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RESEARCH ARTICLE

Dynamics of coherence-induced state ordering under Markovian channels

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We study the dynamics of coherence-induced state ordering under incoherent channels, particularly four specific Markovian channels: amplitude damping channel, phase damping channel, depolarizing channel and bit flit channel for single-qubit states. We show that the amplitude damping channel, phase damping channel, and depolarizing channel do not change the coherence-induced state ordering by l_1 norm of coherence, relative entropy of coherence, geometric measure of coherence, and Tsallis relative α-entropies, while the bit flit channel does change for some special cases.

Keywords l_1-norm of coherence, relative entropy of coherence, geometric measure of coherence, Tsallis relative α-entropies of coherence, ordering state

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1 Introduction

Quantum coherence is a fundamental feature of quantum mechanics, which distinguishes the quantum world from the classical physics realm. It is an important aspect in many research fields such as low-temperature thermodynamics [1–5], quantum biology [6–11], and nanoscale physics [12, 13]. Quantifying the coherence of quantum states [14] has become a topic of interest for researchers. Baumgratz *et al.* have recently proposed a strict framework to quantify quantum coherence [15]. Various coherence measures have been defined based on this framework, such as l_1-norm of coherence, relative entropy of coherence [15], geometric measure of coherence [16] and Tsallis relative α-entropies of coherence measure [17]. Here, the Tsallis relative α-entropy of coherence measure violates the condition of a coherence measure that does not increase under mixing of states, while it satisfies a generalized monotonicity of average coherence under subselection based on measurement.

Different coherence measures employed based on different physical contexts thus give rise to different values of coherence. Questions about ordering states with various coherence measures have also been discussed [17–19]. Whether or not the quantum operators change the coherence-induced state ordering proposed by Zhang *et al.* [18], forms another interesting problem.

Focused on single-qubit states, in this study, we investigate such ordering problems under incoherent channels. In particular, we consider four Markovian channels: − amplitude damping channel, phase damping channel, depolarizing channel, and bit flit channel. Note that for some special cases, Zhang *et al.* have already studied the problem for single-qubit states by using the amplitude damping channel and phase damping channel [18]. Here, we also consider the geometric measure of coherence for more general situations. We extend the results of Ref. [18] to general cases. Furthermore, we show that the depolarizing channel does not change the coherence-induced state ordering while the bit flit channel changes it when $p = \frac{1}{2}$.

2 Preliminaries

In this section, we first recapitulate some concepts related to quantum coherence. Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\{|i\rangle\}_{i=0}^{d-1}$ be an orthonormal basis of $\mathcal{H}$. An incoherent state is defined as $\rho = \Sigma_{i=0}^{d-1} p_i |i\rangle\langle i|$, where $p_i \geq 0, \Sigma_i p_i = 1$. Let $\mathcal{I}$ denote the set of incoherent states. An incoherent operation is defined as $\Lambda(\rho) = \Sigma_n K_n \rho K_n^\dagger$, where $\Sigma_n K_n K_n^\dagger = I$ and $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$. Baumgratz *et al.* proposed a framework to quantify quantum coherence, that is, a function $\mathcal{C}$ can be taken as a coherence measure if it satisfies the following postulates [15]:

(C1) $\mathcal{C}(\rho) \geq 0$, $\mathcal{C}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$;

(C2) $\mathcal{C}(\Lambda(\rho)) \leq \mathcal{C}(\rho)$ for any incoherent operation Λ;

(C3) $\Sigma_n p_n \mathcal{C}(\rho_n) \leq \mathcal{C}(\rho)$, where $p_n = \mathrm{Tr}(K_n \rho K_n^\dagger)$ and $\rho_n = K_n \rho K_n^\dagger / p_n$, $\{K_n\}$ is a set of incoherent Kraus operators;

(C4) $\mathcal{C}(\Sigma_i p_i \rho_i) \leq \Sigma_i p_i \mathcal{C}(\rho_i)$ for any set of quantum states $\{\rho_i\}$ and any probability distribution $\{p_i\}$.

Several coherence measures have been put forward based on this framework. Here, we give the definitions of the following four coherence measures for further use.

Let ρ be a state defined on $\mathcal{H}$, then

$$\mathcal{C}_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \tag{1}$$

is the l_1 norm of coherence, where ρ_{ij} are the entries of ρ. The relative entropy of coherence is defined by

$$\mathcal{C}_r(\rho) = \min_{\sigma \in \mathcal{I}} \mathcal{S}(\rho \| \sigma) = \mathcal{S}(\rho_{diag}) - \mathcal{S}(\rho), \tag{2}$$

where $\mathcal{S}(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$ is the quantum relative entropy, $\mathcal{S}(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy, and $\rho_{diag} = \Sigma_i \rho_{ii} |i\rangle\langle i|$ is the diagonal part of ρ. The geometric measure of coherence is defined by

$$\mathcal{C}_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma), \tag{3}$$

where $F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)^2$ is the fidelity of two density operators ρ and σ. The Tsallis relative α-entropy of coherence is defined by

$$\mathcal{C}_\alpha(\rho) = \min_{\delta \in \mathcal{I}} \mathcal{D}_\alpha(\rho \| \delta) = \frac{r^\alpha - 1}{\alpha - 1}, \tag{4}$$

where $r = \Sigma_i \langle i|\rho^\alpha|i\rangle^{\frac{1}{\alpha}}$ and $\alpha \in (0,1) \cup (1,2]$.

Any single-qubit state can be expressed as

$$\rho = \frac{1}{2}(I + \boldsymbol{k}\boldsymbol{\sigma}) = \frac{1}{2}(I + t\boldsymbol{n}\boldsymbol{\sigma}), \tag{5}$$

where $\boldsymbol{k} = (k_x, k_y, k_z)$ is a real vector satisfying $\|\boldsymbol{k}\| \leq 1$, $t = \|\boldsymbol{k}\|$, $\boldsymbol{n} = (n_x, n_y, n_z)$ is a unit vector, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. Here, we note that n_x, n_y, n_z represent the length of vector $\boldsymbol{k}$ along the direction $\sigma_x, \sigma_y, \sigma_z$, respectively.

A non-coherence-generating channel (NC) $\tilde{\Phi}$ is a completely positive and trace preserving (CPTP) map from an incoherent state to another incoherent state: $\tilde{\Phi}(\mathcal{I}) \subset \mathcal{I}$, where $\mathcal{I}$ denotes the set of incoherent states [20]. Any quantum channel Φ is called an incoherent channel if there exists a Kraus decomposition $\Phi(\cdot) = \Sigma_n K_n(\cdot)K_n^\dagger$ such that $\rho_n = \frac{K_n(\rho)K_n^\dagger}{\mathrm{Tr}(K_n(\rho)K_n^\dagger)}$ is incoherent for any incoherent state ρ.

A rank-2 qubit channel is an NC if and only if it has the Kraus decomposition either as [20]

$$\Phi^{(1)}(\cdot) = E_1^{(1)}(\cdot)E_1^{(1)\dagger} + E_2^{(1)}(\cdot)E_2^{(1)\dagger} \tag{6}$$

with

$$E_1^{(1)} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta}\cos\theta\cos\phi & 0 \\ -\sin\theta\sin\phi & \mathrm{e}^{\mathrm{i}\xi}\cos\phi \end{pmatrix},$$

$$E_2^{(1)} = \begin{pmatrix} \sin\theta\cos\phi & \mathrm{e}^{\mathrm{i}\xi}\sin\phi \\ \mathrm{e}^{-\mathrm{i}\eta}\cos\theta\sin\phi & 0 \end{pmatrix} \tag{7}$$

or as

$$\Phi^{(2)}(\cdot) = E_1^{(2)}(\cdot)E_1^{(2)\dagger} + E_2^{(2)}(\cdot)E_2^{(2)\dagger} \tag{8}$$

with

$$E_1^{(2)} = \begin{pmatrix} \cos\theta & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\xi}\cos\phi \end{pmatrix},$$

$$E_2^{(2)} = \begin{pmatrix} 0 & \sin\phi \\ \mathrm{e}^{\mathrm{i}\xi}\sin\theta & 0 \end{pmatrix}, \tag{9}$$

where θ, ϕ, ξ, and η are all real numbers. Here $\Phi^{(1)}$ is not an incoherent channel unless $\sin\theta\cos\theta\sin\phi\cos\phi = 0$ and $\Phi^{(2)}$ is an incoherent channel.

3 Main results

In this section, we first study the coherence-induced ordering problem under arbitrary incoherent channels for single-qubit states via the coherence measures $\mathcal{C}_{l_1}$, $\mathcal{C}_r$, $\mathcal{C}_\alpha$, and $\mathcal{C}_g$. Then we study the dynamics of coherence-induced state ordering under specific Markovian channels for single-qubit states by four Markovian channels namely, amplitude damping, phase damping channel, depolarizing channel, and bit flit channel.

Suppose that an incoherent channel is defined as in Eq. (6). Let $a = \frac{1-tn_z}{2}$ and $b = \frac{t(n_x - in_y)}{2}$ with $b = |b|\mathrm{e}^{\mathrm{i}\beta}$. Then $\Phi(\rho) = \left(\begin{smallmatrix} A & B \\ B^* & 1-A \end{smallmatrix}\right)$ with $A = a\cos^2\phi + (b^*\mathrm{e}^{\mathrm{i}\xi} + b\mathrm{e}^{-\mathrm{i}\xi})\sin\theta\sin\phi\cos\phi + (1-a)\sin^2\phi$, $B = b\mathrm{e}^{\mathrm{i}\eta - \mathrm{i}\xi}\cos\theta\cos^2\phi + b^*\mathrm{e}^{\mathrm{i}\xi+\mathrm{i}\eta}\cos\theta\sin^2\phi$. Thus, $\mathcal{C}_{l_1}(\Phi(\rho)) = 2|be^{i\eta - i\xi}\cos\theta\cos^2\phi + b^*\mathrm{e}^{\mathrm{i}\xi+\mathrm{i}\eta}\cos\theta\sin^2\phi|$. If $\sin\theta = 0$, then Φ is an incoherent operation and $\mathcal{C}_{l_1}(\Phi(\rho)) = 2|b|\sqrt{\mathrm{e}^{\mathrm{i}\beta - \mathrm{i}\xi}\cos^2\phi + \mathrm{e}^{\mathrm{i}\xi - \mathrm{i}\beta}\sin^2\phi}$. We find that the value of $\mathcal{C}_{l_1}(\rho)$ depends on both b and the channel itself. In other words, there may exist incoherent channels such that $\mathcal{C}_{l_1}(\Phi(\rho_1)) < \mathcal{C}_{l_1}(\Phi(\rho_2))$ though $\mathcal{C}_{l_1}(\rho_1) > \mathcal{C}_{l_1}(\rho_2)$.

Suppose that an incoherent channel is defined as in Eq. (8). Then $\Phi(\rho) = \left(\begin{smallmatrix} C & D \\ D^* & 1-C \end{smallmatrix}\right)$ with $C = a\cos^2\theta + (1-a)\sin^2\phi$ and $D = \mathrm{e}^{\mathrm{i}\xi}(b\cos\theta\cos\phi + b^*\sin\theta\sin\phi)$. Thus, $\mathcal{C}_{l_1}(\rho) = 2|b|\sqrt{\cos^2\beta\cos^2(\theta-\phi) + \sin^2\beta\cos^2(\theta+\phi)}$.

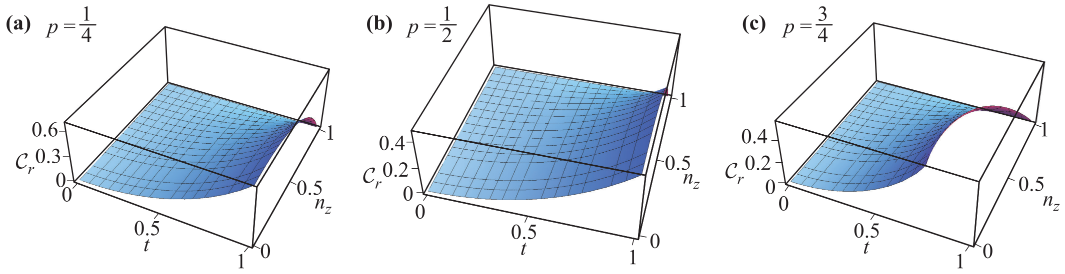

Fig. 1 The variation of $\mathcal{C}_r(\varepsilon(\rho))$ with respect to t and n_z under amplitude damping channel.

Also, we know that the value of $\mathcal{C}_{l_1}(\rho)$ depends on both b and the channel itself. In other words, there may exist incoherent channels such that $\mathcal{C}_{l_1}(\Phi(\rho_1)) < \mathcal{C}_{l_1}(\Phi(\rho_2))$ though $\mathcal{C}_{l_1}(\rho_1) > \mathcal{C}_{l_1}(\rho_2)$.

According to the above discussion, we can conclude that there exist incoherent channels changing the coherence-induced state ordering under the coherence measure $\mathcal{C}_{l_1}$. This is true also for the coherence measure $\mathcal{C}_g$, since $\mathcal{C}_{l_1}$ and $\mathcal{C}_g$ give the same ordering for single-qubit states [21]. For the other coherence measures $\mathcal{C}_r$ and $\mathcal{C}_\alpha$, the issue will become formidably difficult for general incoherent channels. However, we can consider some specific incoherent channels to deal with the problem.

3.1 Amplitude damping channel

The amplitude damping channel is characterized by the Kraus' operators: $K_0 = |0\rangle\langle 0| + \sqrt{p}|1\rangle\langle 1|$, $K_1 = \sqrt{1-p}|0\rangle\langle 1|$, where $p \in [0,1]$. It can be directly verified that [18]

$$\varepsilon(\rho) = \begin{pmatrix} \dfrac{1+tn_z}{2} + \dfrac{p(1-tn_z)}{2} & \dfrac{\sqrt{1-p}t(n_x - in_y)}{2} \\ \dfrac{\sqrt{1-p}t(n_x + in_y)}{2} & \dfrac{(1-p)(1-tn_z)}{2} \end{pmatrix}, \tag{10}$$

$$\mathcal{C}_{l_1}(\varepsilon(\rho)) = (1-p)t\sqrt{1-n_z^2}, \tag{11}$$

$$\mathcal{C}_r(\varepsilon(\rho)) = h\left(\frac{1+t'n_z'}{2}\right) - h\left(\frac{1+t'}{2}\right), \tag{12}$$

$$\mathcal{C}_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1}, \tag{13}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$, $r = \left[\left(\frac{1+t'}{2}\right)^\alpha \frac{1+n_z'}{2} + \left(\frac{1-t'}{2}\right)^\alpha \frac{1-n_z'}{2}\right]^{\frac{1}{\alpha}} + \left[\left(\frac{1+t'}{2}\right)^\alpha \frac{1-n_z'}{2} + \left(\frac{1-t'}{2}\right)^\alpha \frac{1+n_z'}{2}\right]^{\frac{1}{\alpha}}$, $t' = \sqrt{(1-p)t^2(1-n_z^2) + (p+(1-p)n_z t)^2}$, $n_x' = \frac{\sqrt{1-p}n_x t}{t'}$, $n_y' = \frac{\sqrt{1-p}n_y t}{t'}$, and $n_z' = \frac{p+(1-p)n_z t}{t'}$.

It is clear that for the case $p = 1$, the amplitude damping channel transforms any single-qubit state to an incoherent state. For $p = 0$, any single-qubit state is unchanged under amplitude damping channel.

It has been proved that the amplitude damping channel does not change the coherence-induced state ordering under the coherence measure $\mathcal{C}_{l_1}$ [18]. In the following we study the case $p \in (0,1)$ for the coherence measures $\mathcal{C}_r$ and $\mathcal{C}_\alpha$ for $\alpha \in (0,1)\cup(1,2]$. Numerical calculation shows that for any $p \in (0,1)$, the amplitude damping channel does not change the coherence-induced state ordering by $\mathcal{C}_r$ with fixed n_z or fixed t, since $\mathcal{C}_r$ is an increasing function with respect to t for every fixed n_z while a decreasing function with respect to n_z for every fixed t; see figures 1, 2, and 3 for the cases of $p = \frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

For the coherence measure $\mathcal{C}_\alpha$, Zhang *et al* have proved that for $p = \frac{1}{2}$, the amplitude damping channel maintains the coherence-induced state ordering with fixed n_z or fixed t. In fact, we find that it holds for any $p \in (0,1)$

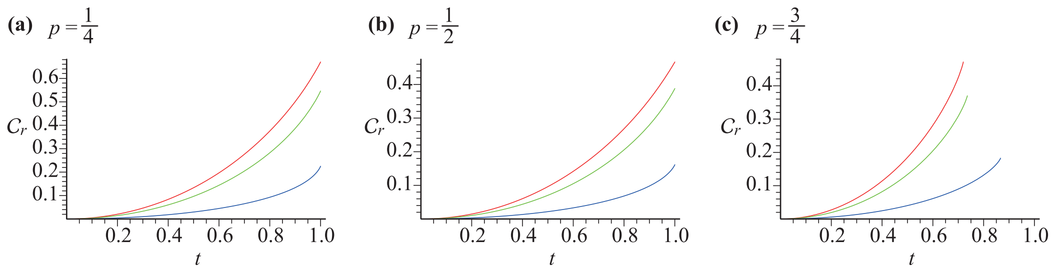

Fig. 2 For $p = \frac{1}{4}$, $p = \frac{1}{2}$ and $p = \frac{3}{4}$, $\mathcal{C}_r(\varepsilon(\rho))$ is an increasing function with respect to t for the cases $n_z = 0.3$ (red line), $n_z = 0.6$ (green line) and $n_z = 0.9$ (blue line).

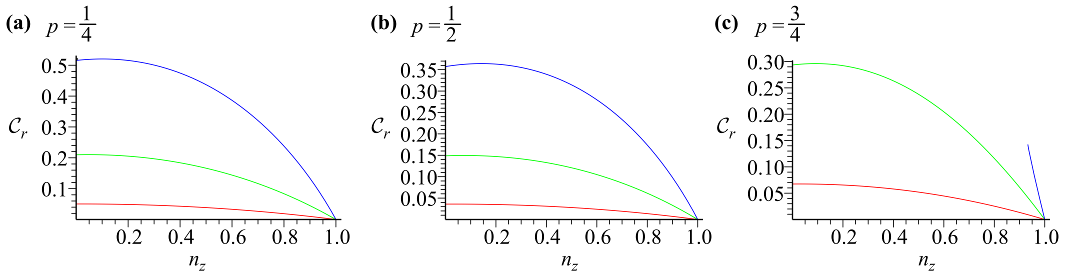

Fig. 3 For $p=\frac{1}{4}$, $p=\frac{1}{2}$, and $p=\frac{3}{4}$, $\mathcal{C}_r(\varepsilon(\rho))$ is a decreasing function with respect to n_z for the cases $t=0.3$ (red line), $t=0.6$ (green line), and $t=0.9$ (blue line).

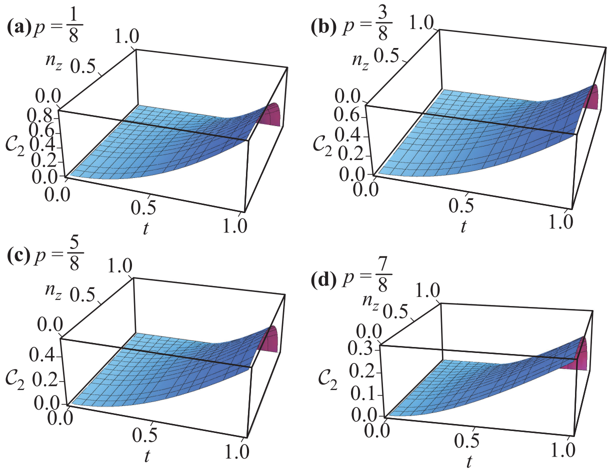

Fig. 4 The variation of $\mathcal{C}_2$ with respect to t and n_z under amplitude damping channel for $p=\frac{1}{8}$, $p=\frac{3}{8}$, $p=\frac{5}{8}$, and $p=\frac{7}{8}$.

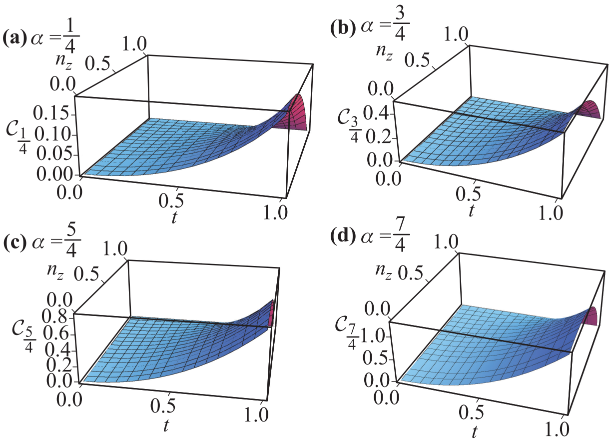

Fig. 5 For fixed $p=\frac{1}{2}$, the variation of $\mathcal{C}_\alpha$ with respect to t and n_z under amplitude damping channel for $\alpha=\frac{1}{4}$, $\alpha=\frac{3}{4}$, $\alpha=\frac{5}{4}$, and $\alpha=\frac{7}{4}$.

and $\alpha\in(0,1)\cup(1,2]$. In Fig. 4, we show the variation of $\mathcal{C}_2$ for $p=\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $p=\frac{7}{8}$. In Fig. 5, we show the variation of $\mathcal{C}_\alpha$ for fixed $p=\frac{1}{2}$ and $\alpha=\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$.

3.2 Phase damping channel

Now we study the dynamics of coherence-induced state ordering under phase damping channel, which can be characterized by the Kraus' operators $K_0=\sqrt{p}I$, $K_1=\sqrt{1-p}|0\rangle\langle 0|$, $K_2=\sqrt{1-p}|1\rangle\langle 1|$, where $0\leq p\leq 1$. By applying the phase damping channel to the state represented by Eq. (5), we get

$$\varepsilon(\rho)=\begin{pmatrix}\dfrac{1+tn_z}{2} & \dfrac{tp(n_x-in_y)}{2}\\ \dfrac{tp(n_x+in_y)}{2} & \dfrac{1-tn_z}{2}\end{pmatrix}. \tag{14}$$

For $p=0$, the phase damping channel transforms a state into an incoherent one. In the following, we study the case $p\neq 0$. For simplicity, we define $A=1+(p^2-1)(1-n_z)^2$, $B=\frac{1+t\sqrt{A}}{2}$, $C=(\sqrt{A}+n_z)^2$, and $D=p^2(1-n_z^2)$. Substituting $\varepsilon(\rho)$ into Eqs. (1), (2), and (4), we have

$$\mathcal{C}_{l_1}(\varepsilon(\rho))=pt\sqrt{1-n_z}=p\mathcal{C}_{l_1}(\rho), \tag{15}$$

$$\mathcal{C}_r(\varepsilon(\rho))=h\left(\frac{1+tn_z}{2}\right)-h(B), \tag{16}$$

$$\mathcal{C}_\alpha(\varepsilon(\rho))=\frac{r^\alpha-1}{\alpha-1}, \tag{17}$$

where $r=[B^\alpha\frac{C}{C+D}+(1-B)^\alpha\frac{D}{C+D}]^{\frac{1}{\alpha}}+[(1-B)^\alpha\frac{C}{C+D}+B^\alpha\frac{D}{C+D}]^{\frac{1}{\alpha}}$. According to Eq. (15) the phase damping channel does not change the coherence-induced state ordering by $\mathcal{C}_{l_1}$ for single-qubit states.

Next we consider the coherence measure $\mathcal{C}_r$. On differentiating Eq. (16) with respect to t, we get

$$\begin{aligned}\frac{\partial\mathcal{C}_r(\varepsilon(\rho))}{\partial t}&=\frac{n_z}{2}\log\frac{1-tn_z}{1+tn_z}+\frac{\sqrt{A}}{2}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}}\\&\geq\frac{n_z}{2}\log\frac{1-tn_z}{1+tn_z}+\frac{n_z}{2}\log\frac{1+tn_z}{1-tn_z}=0,\end{aligned}$$

since $\frac{\sqrt{A}}{2}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}}$ is an increasing function with respect to $p\in(0,1]$. Moreover, since $\frac{\partial\mathcal{C}_r(\rho)}{\partial t}\geq 0$, the phase damping channel does not change the coherence-induced state ordering by $\mathcal{C}_r$ for single-qubit states with fixed n_z. On differentiating Eq. (15) with respect to n_z, we obtain

$$\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial n_z} = \frac{t}{2}\log\frac{1-tn_z}{1+tn_z} - \frac{(p^2-1)n_z t}{2\sqrt{A}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}}.$$

Set $f(p) = \frac{p^2-1}{\sqrt{A}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}}$. Then

$$\begin{aligned} f'(p) &= \frac{p+pA}{\sqrt{A^3}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}} + \frac{2tp(1-n_z^2)(p^2-1)}{A(1-t^2A)\ln 2} \\ &\geq \frac{p+pA}{\sqrt{A^3}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}} + \frac{2tp(1-n_z^2)(p^2-1)}{A(1-A)\ln 2} \\ &= \frac{p}{A}\left(\frac{A+1}{\sqrt{A}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}} - \frac{2t}{\ln 2}\right) \geq 0, \end{aligned}$$

since $\frac{A+1}{\sqrt{A}}\log\frac{1+t\sqrt{A}}{1-t\sqrt{A}} - \frac{2t}{\ln 2}$ is an increasing function with respect to $t \geq 0$. Thus,

$$\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial n_z} \leq \frac{t}{2}\log\frac{1-tn_z}{1+tn_z} - \frac{n_z t}{2}f(0) = 0.$$

Therefore, the phase damping channel keeps the coherence-induced state ordering by $\mathcal{C}_r$ for single-qubit states with fixed t as $\frac{\partial \mathcal{C}_r(\rho)}{\partial n_z} \leq 0$.

According to Eq. (16), for the coherence measure $\mathcal{C}_\alpha$, $\alpha \in (0,1)\cup(1,2]$, we have $\frac{\partial \mathcal{C}_\alpha(\varepsilon(\rho))}{\partial t} = \frac{\alpha}{\alpha-1}r^{\alpha-1}\frac{\partial r}{\partial t}$, where

$$\begin{aligned} \frac{\partial r}{\partial t} = \frac{\sqrt{A}}{2}&\left\{\left[B^\alpha\frac{C}{C+D} + (1-B)^\alpha\frac{D}{C+D}\right]^{\frac{1}{\alpha}-1}\left[B^{\alpha-1}\frac{C}{C+D} - (1-B)^{\alpha-1}\frac{D}{C+D}\right]\right\} \\ &+\frac{\sqrt{A}}{2}\left\{\left[(1-B)^\alpha\frac{C}{C+D} + B^\alpha\frac{D}{C+D}\right]^{\frac{1}{\alpha}-1}\left[B^{\alpha-1}\frac{D}{C+D} - (1-B)^{\alpha-1}\frac{C}{C+D}\right]\right\}. \end{aligned} \tag{18}$$

If $\alpha \in (0,1)$, $\frac{\partial r}{\partial t} \leq [B^\alpha\frac{C}{C+D} + (1-B)^\alpha\frac{D}{C+D}]^{\frac{1}{\alpha}-1}(B^{\alpha-1} - (1-B)^{\alpha-1}) \leq 0$.

If $\alpha \in (1,2]$, $\frac{\partial r}{\partial t} \geq [B^\alpha\frac{C}{C+D} + (1-B)^\alpha\frac{D}{C+D}]^{\frac{1}{\alpha}-1}(B^{\alpha-1} -(1-B)^{\alpha-1}) \geq 0$.

Then $\frac{\partial \mathcal{C}_\alpha(\varepsilon(\rho))}{\partial t} \geq 0$. Since $\frac{\partial \mathcal{C}_\alpha(\rho)}{\partial t} \geq 0$, the phase damping channel does not change the coherence-induced state ordering by $\mathcal{C}_\alpha$ for single-qubit states with fixed n_z. In fact, the phase damping channel does not change the coherence-induced state ordering by $\mathcal{C}_\alpha$ for single-qubit states with fixed t. In general, it is very difficult to discuss the monotony of $\mathcal{C}_\alpha$ for all parameters $\alpha \in (0,1)\cup(1,2]$ and $p \in (0,1]$ with respect to n_z. In Fig. 6 we present the variation of $\mathcal{C}_\alpha$ with fixed $p = \frac{1}{2}$ for $\alpha = \frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\alpha = \frac{7}{4}$.

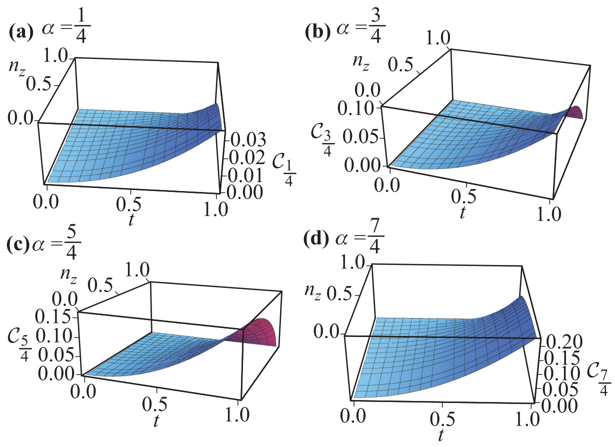

Fig. 6 The variation of $\mathcal{C}_\alpha$ with respect to t and n_z under phase damping channel for fixed $p = \frac{1}{2}$ and $\alpha = \frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\alpha = \frac{7}{4}$.

3.3 Depolarizing channel

Now we study the dynamics of coherence-induced state ordering under depolarizing channel. The state of the quantum system after depolarizing channel is given by $\varepsilon(\rho) = \frac{pI}{2} + (1-p)\rho$,

$$\varepsilon(\rho) = \begin{pmatrix} \frac{1+tn_z(1-p)}{2} & \frac{(1-p)(n_x-\mathrm{i}n_y)t}{2} \\ \frac{(1-p)(n_x+\mathrm{i}n_y)t}{2} & \frac{1-tn_z(1-p)}{2} \end{pmatrix}. \tag{19}$$

Substituting $\varepsilon(\rho)$ into Eqs. (1), (2), and (4), we have

$$\mathcal{C}_{l_1}(\varepsilon(\rho)) = (1-p)t\sqrt{1-n_z} = (1-p)\mathcal{C}_{l_1}(\rho), \tag{20}$$

$$\mathcal{C}_r(\varepsilon(\rho)) = h\left(\frac{1+tn_z(1-p)}{2}\right) - h\left(\frac{1+t(1-p)}{2}\right), \tag{21}$$

$$\mathcal{C}_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1}, \tag{22}$$

where $r = [E^\alpha F + (1-E)^\alpha(1-F)]^{\frac{1}{\alpha}} + [E^\alpha(1-F) + (1-E)^\alpha F]^{\frac{1}{\alpha}}$, and $E = \frac{1+t(1-p)}{2}$, $F = \frac{1+n_z}{2}$.

According to Eq. (20), the depolarizing channel keeps the coherence-induced state ordering under $\mathcal{C}_{l_1}$ for single-qubit states.

Next, we consider the coherence measure $\mathcal{C}_r$. $\mathcal{C}_r(\varepsilon(\rho))$ is clearly a decreasing function with respect to n_z, since $\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial n_z} = \frac{t(1-p)}{2}\log\frac{1-tn_z(1-p)}{1+tn_z(1-p)} \leq 0$. Thus, the depolarizing channel does not change the coherence-induced state ordering by $\mathcal{C}_r$ for single-qubit states with fixed t, due to the fact that $\mathcal{C}_r(\rho)$ is also a decreasing function with respect to n_z. In fact, $\mathcal{C}_r(\varepsilon(\rho))$ is an increasing function with respect to t, as

$$\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial t \partial n_z} = \frac{(1-p)}{2} \log \frac{1-tn_z(1-p)}{1+tn_z(1-p)} - \frac{t(1-p)^2 n_z}{1-t^2 n_z^2 (1-p)^2} \le 0.$$

Therefore,

$$\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial t} = \frac{(1-p)n_z}{2} \log \frac{1-tn_z(1-p)}{1+tn_z(1-p)} + \frac{1-p}{2} \log \frac{1+t(1-p)}{1-t(1-p)} \ge 0.$$

In addition, $\mathcal{C}_r(\rho)$ is also an increasing function with respect to t [18]. Thus, the depolarizing channel does not change the coherence-induced state ordering by $\mathcal{C}_r$ for single-qubit states with fixed n_z.

In the end, we consider the coherence measure $\mathcal{C}_\alpha$, where $\alpha \in (0,1) \cup (1,2]$. First of all, we show $\frac{\partial r}{\partial t} \ge 0$ if $\alpha \in (1,2]$ and $\frac{\partial r}{\partial t} \le 0$ if $\alpha \in (0,1)$. We have

$$\frac{\partial r}{\partial t} = \frac{1-p}{2} \{[E^\alpha F + (1-E)^\alpha (1-F)]^{\frac{1}{\alpha}-1}[E^{\alpha-1}F - (1-E)^{\alpha-1}(1-F)] + [E^\alpha(1-F) + (1-E)^\alpha F]^{\frac{1}{\alpha}-1}[E^{\alpha-1}(1-F) - (1-E)^{\alpha-1}F]\}. \tag{23}$$

Since $x^\alpha y + (1-x)^\alpha (1-y) \ge x^\alpha (1-y) + (1-x)^\alpha y$, where $\alpha > 0, \frac{1}{2} \le x, y \le 1$, we have

$$\frac{\partial r}{\partial t} \ge \frac{1-p}{2}[E^\alpha F + (1-E)^\alpha (1-F)]^{\frac{1}{\alpha}-1}[E^{\alpha-1} - (1-E)^{\alpha-1}] \ge 0 \tag{24}$$

if $\alpha \in (1,2]$. If $\alpha \in (0,1)$, we have

$$\frac{\partial r}{\partial t} \le \frac{1-p}{2}[E^\alpha F + (1-E)^\alpha (1-F)]^{\frac{1}{\alpha}-1}[E^{\alpha-1} - (1-E)^{\alpha-1}] \le 0. \tag{25}$$

Thus, $\frac{\partial \mathcal{C}_\alpha(\varepsilon(\rho))}{\partial t} = \frac{\alpha r^{\alpha-1}}{\alpha-1} \frac{\partial r}{\partial t} \ge 0$. Since $\frac{\partial \mathcal{C}_\alpha(\rho)}{\partial t} \ge 0$, we arrive at the conclusion that the depolarizing channel does not change the coherence-induced state ordering by $\mathcal{C}_\alpha$ for single-qubit states with fixed n_z.

On the other hand, as

$$\frac{\partial r}{\partial n_z} = \frac{1}{2\alpha}[E^\alpha - (1-E)^\alpha]\{[E^{\alpha-1}F - (1-E)^{\frac{1}{\alpha}-1}(1-F)]^{\frac{1}{\alpha}-1} - [E^\alpha(1-F) + (1-E)^\alpha F]^{\frac{1}{\alpha}-1}\}$$

and $x^\alpha y - (1-x)^\alpha (1-y) \ge x^\alpha (1-y) - (1-x)^\alpha y$ for $\alpha \ge 0$ and $\frac{1}{2} \le x, y \le 1$, one has $\frac{\partial \mathcal{C}_\alpha(\varepsilon(\rho))}{\partial n_z} \le 0$. Also, $\frac{\partial \mathcal{C}_\alpha(\rho)}{\partial n_z} \le 0$, therefore, we conclude that the depolarizing channel keeps the coherence-induced state ordering by $\mathcal{C}_\alpha$ for single-qubit states with fixed t.

3.4 Bit flit channel

Now we study the dynamics of coherence-induced state ordering under bit flit channel, which can be characterized by the Kraus' operators $K_0 = \sqrt{p}I$, $K_1 = \sqrt{1-p}\sigma_x$, where $0 \le p \le 1$. Applying the bit flit channel to the state represented by Eq. (5), we get

$$\varepsilon(\rho) = \begin{pmatrix} \frac{1+tn_z(2p-1)}{2} & \frac{tn_x - itn_y(2p-1)}{2} \\ \frac{tn_x + itn_y(2p-1)}{2} & \frac{1-tn_z(2p-1)}{2} \end{pmatrix}. \tag{26}$$

Substituting this $\varepsilon(\rho)$ into Eqs. (1), (2), and (4), we have

$$\mathcal{C}_{l_1}(\varepsilon(\rho)) = \sqrt{t^2 n_x^2 + (2p-1)^2 t^2 n_y^2}$$

$$= \sqrt{(2p-1)^2 \mathcal{C}_{l_1}^2(\rho) + 4(p-p^2)t^2 n_x^2}, \tag{27}$$

$$\mathcal{C}_r(\varepsilon(\rho)) = h(\frac{1+tn_z(2p-1)}{2}) - h(H), \tag{28}$$

$$\mathcal{C}_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1}, \tag{29}$$

where $r = [H^\alpha \frac{M}{M+N} + (1-H)^\alpha \frac{N}{M+N}]^{\frac{1}{\alpha}} + [(1-H)^\alpha \frac{M}{M+N} + H^\alpha \frac{N}{M+N}]^{\frac{1}{\alpha}}$, and $G = \sqrt{1+4(p^2-p)(1-n_x^2)}$, $H = \frac{1+t\sqrt{G}}{2}$, $M = n_x^2 + (2p-1)^2 n_y^2$ and $N = (\sqrt{G} - (2p-1)n_z)^2$.

Let us consider the special case $p = \frac{1}{2}$. Thus,

$$\mathcal{C}_{l_1}(\varepsilon(\rho)) = tn_x, \tag{30}$$

$$\mathcal{C}_r(\varepsilon(\rho)) = 1 - h\left(\frac{1+tn_x}{2}\right), \tag{31}$$

$$\mathcal{C}_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1}, \tag{32}$$

where $r = 2[\frac{1}{2}(\frac{1+tn_x}{2})^\alpha + \frac{1}{2}(\frac{1-tn_x}{2})^\alpha]^{\frac{1}{\alpha}}$. Hence, $\frac{\partial \mathcal{C}_{l_1}(\varepsilon(\rho))}{\partial n_x} = t \ge 0$, $\frac{\partial \mathcal{C}_r(\varepsilon(\rho))}{\partial n_x} = \frac{t}{2} \log \frac{1+tn_x}{1-tn_x} \ge 0$ and

$$\frac{\partial \mathcal{C}_\alpha(\varepsilon(\rho))}{\partial n_x} = \frac{\alpha t}{2(\alpha-1)} r^{\alpha-1} \left[\frac{1}{2}\left(\frac{1+tn_x}{2}\right)^{\alpha} + \frac{1}{2}\left(\frac{1-tn_x}{2}\right)^{\alpha}\right]^{\frac{1}{\alpha}-1} \left[\left(\frac{1+tn_x}{2}\right)^{\alpha-1} - \left(\frac{1-tn_x}{2}\right)^{\alpha-1}\right] \geq 0.$$

Let $\rho_1 = \frac{1}{2}(I + t_1 n_1 \sigma)$ and $\rho_2 = \frac{1}{2}(I + t_2 n_2 \sigma)$, where $n_1 = (n_{1x}, n_{1y}, n_{1z})$, $n_2 = (n_{2x}, n_{2y}, n_{2z})$. Assume $t_1 = t_2$, $n_{1x} < n_{2x}$ and $n_{1z} < n_{2z}$. Then we find that $\mathcal{C}_{l_1}(\rho_1) > \mathcal{C}_{l_1}(\rho_2), \mathcal{C}_{l_1}(\varepsilon(\rho_1)) < \mathcal{C}_{l_1}(\varepsilon(\rho_2))$, $\mathcal{C}_r(\rho_1) > \mathcal{C}_r(\rho_2), \mathcal{C}_r(\varepsilon(\rho_1)) < \mathcal{C}_r(\varepsilon(\rho_2))$, $\mathcal{C}_\alpha(\rho_1) > \mathcal{C}_\alpha(\rho_2)$, and $\mathcal{C}_\alpha(\varepsilon(\rho_1)) < \mathcal{C}_\alpha(\varepsilon(\rho_2))$. Thus, the bit flit channel changes the coherence-induced state ordering by the coherence measures $\mathcal{C}_{l_1}$, $\mathcal{C}_r$, and $\mathcal{C}_\alpha$ for single-qubit states with fixed t, where $\alpha \in (0,1) \cup (1,2]$.

Now assume $t_1 > t_2$, $n_{1x} < n_{2x}$ and $n_{1z} = n_{2z}$ such that $t_1 n_{1x} < t_2 n_{2x}$. Then we find that $\mathcal{C}_{l_1}(\rho_1) > \mathcal{C}_{l_1}(\rho_2), \mathcal{C}_{l_1}(\varepsilon(\rho_1)) < \mathcal{C}_{l_1}(\varepsilon(\rho_2))$, $\mathcal{C}_r(\rho_1) > \mathcal{C}_r(\rho_2)$, $\mathcal{C}_r(\varepsilon(\rho_1)) < \mathcal{C}_r(\varepsilon(\rho_2))$, $\mathcal{C}_\alpha(\rho_1) > \mathcal{C}_r(\rho_2)$ and $\mathcal{C}_\alpha(\varepsilon(\rho_1)) < \mathcal{C}_r(\varepsilon(\rho_2))$, since the coherence measures $\mathcal{C}_{l_1}$, $\mathcal{C}_r$ and $\mathcal{C}_\alpha$ are all increasing functions with respect to tn_x. Thus, bit flit channel changes the coherence-induced state ordering by the coherence measures $\mathcal{C}_{l_1}$, $\mathcal{C}_r$ and $\mathcal{C}_\alpha$ for single-qubit states with fixed n_z, where $\alpha \in (0,1) \cup (1,2]$.

4 Conclusion

We have discussed whether or not a quantum channel changes the coherence-induced state ordering, for four specific Markovian channels: amplitude damping channel, phase flit channel, depolarizing channel, and bit flit channel. We have shown that the depolarizing channel does not change the coherence-induced state ordering by $\mathcal{C}_{l_1}$, $\mathcal{C}_r$, $\mathcal{C}_\alpha$, and $\mathcal{C}_g$. For the bit flit channel, we have shown that it does change the coherence-induced state ordering under these four coherence measures for the case of $p = \frac{1}{2}$. Our results enrich the understanding of coherence-induced state ordering under quantum channels.

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