

The zero active mass condition in Friedmann–Robertson–Walker cosmologies

Fulvio Melia[†]

*Department of Physics, The Applied Math Program, and Department of Astronomy,
The University of Arizona, AZ 85721, USA*

[†]*John Woodruff Simpson Fellow. E-mail: fmelia@email.arizona.edu*

Received July 4, 2016; accepted August 2, 2016

Many cosmological measurements today suggest that the Universe is expanding at a constant rate. This is inferred from the observed age versus redshift relationship and various distance indicators, all of which point to a cosmic equation of state (EoS) $p = -\rho/3$, where ρ and p are, respectively, the total energy density and pressure of the cosmic fluid. It has recently been shown that this result is not a coincidence and simply confirms the fact that the symmetries in the Friedmann–Robertson–Walker (FRW) metric appear to be viable only for a medium with zero active mass, i.e., $\rho + 3p = 0$. In their latest paper, however, Kim, Lasenby and Hobson (2016) have provided what they believe to be a counter argument to this conclusion. Here, we show that these authors are merely repeating the conventional mistake of incorrectly placing the observer *simultaneously* in a comoving frame, where the lapse function g_{tt} is coordinate dependent when $\rho + 3p \neq 0$, and a supposedly different, free-falling frame, in which $g_{tt} = 1$, implying no time dilation. We demonstrate that the Hubble flow is not inertial when $\rho + 3p \neq 0$, so the comoving frame is generally not in free fall, even though in FRW, the comoving and free-falling frames are supposed to be identical at every spacetime point. So this confusion of frames not only constitutes an inconsistency with the fundamental tenets of general relativity but, additionally, there is no possibility of using a gauge transformation to select a set of coordinates for which $g_{tt} = 1$ when $\rho + 3p \neq 0$.

Keywords cosmology, gravitation, spacetime metric

PACS numbers 04.20.Ex, 95.36.+x, 98.80.-k, 98.80.Jk

1 Introduction

The standard model of cosmology (Λ CDM) is quite successful in accounting for many observations, in part due to its rather large number of free parameters. These include the spatial curvature constant k , the Hubble constant H_0 and the scaled matter (Ω_m), radiation (Ω_r), and dark-energy (Ω_{de}) densities. In terms of the critical density $\rho_c \equiv 3c^2 H_0^2 / 8\pi G$, one writes $\Omega_i \equiv \rho_i / \rho_c$, where i represents the energy density ρ_m , ρ_r , or dark energy ρ_{de} , as the case may be. Without invoking priors, one must typically optimize at least five unspecified parameters, all of which can be adjusted to fit the data.

Given this wide latitude of possible outcomes for the expansion history of the Universe, it is therefore very surprising to see that the optimization of model parameters,

especially using anisotropies in the cosmic microwave background (CMB) [1–3], reveals a universal expansion with an average acceleration of zero (within the measurement errors) over a Hubble time H_0^{-1} [4–8]. Another way to characterize this empirical result, in terms of the total pressure $p = p_r + p_m + p_{de}$ and energy density $\rho = \rho_r + \rho_m + \rho_{de}$, is to note that averaged over a Hubble time, the quantity $\langle p/\rho \rangle$ has the value $-1/3$ just at this moment, when we happen to be looking. Yet the combination of ρ_m , ρ_r and ρ_{de} could have produced a wide assortment of epochs with deceleration and acceleration.

This outcome is not merely surprising [9]. In the context of Λ CDM, the condition $\langle p/\rho \rangle = -1/3$ can be achieved only once in the entire (presumably infinite) history of the Universe, making it astonishingly unlikely. There is clearly physics behind this zero active mass condition. In a recent paper [8], we showed that the Friedmann–Robertson–Walker (FRW) metric is in fact

*arXiv: 1602.01435.

valid only for a cosmic fluid with an equation of state (EoS) $\rho + 3p = 0$, so the condition $\langle p/\rho \rangle = -1/3$ is independent of time. In other words, no matter how imperfect or incomplete our parametrization of the standard model happens to be, the optimized parameter values must always yield a zero average acceleration, regardless of when we make the measurements.

Kim *et al.* [9] have challenged this conclusion by providing what they believe to be a counter argument to the zero active mass condition in FRW. In this paper, however, we demonstrate that their claim is based on the conventional mistake of writing the metric from the perspective of an observer who is *simultaneously* in a non-inertial comoving frame, where a time dilation is unavoidable for an accelerated expansion of the spatial coordinates, and in a different, free-falling frame, with no measurable acceleration, thereby “assigning” without justification a constant value of one to the lapse function g_{tt} .

2 The lapse function in FRW

With the adoption of the Cosmological principle, the spacetime metric may be simplified to the FRW form

$$ds^2 = c^2 dt^2 - a^2(t) [dr^2(1 - kr^2)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (1)$$

in terms of the cosmic time t , the comoving radius r , the universal expansion factor $a(t)$, and the angular coordinates θ and ϕ in the comoving frame. The spatial curvature constant k takes on the values $(-1, 0, +1)$, for an open, flat, or closed universe, respectively.

Two rarely discussed issues with this metric are (i) that its lapse function g_{tt} is constant, and (ii) that in deriving its coefficients, one never considers whether the selected components of the energy-momentum tensor $T^{\mu\nu}$ are consistent with the assumption of homogeneity and isotropy. In general relativity (GR), a constant lapse function arises only in an inertial frame, where the observer is not subject to any gravitational influences. So even before we discuss the formal relationship between g_{tt} and $T^{\mu\nu}$, proponents of the use of Eq. (1) to describe the spacetime of an accelerating Universe need to explain why this basic tenet of relativity theory should be violated, i.e., why in the case of FRW an observer in the presumed free-falling frame nonetheless sees a gravitational acceleration without a corresponding time dilation.

The general class of spherically-symmetric spacetime metrics, of which FRW is a special case, may be represented as

$$ds^2 = e^{2\Phi/c^2} c^2 dt^2 - e^\lambda dr^2 - R^2 d\Omega^2, \quad (2)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, and Φ , λ , and R are each

functions of r and t , and are to be determined by solving Einstein’s equations. It is somewhat tedious, though straightforward, to derive the dependence of $g_{tt} \equiv e^{2\Phi/c^2}$ on T^{tt} , T^{rr} , $T^{\theta\theta}$ and $T^{\phi\phi}$. In the conventional approach, however, Φ is set equal to 0 to arrive at Eq. (1) without following this procedure. As such, the FRW form of the metric does not produce any time dilation relative to the proper time in a local inertial frame, even in cases where the observer sees an accelerated expansion of the spatial coordinates, i.e., when $\ddot{a} \neq 0$. This is critical because, as we shall see shortly, the Hubble flow in FRW is not inertial when $\ddot{a} \neq 0$.

But as shown in Ref. [8], g_{tt} is in fact not equal to one when $\rho + 3p \neq 0$, a requirement of $\ddot{a} \neq 0$. Borrowing a result from that paper, we find that

$$e^{2\Phi(t)/c^2} = h\dot{a}^2 e^{\mathcal{I}(t)}, \quad (3)$$

where

$$\mathcal{I}(t) \equiv \int_0^t dt' \frac{8\pi G}{3c^2 H} e^{\Phi/c^2} (\rho + 3p), \quad (4)$$

and $H \equiv e^{-\Phi/c^2} (\dot{a}/a)$ is the Hubble constant. This expression provides the necessary, formal relationship between the lapse function g_{tt} and the active mass $\rho + 3p$ as seen by an observer in the comoving frame. Clearly, Φ changes for different values of the active mass and corresponding expansion history encoded into the integral for \mathcal{I} over cosmic time. In order to achieve a constant Φ , we must have $\mathcal{I} \rightarrow 0$, which is guaranteed only when $\rho + 3p \rightarrow 0$. Then one can show that \dot{a} is also constant, and we may set $g_{tt} = 1$ with an appropriate choice of the initial condition h .

This is where Kim, Lasenby and Hobson [9] interject with a counter claim that one is free to *choose* $\Phi = 0$, in spite of its evident dependence on $\rho + 3p$. They do so without explaining why they view Eq. (3) as allowing Φ to have different values for the same $\rho + 3p$, and without providing a physical justification for making their particular choice leading to $g_{tt} = 1$. In this regard, their approach is no different from the conventional procedure of simply forcing $g_{tt} = 1$ in Eq. (1) before the metric is introduced into Einstein’s equations, thereby removing any possibility of a time dilation. The answer, of course, is that for the same $\rho + 3p$, different values of Φ correspond to different coordinate systems. And in order to force $\Phi = 0$, while keeping $\mathcal{I}(t) \neq 0$, they must consider the observer to be *simultaneously* in the comoving frame where an active mass $\rho + 3p \neq 0$ produces acceleration (according to the second Friedmann equation, in which $\ddot{a} \sim \rho + 3p$) and $\Phi \neq 0$, and in the free-falling frame where $g_{tt} = 1$ by choice. If the introduction of the spherically-symmetric form of the metric [Eq. (2)] into Einstein’s equations were to reduce to the FRW spacetime shown in Eq. (1) irrespective of how one chooses

the active mass, then this should happen automatically and unambiguously, *without* the manual intervention by Kim *et al.* [9] to force $\Phi = 0$ in Eq. (3). As we shall see below, this automatic reduction to Eq. (1) does not happen because the comoving frame is not inertial, except in the special case when $\rho + 3p = 0$.

This is also the reason why a gauge transformation cannot be used to eliminate the coordinate-dependent g_{tt} after the fact. Often misunderstood and incorrectly used, a gauge transformation in GR is a transformation of the coordinates selected in order to alter the metric coefficients $g_{\mu\nu}$ to a desired form. Since Φ is solely a function of t , it is straightforward to find the necessary transformation for FRW but, as discussed in Ref. [8], such a transformation would require the existence of two distinct frames of reference: one (the comoving) frame in which the observer sees $\ddot{a} \neq 0$ when $\rho + 3p \neq 0$, and therefore $g_{tt} \neq 1$, and a second (free-falling) frame in which the observer sees no time dilation and no acceleration. In other words, an observer cannot carry out an actual gauge transformation and stay within the same frame. And since the comoving frame in FRW is generally not inertial, an observer who experiences gravitational effects that lead to a coordinate-dependent lapse function cannot simultaneously also be in a free-falling frame where he does not measure a time dilation.

Ultimately, the issue we have been discussing here, and in Refs. [8, 9], has to do with whether or not one can always clearly distinguish between accelerated and inertial

frames. The answer is yes (see, e.g., Ref. [10], §3.2). In general relativity, acceleration can always be measured absolutely, unlike velocity, which is only measurable in a relative sense. For this reason, we can always find a local diffeomorphism that reduces the chosen manifold's metric to a Minkowski metric in a sufficiently small neighborhood of a given spacetime point when tidal forces are ignored [11, 12]. The FRW metric is not asymptotically flat and has a non-vanishing spacetime curvature tensor $R_{\alpha\beta\gamma\delta}$. Thus, in spite of the fact that its Weyl tensor is zero, which allows Eq. (1) to be written as a conformally-flat metric, the comoving frame used to write the FRW metric is not in free fall.

One seldom sees this property of FRW invoked in the interpretation of measurements made by a Hubble observer (however, see, Refs. [13–15]), but one can easily demonstrate that the Hubble flow in FRW is in fact not inertial when $\rho + 3p \neq 0$. The metric coefficients $g_{\mu\nu}(x)$ and affine connections $\Gamma_{\mu\nu}^\lambda(x)$ at $x^\alpha = (ct, r, \theta, \phi)$ contain enough information for us to determine the local inertial coordinates $\xi^\alpha(x)$ in the neighborhood of x^α . As shown, e.g., in Ref. [10] or Refs. [13, 14], the local inertial coordinates $\xi^\alpha(x)$ satisfy the equation

$$\frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\kappa} = \Gamma_{\lambda\kappa}^\nu(x) \frac{\partial \xi^\mu}{\partial x^\nu}. \tag{5}$$

In their most general form, the non-vanishing affine connections in FRW are

$$\begin{aligned} \Gamma_{00}^0 &= \dot{\Phi}/c^3, \\ \Gamma_{rr}^0 &= \frac{1}{c} \dot{a} a e^{-2\Phi/c^2}, & \Gamma_{\theta\theta}^0 &= \frac{1}{c} \dot{a} a r^2 e^{-2\Phi/c^2}, & \Gamma_{\phi\phi}^0 &= \frac{1}{c} \dot{a} a r^2 \sin^2 \theta e^{-2\Phi/c^2}, \\ \Gamma_{r0}^r &= \frac{1}{c} \frac{\dot{a}}{a}, & \Gamma_{\theta\theta}^r &= -r, & \Gamma_{\phi\phi}^r &= -r \sin^2 \theta, \\ \Gamma_{\theta\theta}^\theta &= \frac{1}{c} \frac{\dot{a}}{a}, & \Gamma_{\theta r}^\theta &= \frac{1}{r}, & \Gamma_{\phi\phi}^\theta &= -\cos \theta \sin \theta, \\ \Gamma_{\phi\phi}^\phi &= \frac{1}{c} \frac{\dot{a}}{a}, & \Gamma_{\phi r}^\phi &= \frac{1}{r}, & \Gamma_{\phi\theta}^\phi &= \cot \theta. \end{aligned}$$

Introducing these into Eq. (5), we see that the inertial coordinates $\xi^\mu = (c\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$ must satisfy the following expressions:

$$\frac{\partial^2 \tilde{t}}{\partial t^2} = \frac{\dot{\Phi}}{c^2} \frac{\partial \tilde{t}}{t}, \tag{6}$$

$$\frac{\partial^2 \tilde{t}}{\partial t \partial r} = \frac{\dot{a}}{a} \frac{\partial \tilde{t}}{\partial r}, \tag{7}$$

$$\frac{\partial^2 \tilde{t}}{\partial r^2} = \frac{1}{c^2} \dot{a} a e^{-2\Phi/c^2} \frac{\partial \tilde{t}}{\partial t}, \tag{8}$$

$$\frac{\partial^2 \tilde{r}}{\partial t^2} = \frac{\dot{\Phi}}{c^2} \frac{\partial \tilde{r}}{\partial t}, \tag{9}$$

and

$$\frac{\partial^2 \tilde{r}}{\partial r \partial t} = \frac{1}{c} \frac{\dot{a}}{a} \frac{\partial \tilde{r}}{\partial r}. \tag{10}$$

Let us first consider the local inertial frame for an FRW cosmology with zero active mass (i.e., $\rho + 3p = 0$), corresponding to $a(t) = t/t_0$ (normalized such that $a = 1$ today). We will place the observer near the origin of his coordinates, so that $r/(ct) \ll 1$. For this equation of state, we also have $\Phi = 0$ [see Eq. (3)]. The solution to Eqs. (6)–(10) is

$$\tilde{r} = a(t)r, \tag{11}$$

and

$$\tilde{t} \approx t \left(1 + \frac{1}{2} \left[\frac{\tilde{r}}{ct} \right]^2 \right), \quad (12)$$

correct to second order in r/ct . Thus, according to these expressions, the local inertial frame in the vicinity of $r = 0$ coincides with the Hubble flow and, in this situation, one has $\ddot{a} = 0$ with $g_{tt} = 1$. Thus, for this case (*and this case only*), it is legitimate to consider the comoving and free-falling frames to be identical, and for the observer to see no acceleration and no time dilation, so that $\Phi = 0$.

When $\rho + 3p \neq 0$, however, $a(t)$ is no longer linear in t , and \tilde{r} in Eq. (11) no longer satisfies Eqs. (9) and (10), regardless of whether or not one uses $\Phi = 0$. For example, in order for $\tilde{r} = ar$ to be consistent with Eq. (9), we would need either $\ddot{a} = 0$ (if $\Phi = 0$), or $\ddot{a} - \dot{a}\dot{\Phi}/c = 0$ (if $\Phi \neq 0$), both of which occur only when [8] $\rho + 3p = 0$. Of course, one can still find a local inertial frame when $\rho + 3p \neq 0$, but it is not coincident with the Hubble flow [11–15]. In this instance, the solution may be written as a polynomial to second order in r , as demonstrated in §3.2 of Ref. [10], or [13, 14], but we do not need to reproduce its coefficients here. The fact that the Hubble flow in FRW is not an inertial frame when $\rho + 3p \neq 0$ is sufficient to prove our argument. Therefore, setting $\Phi = 0$ when $\rho + 3p \neq 0$ in the comoving frame is not consistent with the fact that a time dilation should be measurable as a result of the acceleration seen by the observer.

The confusion about the measurability of a time dilation due to acceleration in FRW may be due to the fact that, because of homogeneity, g_{tt} can only be a function of t , not r , unlike the situation in the Schwarzschild spacetime, where it is a function of r and not of t . But this difference in coordinates does not change the requirement that an acceleration must always produce a time dilation that is measurable relative to the passage of proper time in an inertial frame. Therefore, regardless of whether g_{tt} is a function of t or r , the time dilation resulting from an accelerated expansion of the spatial coordinates cannot be “hidden” from the observer. Putting $g_{tt} = 1$ in Eq. (1), even when $\ddot{a} \neq 0$, suggests otherwise, and is therefore inconsistent with basic relativity theory. To guarantee that $\Phi = 0$ in Eq. (3), one must have $\rho + 3p = 0$.

Concerned by “gauge ambiguities” in their first attempt at challenging the zero active mass condition, Kim *et al.* [16] extended their argument by attempting to derive the FRW solution using a tetrad, which they claim constitutes a superior method that avoids such gauge ambiguities. They apparently believe that the physics of a problem may be changed merely by altering the calculational technique. But this is clearly false; if a gauge ambiguity is present with one approach, it is present for

all approaches because, as we have emphasized all along, a gauge transformation is not arbitrary – it constitutes a transformation of coordinates from one frame to another. Kim *et al.* believe that the sophistication of the tetrad approach somehow selects a unique frame of reference, creating their sought-after merger of a non-inertial comoving frame with a free-falling frame. This thinking, however, is flawed. The tetrad and its inverse simply package the metric coefficients and, once a solution is found, they must be unpacked to restore the spacetime coordinates. One therefore gets out what one puts in. In this regard, their approach is similar to that of Tupper [17], who explicitly set $g_{tt} = 1$ in his use of the tetrad field equations to derive a generalized Friedmann equation. Kim *et al.*’s use of the tetrad is more subtle, but they too invoked a special choice of coordinates – specifically, a particular choice of time – in order to simplify their tetrad ansatz *before* using the field equations to derive a solution.

Finally, we briefly comment on Kim *et al.*’s attempt to discredit the role of R_h as a true gravitational horizon. In so doing, they follow the lead of van Oirschot *et al.* [18] and Lewis *et al.* [19], who failed to realize that a horizon is observer dependent. An observer “sees” a horizon only in terms of null geodesics that actually reach him. Thus, to correctly interpret the role played by the surface at R_h , one must actually solve the null geodesic equations, as demonstrated by Bikwa *et al.* [20] and Melia [21], not simply rely on how or when R_h changes with time. A complete formal discussion of how R_h delimits the portion of the Universe visible to us today is given in Melia [22].

3 Concluding remarks

In this paper we have shown that the counter argument raised by Kim, Lasenby and Hobson [9] is simply based on the conventional approach of forcing the lapse function of the metric to be constant, without providing a physical justification for this particular “choice,” even when the formal relationship between $g_{tt} = e^{2\Phi/c^2}$ and $T^{\mu\nu}$ permits a range of possibilities for the same active mass $\rho + 3p$. In reality, a specific selection of Φ corresponds to a particular choice of coordinates. In FRW, the Hubble flow is non-inertial when $\rho + 3p \neq 0$, so the comoving frame cannot be identical to the free-falling frame. Therefore one cannot measure an acceleration without a corresponding time dilation.

Acknowledgements I am grateful to PMO in Nanjing, China, for its hospitality while this work was being carried out. This work was partially supported by grant 2012T1J0011 from The Chinese Academy of Sciences Visiting Professorships for Senior Inter-

national Scientists, and grant GDJ20120491013 from the Chinese State Administration of Foreign Experts Affairs.

References

1. C. L. Bennett, R. S. Hill, G. Hinshaw, M. R. Nolta, N. Odegard, L. Page, D. N. Spergel, J. L. Weiland, E. L. Wright, M. Halpern, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, G. S. Tucker, and E. Wollack, First-Year *Wilkinson Microwave Anisotropy Probe (WMAP)* Observations: Foreground Emission, *Astrophys. J. Suppl.* 148(1), 97 (2003)
2. D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, First-Year *Wilkinson Microwave Anisotropy Probe (WMAP)* Observations: Determination of cosmological parameters, *Astrophys. J. Suppl.* 148(1), 175 (2003)
3. P. A. R. Ade, et al. (Planck Collaboration), Planck 2013 results. XXIII. Isotropy and statistics of the CMB, *A&A* 571, A23 (2014)
4. F. Melia, *The Edge of Infinity: Supermassive Black Holes in the Universe*, Cambridge: Cambridge University Press, 1972, p. 119
5. F. Melia, The cosmic horizon, *Mon. Not. R. Astron. Soc.* 382(4), 1917 (2007)
6. F. Melia and M. Abdelqader, The cosmological space-time, *Int. J. Mod. Phys. D* 18(12), 1889 (2009)
7. F. Melia and A. S. H. Shevchuk, The $R_h = ct$ universe, *Mon. Not. R. Astron. Soc.* 419(3), 2579 (2012)
8. F. Melia, Physical basis for the symmetries in the Friedmann–Robertson–Walker metric, *Front. Phys.* 11(4), 119801 (2016)
9. D. Y. Kim, A. N. Lasenby, and M. P. Hobson, Friedmann–Robertson–Walker models do not require zero active mass, *Mon. Not. R. Astron. Soc.* 460(1), L119 (2016), arXiv: 1601.07890
10. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, New York: Wiley, 1972
11. A. Harvey, The principle of equivalence, *Ann. Phys.* 29(3), 383 (1964)
12. M. Carrera and D. Giulini, Influence of global cosmological expansion on local dynamics and kinematics, *Rev. Mod. Phys.* 82, 169 (2010), arXiv: 0810.2712v2
13. H. Liu, Nonlinear resonance for quasilinear hyperbolic equation, *J. Math. Phys.* 28(11), 1920 (1987)
14. H. Liu, Nonlinear resonance for quasilinear hyperbolic equation, *J. Math. Phys.* 28(11), 1924 (1987)
15. S. M. Kopeikin, Local gravitational physics of the Hubble expansion, *Eur. Phys. J. Plus* 130(1), 11 (2015)
16. D. Y. Kim, A. N. Lasenby, and M. P. Hobson, Spherically-symmetric solutions in general relativity, *Phys. Rev. D* (2016) (submitted), arXiv: 1604.06365
17. B. O. J. Tupper, Tetrad field equations and a generalized Friedmann equation, *Astrophys. Space Sci.* 28(1), 225 (1974)
18. P. van Oirschot, J. Kwan, and G. F. Lewis, Through the looking glass: Why the “Cosmic Horizon” is not a horizon, *Mon. Not. R. Astron. Soc.* 404, 1633 (2010), arXiv: 1001.4795
19. G. F. Lewis and P. van Oirschot, How does the Hubble sphere limit our view of the Universe? *Mon. Not. R. Astron. Soc.* 423(1), L26 (2012)
20. O. Bikwa, F. Melia, and A. S. H. Shevchuk, Photon geodesics in Friedmann–Robertson–Walker cosmologies, *Mon. Not. R. Astron. Soc.* 421(4), 3356 (2012)
21. F. Melia, The cosmic horizon for a universe with Phantom energy, *J. Cosmol. Astropart. Phys.* 09, 029 (2012)
22. F. Melia, Proper size of the visible Universe in FRW metrics with a constant space-time curvature, *Class. Quantum Gravity* 30(15), 155007 (2013)