

# Uncertainty relations for quantum coherence with respect to mutually unbiased bases

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The concept of quantum coherence, including various ways to quantify the degree of coherence with respect to the prescribed basis, is currently the subject of active research. The complementarity of quantum coherence in different bases was studied by deriving upper bounds on the sum of the corresponding measures. To obtain a two-sided estimate, lower bounds on the coherence quantifiers are also of interest. Such bounds are naturally referred to as uncertainty relations for quantum coherence. We obtain new uncertainty relations for coherence quantifiers averaged with respect to a set of mutually unbiased bases (MUBs). To quantify the degree of coherence, the relative entropy of coherence and the geometric coherence are used. Further, we also derive novel state-independent uncertainty relations for a set of MUBs in terms of the min-entropy.

**Keywords** coherence, complementarity, uncertainty, mutually unbiased bases

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## 1 Introduction

The principle of state superposition is one of the cornerstones of quantum theory. It is closely related to the problem of properly quantifying coherence at the quantum level, which has attracted considerable attention [1, 2]. This line of study is part of more general efforts to understand the strengths and limitations of nonclassical correlations [3]. In principle, a given quantum state can be represented with respect to an arbitrary basis. However, only a few bases may actually be preferred for fundamental physical reasons. When we deal with the measurement problem, the existence of some preferable basis is one of the principal questions to be resolved [4]. Studies of the thermodynamic properties of small systems at low temperatures also imply the use of a quite concrete representation of possible mixed states [5–7]. Analysis of protocols and algorithms for quantum information processing assumes that transformations of quantum carriers will be related to the prescribed orthonormal bases. Thus, the role of quantum coherence should be understood in order to realize efficient quantum computation. Results of recent studies [8, 9] have supported this opin-

ion. Duality relations between the coherence and path information were examined in Refs. [10, 11].

Some complementarity relations for quantum coherence were studied in Ref. [12]. They claimed upper bounds on the sum of coherence quantifiers taken with respect to mutually unbiased bases (MUBs). Uncertainty bounds for coherence can be obtained using previously obtained uncertainty relations of the usual form. Even in the case of conjugate variables, uncertainty relations still attract attention [13–15]. Basic developments within the entropic approach to quantum uncertainty are reviewed in [16–18]. The famous Maassen–Uffink result [19] has been applied in many problems including entropic relations in the presence of quantum memory [20]. These results were used in deriving uncertainty relations for quantum coherence [21–23]. In this work, we obtain new uncertainty bounds for the averaged quantum coherence taken with respect to MUBs. The relevant entropic uncertainty relations will be mentioned in appropriate places in the text. The novelty of our results is threefold. First, lower bounds on coherence measures only for MUBs had not been addressed separately. Second, we obtain some uncertainty relations for the averaged geometric coherence. Third, we present new state-independent uncertainty relations for a set of MUBs in terms of the averaged min-entropy.

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## 2 Preliminaries

Let  $\mathcal{L}(\mathcal{H})$  be the space of linear operators on a  $d$ -dimensional Hilbert space  $\mathcal{H}$ . The set of positive semidefinite operators will be denoted by  $\mathcal{L}_+(\mathcal{H})$ . By  $\text{ran}(X)$ , we mean the range of operator  $X$ . For each  $X \in \mathcal{L}(\mathcal{H})$ , we define  $|X| \in \mathcal{L}_+(\mathcal{H})$  as the unique square root of  $X^\dagger X$ . The eigenvalues of  $|X|$  counted with multiplicities are the singular values  $\sigma_i(X)$  of  $X$ . We will further refer to the following two norms,

$$\|X\|_1 := \sum_{i=1}^d \sigma_i(X), \quad (1)$$

$$\|X\|_\infty := \max\{\sigma_i(X) : 1 \leq i \leq d\}, \quad (2)$$

which are known as the trace and spectral norms, respectively. The state of a quantum system is described by the density matrix  $\rho \in \mathcal{L}_+(\mathcal{H})$  normalized as  $\text{tr}(\rho) = 1$ . Then the von Neumann entropy of  $\rho$  is defined as

$$S_1(\rho) := -\text{tr}(\rho \ln \rho). \quad (3)$$

Let us consider orthonormal bases  $\mathcal{B} = \{|b_i\rangle\}$  and  $\mathcal{B}' = \{|b'_j\rangle\}$  in the  $d$ -dimensional space  $\mathcal{H}$ . They are said to be mutually unbiased if and only if for all  $i$  and  $j$ ,

$$|\langle b_i | b'_j \rangle| = \frac{1}{\sqrt{d}}. \quad (4)$$

Several orthonormal bases form a set of MUBs when each pair in them is mutually unbiased. MUBs are applied in many quantum information problems. They also form a structure with interesting properties and links to discrete mathematics. As even a listing of all such topics may be incomplete, we refer to the literature (see Ref. [24] and references therein). In general, the maximal number of MUBs in  $d$  dimensions is an open problem. For a prime power  $d$ , we can certainly construct  $d + 1$  MUBs.

A rigorous framework for the quantification of coherence was developed in Ref. [1]. To the given orthonormal basis  $\mathcal{B} = \{|b_i\rangle\}$ , we assign the set of all density matrices that are diagonal in this basis,

$$\delta = \sum_{i=1}^d \delta_i |b_i\rangle\langle b_i|. \quad (5)$$

Density matrices of the form (5) constitute the set  $\mathcal{I}_{\mathcal{B}}$  of states incoherent with respect to  $\mathcal{B}$ . Keeping in mind the chosen basis, we ask how far the given state is from those states that are completely incoherent in this basis. The authors of [1] listed general conditions for quantifiers of coherence. Additional conditions imposed on coherence measures were considered in Ref. [2]. The imposed conditions allow us to identify classes of functionals that can

be used as proper coherence measures. In this paper, we will use the relative entropy of coherence and the geometric coherence.

The concept of quantum relative entropy, or divergence, is basic in quantum information theory [25, 26]. For density matrices  $\rho$  and  $\omega$ , this quantity is expressed as [25]

$$D_1(\rho||\omega) := \begin{cases} \text{tr}(\rho \ln \rho - \rho \ln \omega), & \text{if } \text{ran}(\rho) \subseteq \text{ran}(\omega), \\ +\infty, & \text{otherwise.} \end{cases} \quad (6)$$

Although the relative entropy cannot be treated as a metric, it provides a very natural measure of distinguishability of quantum states. Following [1], we define the coherence measure

$$C_1(\mathcal{B}|\rho) := \min_{\delta \in \mathcal{I}_{\mathcal{B}}} D_1(\rho||\delta). \quad (7)$$

The minimization is easy and results in the formula [1]

$$C_1(\mathcal{B}|\rho) = S_1(\rho_{\text{diag}}) - S_1(\rho), \quad (8)$$

where the diagonal state

$$\rho_{\text{diag}} := \text{diag}(\langle b_1 | \rho | b_1 \rangle, \dots, \langle b_d | \rho | b_d \rangle). \quad (9)$$

We can represent  $S(\rho_{\text{diag}})$  as the Shannon entropy calculated with probabilities  $p_i(\mathcal{B}|\rho) = \langle b_i | \rho | b_i \rangle$ , namely,

$$S_1(\rho_{\text{diag}}) = H_1(\mathcal{B}|\rho) := -\sum_{i=1}^d p_i(\mathcal{B}|\rho) \ln p_i(\mathcal{B}|\rho). \quad (10)$$

For the general properties of (7), see the relevant sections in Refs. [1, 2]. It seems that the relative entropy of coherence is the most justifiable measure. Together with (6), other quantum divergences were considered, including the quasi-entropies of Petz [27]. It is for this reason that we designate the considered entropic quantities by the subscript 1. Coherence quantifiers induced by quantum divergences of the Tsallis type were addressed in Ref. [28]. It was shown that such quantifiers do not support a simple form similar to (8). Coherence monotones based on Rényi divergences were considered in Refs. [29–31].

There are also several distance-based quantifiers of coherence [1, 2]. We will use the geometric coherence, which was introduced in terms of the quantum fidelity [32, 33]. The fidelity of density matrices  $\rho$  and  $\omega$  is expressed as

$$F(\rho, \omega) = \|\sqrt{\rho}\sqrt{\omega}\|_1^2. \quad (11)$$

This definition follows Jozsa [33]. Another way is to set the fidelity as the square root of (11) [25]. The fidelity

ranges between 0 and 1, taking the value 1 for two identical states. Hence, the difference  $1 - F(\rho, \omega)$  can be treated as a distance measure. Strictly speaking, a legitimate metric is obtained after the root of this difference is extracted and is usually called the sine distance [34, 35]. Following the general approach, the geometric coherence of  $\rho$  with respect to the basis  $\mathcal{B}$  is defined as [2]

$$C_g(\mathcal{B}|\rho) := 1 - \max_{\delta \in \mathcal{L}_{\mathcal{B}}} F(\rho, \delta). \tag{12}$$

For the properties of this coherence quantifier, see Section III.C.3 of Ref. [2]. For a pure state, the geometric coherence has a very convenient form:

$$C_g(\mathcal{B}|\psi) = 1 - \max_i |b_i|\psi|^2. \tag{13}$$

For impure states, we have only a two-sided estimate on the geometric coherence. For the given density matrix  $\rho$  and orthonormal basis  $\mathcal{B}$ , the index of coincidence is introduced as

$$J(\mathcal{B}|\rho) := \sum_{i=1}^d p_i(\mathcal{B}|\rho)^2, \tag{14}$$

where  $p_i(\mathcal{B}|\rho) = \langle b_i|\rho|b_i \rangle$ . The authors of Ref. [36] have proved that

$$\frac{d-1}{d} \left\{ 1 - \sqrt{1 + \frac{d}{d-1} [J(\mathcal{B}|\rho) - \text{tr}(\rho^2)]} \right\} \leq C_g(\mathcal{B}|\rho) \leq 1 - \max_i p_i(\mathcal{B}|\rho). \tag{15}$$

This two-sided estimate is based on the concept of sub- and super-fidelities proposed in Ref. [37]. It seems that the quantifier (12) deserves to be studied more widely. For this reason, we will formulate uncertainty relations in terms of the geometric coherence.

Additionally, we recall some results concerning special forms of the uncertainty relations. Two of our relations are based on the following inequality derived for  $M$  MUBs in Ref. [38]:

$$\sum_{t=1}^M J(\mathcal{B}_t|\rho) \leq \text{tr}(\rho^2) + \frac{M-1}{d} \leq 1 + \frac{M-1}{d}. \tag{16}$$

Assuming the existence of  $d+1$  MUBs, one can show that inequality (16) is saturated here. The authors of [38] further used (16) to obtain uncertainty relations in terms of the Shannon entropy. In Ref. [39], we extended this approach to the Rényi and Tsallis entropies.

Let us recall also one result of Ref. [40]. Any quantum measurement can be described by the set  $\mathcal{A}$  of elements  $A_j \in \mathcal{L}_+(\mathcal{H})$  such that the completeness relation holds:

$$\sum_j A_j = \mathbf{1}. \tag{17}$$

The set  $\mathcal{A}$  is a positive operator-valued measure (POVM) [25]. We do not specify the range of summation in (17), as the number of different outcomes in a POVM measurement can exceed the dimensionality  $d$ . The trace  $\text{tr}(A_j\rho)$  gives the probability of the  $j$ -th outcome. Let  $\{\mathcal{A}_1, \dots, \mathcal{A}_M\}$  be a set of  $M$  POVMs, and let some index  $j(t)$  be assigned to each  $t = 1, \dots, M$ . For an arbitrary state  $\rho$ , it holds that [40]

$$\sum_{t=1}^M p_{j(t)}(\mathcal{A}_t|\rho) \leq 1 + \left( \sum_{s \neq t} \left\| \sqrt{A_{j(s)}^{(s)}} \sqrt{A_{j(t)}^{(t)}} \right\|_{\infty}^2 \right)^{1/2}. \tag{18}$$

This upper bound leads to uncertainty relations of the Landau–Pollak type for more than two measurements. Indeed, we can choose the indices  $j(t)$  so that the left-hand side of (18) will include the maximal probability for each measurement. Formula (18) provides a nontrivial bound for  $M \geq 2$ . Further studies of the Landau–Pollak uncertainty relations for POVM measurements were reported in Ref. [41].

### 3 Main results

We are now ready to formulate the uncertainty relations for quantum coherence taken with respect to a set of MUBs. Let us begin the presentation with lower bounds on the averaged relative entropy of coherence. The following statement is presented.

**Proposition 1** *Let  $\mathbb{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_M\}$  be a set of MUBs in the  $d$ -dimensional Hilbert space  $\mathcal{H}$ . For any state  $\rho$ , the averaged relative entropy of coherence obeys*

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_1(\mathcal{B}|\rho) \geq \ln \left( \frac{Md}{\text{tr}(\rho^2)d + M - 1} \right) - S_1(\rho). \tag{19}$$

**Proof.** According to (8), the left-hand side of (19) is represented as

$$\begin{aligned} & -S_1(\rho) + \frac{1}{M} \sum_{t=1}^M H_1(\mathcal{B}_t|\rho) \geq \\ & -S_1(\rho) + \ln \left( \frac{Md}{\text{tr}(\rho^2)d + M - 1} \right). \end{aligned} \tag{20}$$

Here, we used the lower bound on the averaged Shannon entropy proved in Refs. [38, 39]. ■

The statement of Proposition 1 provides a state-dependent lower bound on the averaged relative entropy of coherence taken with respect to a set of MUBs. For any pure state  $|\psi\rangle$ , it reduces to

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_1(\mathcal{B}|\psi) \geq \ln \left( \frac{Md}{d + M - 1} \right). \tag{21}$$

If  $d$  is a prime power, we certainly have  $d + 1$  MUBs. In this case, the lower bound (19) reads as

$$\frac{1}{d+1} \sum_{t=1}^{d+1} C_1(\mathcal{B}_t|\rho) \geq \ln \left( \frac{d+1}{\text{tr}(\rho^2)+1} \right) - S_1(\rho). \quad (22)$$

It is essential that arguments of the logarithm in (19) and (22) depend on the purity  $\text{tr}(\rho^2)$ . For the completely mixed state  $\rho_* = \mathbf{1}/d$ , these bounds are obviously saturated.

The authors of Refs. [21, 22] derived uncertainty relations for the relative entropy of coherence taken with respect to different bases. These results follow from the entropic uncertainty relation obtained in Ref. [42]. The method of that paper was inspired in turn by the relative entropy approach to entropic uncertainties proposed in Ref. [43]. The authors of [23] dealt mainly with relations for the relative entropy of coherence in two measurement bases. In the qubit case, they also considered the uncertainty bounds on the  $\ell_1$ -norm of coherence. To justify the significance of relations of the form (19), we should compare them with the results of Refs. [21, 22]. Let us begin with the case of pure states. For the averaged coherence, the corresponding inequality of Refs. [21, 22] can be written as

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_1(\mathcal{B}|\psi) \geq -\frac{\ln m(\mathbb{B})}{M}. \quad (23)$$

The principal quantity  $m(\mathbb{B})$  should be found by solving a certain problem of two-stage maximization [42]. This sufficiently difficult problem is essentially simplified for MUBs. We then deal with a quantity obtained from the multiple sum with  $(M - 2)$  indices, each of which runs  $d$  different values. The factor to be added is independent of these indices and equal to  $d^{1-M}$ . Using (23) for a set of MUBs, we substitute

$$m(\mathbb{B}) = d^{M-2} d^{1-M} = \frac{1}{d}. \quad (24)$$

Thus, the right-hand side of (23) becomes  $\ln d/M$ . The latter may compete with (21) only when the number  $M$  is sufficiently small compared with  $d$ . For a prime power  $d$ , the maximal number of MUBs is certainly equal to  $d + 1$ . For  $M = d + 1$ , the right-hand side of (21) is  $\ln(d + 1) - \ln 2$ , whereas the right-hand side of (23) is  $\ln d/(d + 1)$ . To each  $d$ , we can assign some value  $M_1$  such that for all allowed  $M \geq M_1$ , the result (21) is stronger than (23). For the given  $M_1$ , we actually have the corresponding interval of dimensionality. Table 1 lists such intervals for  $M_1 = 3, 4, 5, 6, 7$ . Thus, our uncertainty relations for quantum coherence are relevant, at least for pure states.

Our second result concerns the uncertainty relations for the geometric coherence taken with respect to a set of MUBs. Lower bounds on the averaged geometric coherence of a pure state are posed as follows.

**Table 1** Intervals of values of  $d$  for which formula (21) gives a stronger bound.

| $M_1 = 3$   | $M_1 = 4$     | $M_1 = 5$       | $M_1 = 6$         | $M_1 = 7$           |
|-------------|---------------|-----------------|-------------------|---------------------|
| $2 \div 20$ | $21 \div 243$ | $244 \div 3104$ | $3105 \div 46625$ | $46626 \div 823500$ |

**Proposition 2** Let  $\mathbb{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_M\}$  be a set of MUBs in the  $d$ -dimensional Hilbert space  $\mathcal{H}$ . For any state  $\rho$ , the averaged geometric coherence obeys

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_g(\mathcal{B}|\rho) \geq \frac{d-1}{d} - \frac{\sqrt{d-1}}{d\sqrt{M}} \sqrt{Md-1-(Md-d)\text{tr}(\rho^2)}. \quad (25)$$

For a pure state  $|\psi\rangle$ , we also have

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_g(\mathcal{B}|\psi) \geq 1 - \frac{1}{M} \left( 1 + \sqrt{\frac{M^2 - M}{d}} \right). \quad (26)$$

**Proof.** The left-hand side of (15) is a convex and decreasing function of  $J(\mathcal{B}|\rho)$ . By Jensen's inequality, we obtain

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_g(\mathcal{B}|\rho) \geq \frac{d-1}{d} \left\{ 1 - \sqrt{1 - \frac{\text{tr}(\rho^2)d}{d-1} + \frac{d}{(d-1)M} \sum_{t=1}^M J(\mathcal{B}_t|\rho)} \right\}. \quad (27)$$

Combining (16) with (27) finally gives (25).

Let us prove the claim (26). It follows from (13) that

$$\frac{1}{M} \sum_{t=1}^M C_g(\mathcal{B}_t|\psi) = 1 - \frac{1}{M} \sum_{t=1}^M p_{\max}(\mathcal{B}_t|\psi), \quad (28)$$

where  $p_{\max}(\mathcal{B}|\psi) := \max\{p_i(\mathcal{B}|\psi) : 1 \leq i \leq d\}$ . The sum of the maximal probabilities on the right-hand side of (28) can be estimated by applying (18) to the case of MUBs. The right-hand side of (18) takes a simple form, so

$$\sum_{t=1}^M p_{\max}(\mathcal{B}_t|\psi) \leq 1 + \sqrt{\frac{M^2 - M}{d}}. \quad (29)$$

Combining (28) with (29) completes the proof of (26). ■

The statement of Proposition 2 gives lower bounds on the geometric coherence averaged with respect to several MUBs. For pure states, we have arrived at two different formulas. Indeed, the relation (25) then reduces to

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} C_g(\mathcal{B}|\psi) \geq \frac{d-1}{d} \left( 1 - \frac{1}{\sqrt{M}} \right). \quad (30)$$

Inspection reveals that neither of the lower bounds [(26) and (30)] should be disregarded. For the prime power  $d$ , we can use up to  $d + 1$  MUBs. It can be checked that

$$\frac{1}{d} \left( 1 + \frac{d-1}{\sqrt{d+1}} \right) < \frac{1}{d+1} + \frac{1}{\sqrt{d+1}}.$$

Here, the lower bound (30) is stronger. The same fact is obvious for  $M = d$ , when  $d$  MUBs exist in  $d$  dimensions. For sufficiently small  $M$ , however, the lower bound (26) is better than (30). Such behavior was already mentioned in the context of the separability conditions in terms of the maximal probabilities [44]. On the other hand, the distinctions between the lower bounds (26) and (30) are relatively small for all  $M$  that could be allowed here.

At this stage, we ask whether inequality (29) can lead to some analog of (19). The answer is positive in the sense that state-independent lower bounds on the averaged relative entropy of coherence actually follow from (29). To obtain such bounds, we use convexity and the decrease of the function  $x \mapsto -\ln x$ , together with  $H_1(\mathcal{B}|\rho) \geq -\ln J(\mathcal{B}|\rho)$  and  $J(\mathcal{B}|\rho) \leq p_{\max}(\mathcal{B}|\rho)$ . Hence, we obtain

$$\begin{aligned} \frac{1}{M} \sum_{t=1}^M H_1(\mathcal{B}_t|\rho) &\geq -\ln \left( \frac{1}{M} \sum_t J_t \right) \\ &\geq \ln \left( \frac{M\sqrt{d}}{\sqrt{d} + \sqrt{M^2 - M}} \right). \end{aligned} \quad (31)$$

Simple calculations show that the right-hand side of (31) is less than the right-hand side of (21). This result is evidence that inequality (29) is especially useful in estimates directly expressed via the maximal probabilities. To illustrate this conclusion, we will obtain new uncertainty relations for MUBs in terms of the averaged min-entropy.

By taking the maximum among the probabilities  $p_i(\mathcal{B}|\rho)$ , the min-entropy is defined as

$$H_\infty(\mathcal{B}|\rho) := -\ln p_{\max}(\mathcal{B}|\rho). \quad (32)$$

The notation used here reflects the fact that (32) is a particular case of the Rényi  $\alpha$ -entropy [18]. Combining (29) with convexity and the decrease of the function  $x \mapsto -\ln x$ , we claim the following.

**Proposition 3** *Let  $\mathbb{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_M\}$  be a set of MUBs in the  $d$ -dimensional Hilbert space  $\mathcal{H}$ . For any state  $\rho$ , the averaged min-entropy obeys*

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} H_\infty(\mathcal{B}|\rho) \geq \ln \left( \frac{M\sqrt{d}}{\sqrt{d} + \sqrt{M^2 - M}} \right). \quad (33)$$

This new form of the uncertainty relations for MUBs should be compared with the state-independent relation

based on (16) and lemma 3 of [39]. Namely, we have [39]

$$\frac{1}{M} \sum_{\mathcal{B} \in \mathbb{B}} H_\infty(\mathcal{B}|\rho) \geq \ln \left( \frac{\sqrt{Md}}{d + \sqrt{M} - 1} \right). \quad (34)$$

If we are interested in state-independent lower bounds that hold for all states, then formula (33) can overcome (34). In general, the competition between (33) and (34) is similar to that between relation (26) and (30). For sufficiently small  $M$ , lower bound (33) is stronger.

## 4 Conclusions

Using the relative entropy of coherence and the geometric coherence, we obtained the uncertainty relations for the averaged quantifiers taken with respect to several MUBs. The uncertainty relations for the averaged relative entropy of coherence are state-dependent via the purity and the von Neumann entropy calculated with the given state. Further, the derived lower bounds are explicitly expressed in terms of the number of MUBs and the dimensionality. The presented bounds differ from the uncertainty relations for the relative entropy of coherence derived in Refs. [21, 22]. The authors of Refs. [21, 22] used the entropic uncertainty relations obtained in Ref. [42]. The method of Ref. [42] leads to the appearance of the entropic bound as a result of solving a certain maximization problem. For MUBs, this general lower bound is not always better than the bounds derived just for such bases. The uncertainty relations for the averaged geometric coherence were also presented. We examined two lower bounds, neither of which overcomes the other everywhere. To the best of our knowledge, the uncertainty relations for the geometric coherence have not been addressed previously. Finally, we considered novel state-independent uncertainty relations for several MUBs in terms of the averaged min-entropy.

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