mathrm{d}\Omega\\&=\int\delta\chi^\mu_{sp}\mathrm{d}\tau\int\left\{\frac{\partial\hat{\mathcal{L}}_P}{\partial\chi^\mu_{sp}}-\frac{D}{D\tau}\left[\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\mu_{sp}}-\frac{1}{c^2}\left(\chi^\nu_{sp}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\nu_{sp}}-\hat{\mathcal{L}}_P\right)\dot{\chi}_{sp\mu}\right]\right\}\mathrm{d}\Omega.\end{aligned} \tag{66}$$

The second and third terms on the right-hand side of Eq. (65) are actually

$$\int\left\{\frac{\partial\mathcal{L}}{\partial A_\mu}-\frac{D}{D\chi^\nu}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\right]\right\}\delta A_\mu\mathrm{d}\Omega. \tag{67}$$

If we define

$$E^\mu_A=\frac{\partial\mathcal{L}}{\partial A_\mu}-\frac{D}{D\chi^\nu}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\right], \tag{68}$$

$$E_{\chi_{sp}\mu}=\frac{\partial\hat{\mathcal{L}}_P}{\partial\chi^\mu_{sp}}-\frac{D}{D\tau}\left[\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\mu_{sp}}-\frac{1}{c^2}\left(\chi^\nu_{sp}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\nu_{sp}}-\hat{\mathcal{L}}_P\right)\dot{\chi}_{sp\mu}\right] \tag{69}$$

and substitute Eqs. (66)–(69) into Eq. (65), then

$$\delta\mathcal{A}=\int\left[\int E_{\chi_{sp}\mu}\mathrm{d}\Omega\right]\delta\chi^\mu_{sp}\mathrm{d}\tau+\int E^\mu_A\delta A_\mu\mathrm{d}\Omega. \tag{70}$$

Thus,

$$E^\mu_A=\frac{\partial\mathcal{L}}{\partial A_\mu}-\frac{D}{D\chi^\nu}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\right]=0, \tag{71}$$

$$\int E_{\chi_{sp}\mu}\mathrm{d}\Omega\equiv\int\left\{\frac{\partial\hat{\mathcal{L}}_P}{\partial\chi^\mu_{sp}}-\frac{D}{D\tau}\left[\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\mu_{sp}}-\frac{1}{c^2}\left(\chi^\nu_{sp}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}^\nu_{sp}}-\hat{\mathcal{L}}_P\right)\dot{\chi}_{sp\mu}\right]\right\}\mathrm{d}\Omega=0 \tag{72}$$

by the arbitrariness of $\delta\chi_{sp}^{\mu}$ and δA_μ.

Equation (71) is the EL equation of the 4-potential of the electromagnetic field. Substituting Eq. (60) into Eq. (71) gives the Maxwell equation,

$$\partial^\nu F_{\nu\mu} = \frac{4\pi}{c}\sum_{s,p} q_s \int \dot{\chi}_{sp\mu}\delta\left(\chi-\chi_{sp}\right)\mathrm{d}s, \tag{73}$$

where $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, and s is the length of the world line; i.e.,

$$\mathrm{d}s = c\mathrm{d}\tau. \tag{74}$$

The 4-current is

$$J_\mu = \sum_{s,p} q_s \int \dot{\chi}_{sp\mu}\delta\left(\chi-\chi_{sp}\right)\mathrm{d}s. \tag{75}$$

Equation (72) will be called the geometric submanifold EL equation because it is defined only on the world line after integration over the space-time variable [13]. If χ_{sp} were a function of the entire space-time domain, then $E_{\chi_{sp}}$ would vanish everywhere, as in the case of the 4-potential A. In general, we expect that $E_{\chi_{sp}} \neq 0$.

We now derive an expression for $E_{\chi_{sp}}$ by substituting Eq. (62) into Eq. (69). For the first term in $E_{\chi_{sp}}$,

$$\frac{\partial\hat{\mathcal{L}}_P}{\partial\chi_{sp}^{\mu}} = \left(-m_s c^2 - \frac{q_s}{c}A_\nu\dot{\chi}_{sp}^{\nu}\right)\frac{\partial\delta_2}{\partial\chi_{sp}^{\mu}} = -\frac{q_s}{c}\frac{\partial A_\nu}{\partial\chi^\mu}\dot{\chi}_{sp}^{\nu}\delta_2 + \frac{D}{D\chi^\mu}\left[\left(m_s c^2 + \frac{q_s}{c}A_\nu\dot{\chi}_{sp}^{\nu}\right)\delta_2\right], \tag{76}$$

where $\delta_2 \equiv \delta\left(\chi-\chi_{sp}\right)$. The second term of $E_{\chi_{sp}}$ is given by

$$-\frac{D}{D\tau}\left(\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}_{sp}^{\mu}}\right) = \frac{q_s}{c}A_\mu\frac{D\delta_2}{D\tau} = -\frac{q_s}{c}A_\mu\dot{\chi}_{sp}^{\nu}\frac{\partial\delta_2}{\partial\chi^\nu} = -\frac{D}{D\chi^\nu}\left[\frac{q_s}{c}A_\mu\dot{\chi}_{sp}^{\nu}\delta_2\right] + \frac{q_s}{c}\dot{\chi}_{sp}^{\nu}\frac{\partial A_\mu}{\partial\chi^\nu}\delta_2. \tag{77}$$

The third and fourth terms come from the mass-shell constraint and can be rewritten as

$$\begin{aligned}-\frac{D}{D\tau}\left(-\frac{1}{c^2}\dot{\chi}_{sp}^{\nu}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}_{sp}^{\nu}}\dot{\chi}_{sp\mu}\right) &= -\frac{q_s}{c^3}A_\nu\ddot{\chi}_{sp}^{\nu}\dot{\chi}_{sp\mu}\delta_2 - \frac{q_s}{c^3}A_\nu\dot{\chi}_{sp}^{\nu}\ddot{\chi}_{sp\mu}\delta_2 - \frac{q_s}{c^3}A_\nu\dot{\chi}_{sp}^{\nu}\dot{\chi}_{sp\mu}\frac{D\delta_2}{D\tau}\\ &= -\frac{q_s}{c^3}A_\nu\ddot{\chi}_{sp}^{\nu}\dot{\chi}_{sp\mu}\delta_2 - \frac{q_s}{c^3}A_\nu\dot{\chi}_{sp}^{\nu}\ddot{\chi}_{sp\mu}\delta_2 + \frac{D}{D\chi^\sigma}\left(\frac{q_s}{c^3}A_\nu\dot{\chi}_{sp}^{\nu}\dot{\chi}_{sp\mu}\dot{\chi}_{sp}^{\sigma}\delta_2\right) - \frac{q_s}{c^3}\dot{\chi}_{sp}^{\nu}\frac{\partial A_\nu}{\partial\chi^\sigma}\dot{\chi}_{sp}^{\sigma}\dot{\chi}_{sp\mu}\delta_2\end{aligned} \tag{78}$$

and

$$\begin{aligned}-\frac{D}{D\tau}\left(\frac{1}{c^2}\hat{\mathcal{L}}_P\dot{\chi}_{sp\mu}\right) &= -\frac{D}{D\chi^\nu}\left[\left(m_s + \frac{q_s}{c^3}A_\sigma\dot{\chi}_{sp}^{\sigma}\right)\dot{\chi}_{sp\mu}\dot{\chi}_{sp}^{\nu}\delta_2\right]\\ &\quad + \frac{q_s}{c^3}\dot{\chi}_{sp}^{\nu}\frac{\partial A_\sigma}{\partial\chi^\nu}\dot{\chi}_{sp}^{\sigma}\dot{\chi}_{sp\mu}\delta_2 + m_s\ddot{\chi}_{sp\mu}\delta_2 + \frac{q_s}{c^3}A_\sigma\dot{\chi}_{sp}^{\sigma}\ddot{\chi}_{sp\mu}\delta_2 + \frac{q_s}{c^3}A_\sigma\ddot{\chi}_{sp}^{\sigma}\dot{\chi}_{sp\mu}\delta_2.\end{aligned} \tag{79}$$

Therefore,

$$E_{\chi_{sp}\mu} = \frac{D}{D\chi^\nu}\left\{\left[\left(m_s c^2 + \frac{q_s}{c}A_\sigma\dot{\chi}_{sp}^{\sigma}\right)\eta_\mu^\nu - \left(\frac{q_s}{c}A_\mu + m_s\dot{\chi}_{sp\mu}\right)\dot{\chi}_{sp}^{\nu}\right]\delta_2\right\} + \left[m_s\ddot{\chi}_{sp\mu} - \frac{q_s}{c}\dot{\chi}_{sp}^{\nu}\left(\frac{\partial A_\nu}{\partial\chi^\mu} - \frac{\partial A_\mu}{\partial\chi^\nu}\right)\right]\delta_2. \tag{80}$$

Substituting Eq. (80) into the geometric submanifold EL equation, (72), we immediately obtain the equation of motion of the particles:

$$m_s\ddot{\chi}_{sp\mu} = \frac{q_s}{c}\left(\frac{\partial A_\nu}{\partial\chi^\mu} - \frac{\partial A_\mu}{\partial\chi^\nu}\right)\dot{\chi}_{sp}^{\nu}. \tag{81}$$

The right-hand term in Eq. (81) is the 4-Lorentzian force on the particles. Then, $E_{\chi_{sp}}$ is reduced to

$$\begin{aligned}E_{\chi_{sp}\mu} &= \frac{\partial\hat{\mathcal{L}}_P}{\partial\chi_{sp}^{\mu}} - \frac{D}{D\tau}\left[\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}_{sp}^{\mu}} - \frac{1}{c^2}\left(\dot{\chi}_{sp}^{\nu}\frac{\partial\hat{\mathcal{L}}_P}{\partial\dot{\chi}_{sp}^{\nu}} - \hat{\mathcal{L}}_P\right)\dot{\chi}_{sp\mu}\right]\\ &= \frac{D}{D\chi^\nu}\left\{\left[\left(m_s c^2 + \frac{q_s}{c}A_\sigma\dot{\chi}_{sp}^{\sigma}\right)\eta_\mu^\nu - \left(\frac{q_s}{c}A_\mu + m_s\dot{\chi}_{sp\mu}\right)\dot{\chi}_{sp}^{\nu}\right]\delta_2\right\}.\end{aligned} \tag{82}$$

As expected, $E_{\chi_{sp}} \neq 0$. We will refer to Eq. (82), which plays a significant role in subsequent analysis of the local conservation laws, as the geometric weak EL equation [13]. The qualifier "weak" indicates that only the space-time integral of $E_{\chi_{sp}}$ is zero. Unlike the non-relativistic case [see Eq. (30)], the geometric weak EL equation is written here as a differential equation about $\hat{\mathcal{L}_P}$, instead of the Lagrangian density.

Another crucial step in systematically obtaining the conservation laws is to find the symmetries. A symmetry group of the action $\mathcal{A}$ is defined by a continuous transformation,

$$(\chi, [\chi_{sp}], [\dot{\chi}_{sp}], A, \partial A) \longmapsto (\widetilde{\chi}, [\widetilde{\chi}_{sp}], [\widetilde{\dot{\chi}}_{sp}], \widetilde{A}, \widetilde{\partial A}), \quad (83)$$

such that

$$\int \mathcal{L}(\chi, [\chi_{sp}], [\dot{\chi}_{sp}], A, \partial A)\mathrm{d}\Omega = \int \tilde{\mathcal{L}}(\widetilde{\chi}, [\widetilde{\chi}_{sp}], [\widetilde{\dot{\chi}}_{sp}], \widetilde{A}, \widetilde{\partial A})\mathrm{d}\Omega, \quad (84)$$

where the indexes of the physical quantities have been omitted to simplify the notation. The symbol $[\beta](\beta = \chi_{sp}, \dot{\chi}_{sp}, \widetilde{\chi}_{sp}, \widetilde{\dot{\chi}}_{sp})$ indicates that the geometric Lagrangian density $\mathcal{L}$ is a functional of β. For the following transformation defined by

$$(\widetilde{\chi}, [\widetilde{\chi}_{sp}], [\widetilde{\dot{\chi}}_{sp}], \widetilde{A}, \widetilde{\partial A}) = (\chi + \epsilon X, [\chi_{sp} + \epsilon X], [\dot{\chi}_{sp}], A, \partial A), \quad \epsilon \in \mathbb{R}, \quad (85)$$

where X is a given constant 4-vector field, the condition in (84) is satisfied because

$$\begin{aligned} \mathcal{L} &= \int \hat{\mathcal{L}}_P\left(\chi_{sp}, \dot{\chi}_{sp}, \chi, A\right)\mathrm{d}\tau + \mathcal{L}_F \\ &= \int \hat{\mathcal{L}}_P\left(\chi + \epsilon X, \chi_{sp} + \epsilon X, \dot{\chi}_{sp}, A\right)\mathrm{d}\tau + \mathcal{L}_F, \end{aligned} \quad (86)$$

with $\widetilde{A}(\widetilde{\chi}) = A(\chi) = A(\widetilde{\chi} - \epsilon X)$ and $\widetilde{\partial A}(\widetilde{\chi}) = \partial A(\chi) = \partial A(\widetilde{\chi} - \epsilon X)$ [13]. Equation (86) should be transformed into a partial differential equation before we apply it to derivation of the conservation laws. The symmetry condition is

$$\left[\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\epsilon}\right]_{\epsilon=0} \equiv 0, \quad (87)$$

which is equivalent to

$$\begin{aligned} 0 &= \int \mathrm{d}\tau \left[\frac{d\left(\chi^\mu + \epsilon X^\mu\right)}{d\epsilon}\frac{\partial \hat{\mathcal{L}}_P}{\partial\left(\chi^\mu + \epsilon X^\mu\right)}\right. \\ &\quad \left. + \sum_{s,p}\frac{\mathrm{d}\left(\chi_{sp}^\mu + \epsilon X^\mu\right)}{\mathrm{d}\epsilon}\frac{\partial \hat{\mathcal{L}}_P}{\partial\left(\chi_{sp}^\mu + \epsilon X^\mu\right)}\right]_{\epsilon=0} \\ &= X^\mu \int \left(\frac{\partial \hat{\mathcal{L}}_P}{\partial \chi^\mu} + \sum_{s,p}\frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu}\right)\mathrm{d}\tau. \end{aligned} \quad (88)$$

Therefore,

$$\int \left[\frac{\partial \hat{\mathcal{L}}_P}{\partial \chi^\mu} + \sum_{s,p}\frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu}\right]\mathrm{d}\tau = 0, \quad (89)$$

or

$$\frac{\partial \mathcal{L}}{\partial \chi^\mu} + \sum_{s,p}\int \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu}\mathrm{d}\tau = 0. \quad (90)$$

Owing to the integral, Eq. (90) indicates that the corresponding vector field of this symmetry for particle-field systems is infinite dimensional.

Next, we establish the connection between the symmetry of the particle-field system, i.e., Eq. (90), and the geometric conservation laws. The first term in Eq. (90) is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \chi^\mu} &= \frac{D\mathcal{L}}{D\chi^\mu} - \frac{DA_\nu}{D\chi^\mu}\frac{\partial \mathcal{L}}{\partial A_\nu} - \frac{D\left(\partial_\sigma A_\nu\right)}{D\chi^\mu}\frac{\partial \mathcal{L}}{\partial\left(\partial_\sigma A_\nu\right)} \\ &= \frac{D}{D\chi^\nu}\left[\mathcal{L}\eta_\mu^\nu - \partial_\mu A_\sigma \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu A_\sigma\right)}\right], \end{aligned} \quad (91)$$

where we used the EL equation, i.e., Eq. (71). For the second term in Eq. (90), we used the geometric weak EL equation, (82), to obtain

$$\begin{aligned} \int_{p_1}^{p_2}\frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu}\mathrm{d}\tau &= \frac{D}{D\chi^\nu}\left\{\int \left[\left(m_s c^2 + \frac{q_s}{c}A_\sigma \dot{\chi}^\sigma\right)\eta_\mu^\nu \delta_2 - \left(\frac{q_s}{c}A_\mu + m_s \dot{\chi}_{sp\mu}\right)\dot{\chi}_{sp}^\nu \delta_2\right]\mathrm{d}\tau\right\} \\ &\quad + \left[\frac{\partial \hat{\mathcal{L}}_P}{\partial \dot{\chi}_{sp}^\mu} - \frac{1}{c^2}\left(\chi_{sp}^\nu \frac{\partial \hat{\mathcal{L}}_P}{\partial \dot{\chi}_{sp}^\nu} - \hat{\mathcal{L}_P}\right)\dot{\chi}_{sp\mu}\right]_{p_1}^{p_2}. \end{aligned} \quad (92)$$

The boundaries, i.e., p_1 and p_2, of the integral in Eq. (92) should be extended to infinity, which will make the last term vanish because of the existence of the Dirac delta function in $\hat{\mathcal{L}_P}$.

Substituting Eqs. (91), (92), and (60) into Eq. (90), we obtain

$$\frac{D}{D\chi^{\nu}}\left[\sum_{s,p} m_s \int \dot{\chi}_{sp\mu}\dot{\chi}_{sp}^{\nu}\delta_2 ds - \frac{1}{2\pi}\partial_\mu A_\sigma \partial^{[\nu}A^{\sigma]} + \frac{1}{4\pi}A_\mu \partial^\sigma F_\sigma^\nu + \frac{1}{16\pi}F_{\sigma\rho}F^{\sigma\rho}\eta_\mu^\nu\right] = 0, \tag{93}$$

where we used Maxwell's equation, i.e., Eq. (73). Equation (93) is equivalent to

$$\frac{D}{D\chi^{\nu}}\left\{\sum_{s,p} m_s \int \dot{\chi}_{sp\mu}\dot{\chi}_{sp}^{\nu}\delta_2 d\tau + \frac{1}{4\pi}\left(\frac{1}{4}\eta_\mu^\nu F_{\sigma\rho}F^{\sigma\rho} - F_\mu^\sigma F_\sigma^\nu\right) - \frac{1}{4\pi}\left[\partial_\sigma\left(A_\mu F^{\nu\sigma}\right)\right]\right\} = 0 \tag{94}$$

with the identity

$$-\frac{1}{2\pi}\partial_\mu A_\sigma \partial^{[\nu}A^{\sigma]} + \frac{1}{4\pi}A_\mu\partial^\sigma F_\sigma^\nu = -\frac{1}{4\pi}F_{\mu\sigma}F^{\nu\sigma} - \frac{1}{4\pi}\partial_\sigma\left(A_\mu F^{\nu\sigma}\right). \tag{95}$$

The last term in Eq. (94) is zero because

$$\frac{D}{D\chi^{\nu}}\partial_\sigma\left(A_\mu F^{\nu\sigma}\right) = \frac{D}{D\chi^{(\nu}}\frac{D}{D\chi^{\sigma)}}\left(A_\mu F^{[\nu\sigma]}\right) \equiv 0, \tag{96}$$

where $(\nu\ \mu)$ is the total symmetrization of the indexes of ν and μ [4]. Finally, we arrive at the geometric, or manifestly covariant, energy-momentum conservation laws,

$$\frac{\partial}{\partial\chi^{\nu}}\left[\sum_{s,p} m_s \int \dot{\chi}_s^{\mu}\dot{\chi}_{sp}^{\nu}\delta_2 ds + \frac{1}{4\pi}\left(\frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F^{\mu\sigma}F_\sigma^\nu\right)\right], \tag{97}$$

where we regard the field as defined on χ, and $\partial/\partial\chi^\nu \equiv D/D\chi^\nu$. In terms of the energy-momentum tensor, Eq. (97) is

$$\partial_\nu T^{\mu\nu} = 0, \tag{98}$$

where

$$T^{\mu\nu} = T_P^{\mu\nu} + T_F^{\mu\nu}, \tag{99}$$

$$T_P^{\mu\nu} = \sum_{s,p} m_s \int \dot{\chi}_{sp}^{\mu}\dot{\chi}_{sp}^{\nu}\delta_2 ds, \tag{100}$$

$$T_F^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\sigma}F_\sigma^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right). \tag{101}$$

Here, $T_P^{\mu\nu}$, $T_F^{\mu\nu}$, and $T^{\mu\nu}$ are the energy-momentum tensors of the particles, electromagnetic field, and particle-field system, respectively, written in the geometric form, and it is easy to check that all of these tensors are symmetric. The energy-momentum tensor $T_F^{\mu\nu}$ for the electromagnetic field described by Eq. (101) is well-known. However, to the best of our knowledge, the geometric particle energy-momentum tensor given in Eq. (100) has not been derived previously. Instead, the particle energy-momentum tensor is typically written in the literature [6] as

$$T_P^{\mu\nu} = \sum_{s,p} m_s \delta\left(\boldsymbol{x} - \boldsymbol{X}_{sp}(t)\right)\dot{\chi}_{sp}^{\mu}(t)\dot{\chi}_{sp}^{\nu}(t)\frac{d\tau(t)}{dt}, \tag{102}$$

where a Lorentzian coordinate system $\{t, \boldsymbol{x}\}$ is chosen. Obviously, Eq. (102) is not a geometric form of Eq. (100). We can prove that Eq. (100) becomes Eq. (102) when a coordinate system is chosen. For this purpose, we first show that

$$\int \delta(\chi - \chi_{sp})g(\chi_{sp})ds = \gamma_{sp}^{-1}(t)\delta\left(\boldsymbol{x} - \boldsymbol{X}_{sp}(t)\right)g\left(\chi_{sp}(t)\right), \tag{103}$$

where $\gamma_{sp}^{-1}(t) \equiv \sqrt{1 - \dot{\boldsymbol{X}}_{sp}^2(t)/c^2}$, and $g(\chi_{sp})$ is an arbitrary field. The left-hand side of Eq. (103) is

$$\begin{aligned}\int \delta\left(\chi - \chi_{sp}\right)g\left(\chi_{sp}\right)ds &\equiv \int \delta\left[\chi - \chi_{sp}\left(s_{sp}\right)\right]g\left[\chi_{sp}\left(s_{sp}\right)\right]ds_{sp} = \delta\left[\chi - \chi_{sp}\left(s_{sp}\left(t_{sp}\right)\right)\right]g\left[\chi_{sp}\left(s_{sp}\left(t_{sp}\right)\right)\right]\frac{ds_{sp}}{dt_{sp}}dt_{sp} \\ &= \int dt_{sp}\delta\left[c\left(t - t_{sp}\right)\right]\delta\left[\boldsymbol{x} - \boldsymbol{X}_{sp}\left(t_{sp}\right)\right]g\left[\chi_{sp}\left(t_{sp}\right)\right]\frac{cd\tau_{sp}\left(t_{sp}\right)}{dt_{sp}},\end{aligned} \tag{104}$$

where t_{sp} is the time parameter for each world line, and $\boldsymbol{X}_{sp}(t_{sp})$ is the space position of sp-particle at time t_{sp}. Because

$$\frac{\mathrm{d}\tau_{sp}(t_{sp})}{\mathrm{d}t_{sp}} = \gamma_{sp}^{-1}(t_{sp}) \tag{105}$$

and

$$\delta\left[c\left(t-t_{sp}\right)\right] = \frac{1}{c}\delta\left(t-t_{sp}\right), \tag{106}$$

we have

$$\begin{aligned}&\int \delta\left(\chi-\chi_{sp}\right) g\left(\chi_{sp}\right) \mathrm{d}s\\ &= \int \gamma_{sp}^{-1}(t_{sp})\delta\left[\boldsymbol{x}-\boldsymbol{X}_{sp}\left(t_{sp}\right)\right] g\left[\chi_{sp}\left(t_{sp}\right)\right]\delta\left(t-t_{sp}\right)\mathrm{d}t_{sp}\\ &= \gamma_{sp}^{-1}(t)\delta\left[\boldsymbol{x}-\boldsymbol{X}_{sp}(t)\right] g\left[\chi_{sp}(t)\right],\end{aligned} \tag{107}$$

which is Eq. (103). If we take $g\left(\chi_{sp}\right) = m_s\dot{\chi}_{sp}^{\mu}\dot{\chi}_{sp}^{\nu}$, then the geometric particle energy-momentum tensor is

$$\begin{aligned}T_P^{\mu\nu} &= \sum_{s,p} m_s \int \delta\left(\chi-\chi_{sp}\right)\dot{\chi}_{sp}^{\mu}\dot{\chi}_{sp}^{\nu}\mathrm{d}s\\ &= m_s\gamma_{sp}^{-1}(t)\delta\left[\boldsymbol{x}-\boldsymbol{X}_{sp}(t)\right]\dot{\chi}_{sp}^{\mu}(t)\,\dot{\chi}_{sp}^{\nu}(t)\\ &= \sum_{s,p} m_s\delta\left[\boldsymbol{x}-\boldsymbol{X}_{sp}(t)\right]\dot{\chi}_{sp}^{\mu}(t)\,\dot{\chi}_{sp}^{\nu}(t)\,\frac{\mathrm{d}\tau_{sp}(t)}{\mathrm{d}t}.\end{aligned} \tag{108}$$

This confirms that the geometric particle energy-momentum tensor recovers the non-geometric form in a chosen coordinate system.

6 Conclusions

In this study, we developed a manifestly covariant, or geometric, field theory of the relativistic classical particle-field systems often encountered in plasma physics, accelerator physics, and astrophysics. The connection between the space-time symmetry and energy-momentum conservation laws was demonstrated geometrically. In our theoretical formalism, space and time are placed on an equal footing; i.e., space-time is treated as one entity without choosing a coordinate system. This is different from existing field theories, where it is necessary to split the space and time coordinates at a certain stage, and thus the manifestly covariant property is lost.

There are several unique features of the developed geometric field theory. The first is the mass-shell constraint, which produces two new terms in the geometric EL equation for particles, (55). The geometric Lagrangian density of particle-field systems is a functional of the particles' world lines [see Eq. (60)], and thus the symmetry vector field of the systems lies on infinite dimensional space [see Eq. (92)]. Another feature of the theory is that the particles and fields reside on different manifolds. The domain of the particle field can be the proper time or any other parameterization of the world lines; on the other hand, the electromagnetic field is defined on the space-time. To establish geometrically the connection between the symmetries and energy-momentum conservation laws, a geometric weak EL equation for particles, (81), was derived. Combining the EL equation, (71), for the field and the geometric weak EL equation, (81), for the particles, the symmetries and conservation laws can be established geometrically. Using the theory, we derived for the first time a geometric expression for the particle energy-momentum tensor in Eq. (100), which recovers the non-geometric form in Ref. [6] when a coordinate system is chosen.

In this study, we used proper time. We note that different particles have different proper times, which are not synchronized in the laboratory frame. This causes difficulties if one would like to use proper time for particle-in-cell (PIC) simulations. However, proper time can be useful in certain situations. For example, proper time has been used to construct explicit symplectic integrators for studying the relativistic dynamics of charged particles [24, 29]. In fact, the geometric field theory of classical particle-field systems in this study can be established only with the help of proper time. Regarding the specific application of proper time in PIC simulations, more investigation is needed. The facts that proper time can be used to construct explicit symplectic integrators and that it is essential in establishing the geometric field theory of classical particle-field systems suggest that proper time could play a role in developing advanced PIC algorithms [30–32]. For example, we can investigate the possibility of using different proper time steps in the symplectic integrators for different particles such that they are synchronized in the laboratory frame. This topic will be explored in a future study.

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