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RESEARCH ARTICLE

Effects of frustration on explosive synchronization

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In this study, we consider the emergence of explosive synchronization in scale-free networks by considering the Kuramoto model of coupled phase oscillators. The natural frequencies of oscillators are assumed to be correlated with their degrees and frustration is included in the system. This assumption can enhance or delay the explosive transition to synchronization. Interestingly, a de-synchronization phenomenon occurs and the type of phase transition is also changed. Furthermore, we provide an analytical treatment based on a star graph, which resembles that obtained in scale-free networks. Finally, a self-consistent approach is implemented to study the de-synchronization regime. Our findings have important implications for controlling synchronization in complex networks because frustration is a controllable parameter in experiments and a discontinuous abrupt phase transition is always dangerous in engineering in the real world.

Keywords coupled phase oscillator, explosive synchronization, frustration

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1 Introduction

Synchronization is one of the most captivating cooperative phenomena and a universal concept in the nonlinear sciences and non-equilibrium physics, where it is found widely in biological, chemical, physical, and social systems [1–6]. Understanding the intrinsic mechanism of synchronization has attracted much attention during the last decades due to its theoretical importance and relevance to real applications. In the 1970s, Kuramoto proposed a mathematically tractable model as a phenomenological theory, which comprises an ensemble of coupled phase oscillators where their natural frequencies $\{\omega_i\}$ and distribution vary according to some specific probability density function $g(\omega)$ (such as a unimodal symmetrical distribution for the sake of simplicity) [7–11]. The dynamic equation of each oscillator θ_i is defined as

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N, \quad (1)$$

where λ denotes the coupling strength, ω_i are the intrinsic frequencies, and A_{ij} are the elements of the adjacency matrix A, which represents the topological structure of a network. The elements $A_{ij} = 1$ if two nodes i and j are connected, whereas $A_{ij} = 0$ when nodes i and j have no physical connections. The original Kuramoto model is a mean-field type, and thus it is an all-to-all connected network. It has been proved that under these conditions, when the coupling strength is higher than a threshold λ_c, the system will collectively lead to the onset of synchronization, which emerges as a continuous phase transition from an incoherent state to a synchronous state. The mean-field approach is used to measure the degree of synchronization in the system, and the complex-valued order parameter is described as

$$r(t)\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j}, \quad (2)$$

where $0 \le r(t) \le 1$ measures the coherence of the oscillator population and $\psi(t)$ is the average phase. When $r(t) = 1$, the system reaches a completely synchronous state; otherwise, when $r(t) \approx 0$, the oscillator ensemble exhibits an incoherent state, where the oscillators have incoherent and almost independent behaviors.

Recent studies have shown that the topological structures of networks strongly influence the synchronization process [12–40]. A seminal study [12] investigated the Kuramoto model based on complex networks and showed that the positive correlation between the degree and frequencies of nodes may lead to a first-order phase transition in scale-free networks. By contrast, Liu *et al.* [13] considered the negative microscopic correlation between the frequencies and degree of nodes, which showed that explosive synchronization is replaced by a type of hierarchical synchronization. When time delays exist among the coupling of oscillators, the explosive synchronization can be enhanced [14]. Thus, Leyva *et al.* [15] investigated the emergence of dynamical abrupt transitions in a general network and set appropriate conditions for the transition from a continuous second-order transition to a sharp and discontinuous first-order phase transition. Zhu *et al.* [16] showed that explosive synchronization can only occur when both the degrees and frequencies of the network's nodes are disassortative. Furthermore, the frequency-weighted phase oscillator model can also lead to explosive synchronization [17–20].

Previous investigations of explosive synchronization have focused on the topological configuration of networks and the coupled forms among oscillators [21–35], whereas the effects of frustration on the synchronization process have not been considered in detail [36, 41]. Frustration has important effects on the spin Heisenberg XY model [42], coupled Josephson-junction arrays or ladders, and other systems with potentially useful applications [43, 44]. Experimentally, frustration can be implemented by adding a magnetic field [45]. In the present study, we consider explosive synchronization in scale-free networks by considering the effects of frustration from two different aspects, i.e., the order parameter and the effective frequency. We show that frustration influences the synchronization process dramatically, where the natural frequencies of the nodes are positively correlated with their degrees but without a lack of generality.

The remainder of this paper is organized as follows. In Section 2, we introduce the model of coupled oscillators by considering the effects of frustration in scale-free networks and we describe the main results obtained from numerical simulations. It has been shown that synchronization may be enhanced when the frustration is small, whereas in some phase interval, frustration can induce de-synchronization and even complete incoherence. Moreover, the continuity and critical coupling strength required for the onset of a phase transition can also be changed dramatically. In Section 3, we provide an analytical treatment using a star graph, which is similar to scale-free networks. Furthermore, a self-consistent model is proposed to explain the mechanism of de-synchronization. Our theoretical predictions are supported by extensive numerical simulations. In Section 4, we give our conclusions and interpret the potential implications of this problem.

2 Coupled phase oscillators with frustration

We start by considering an undirected and unweighted scale-free network of N coupled phase oscillators defined by the following equations:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i - \alpha), i = 1, \ldots, N, \quad (3)$$

where α is the phase shift that represents the frustration factor. According to the symmetry of the trigonometric function, we only need to deal with the interactions of the phase in $[0, \pi]$. In addition, we set $\omega_i = k_i$, and the distribution of the degree obeys a power law, with $p(k) \sim k^{-\nu}$, $\nu = 3$. In this study, we consider the case with an average degree $\langle k \rangle = 6$.

Figure 1 shows the synchronization diagram $r(\lambda)$ for a scale-free topology with $N = 100$ and $N = 400$. To understand the synchronization process at a microscopic level, we compute the effective frequency, which is defined as

$$\omega_i^{\mathrm{eff}} = \frac{1}{T} \int_{\tau}^{\tau+T} \dot{\theta}_i(\tau) \mathrm{d}\tau, \quad (4)$$

with a transient time $T = 1000$. For simplicity, we select several typical nodes where the size of the network is $N = 100$ because the results described in the following do not change qualitatively for larger sizes of N. The numerical integrations of coupled ordinary differential equations are performed using the fourth-order Runge–Kutta method with a time step of 0.001, where the initial conditions for the phase oscillators are random. According to the panels in Figs. 1 and 2, it is clear that the presence of frustration in small phase interactions ($\alpha < 0.5\pi$) allows synchronization to be reached with a smaller critical coupling strength, which is similar to the case considered with time-delay coupling by Peron and Rodrigues [14]. The continuity of the phase transition is also changed. However, the system has difficulty reaching synchronization with large frustration ($\alpha > 0.5\pi + \alpha_0$), where α_0 is a specific phase, as determined in the next section. In particular, the system will never be synchronous when the frustration is $\alpha = 0.5\pi$, as shown in Fig. 1(d) and Fig. 2(d). Moreover, as a non-trivial phenomenon, the system exhibits de-synchronization when the frustration lies in the interval of $\alpha \in (0.5\pi - \alpha_0, 0.5\pi)$, which implies that the order parameter will decline from the high value in Fig. 1(c), and thus the synchronous effective average frequency will disperse with increases in the coupling strength, as shown in Fig. 2(c).

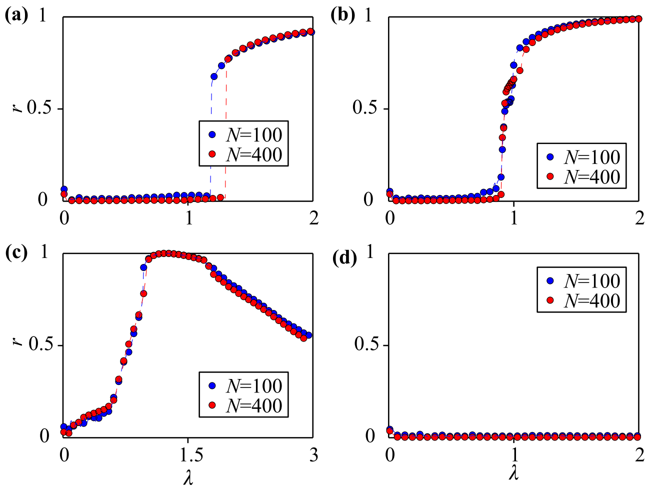

Fig. 1 The panels show the order parameter in a scale-free network with different frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.1\pi$, **(c)** $\alpha = 0.3\pi$, and **(d)** $\alpha = 0.5\pi$, where the size of the system is $N = 100$ or $N = 400$.

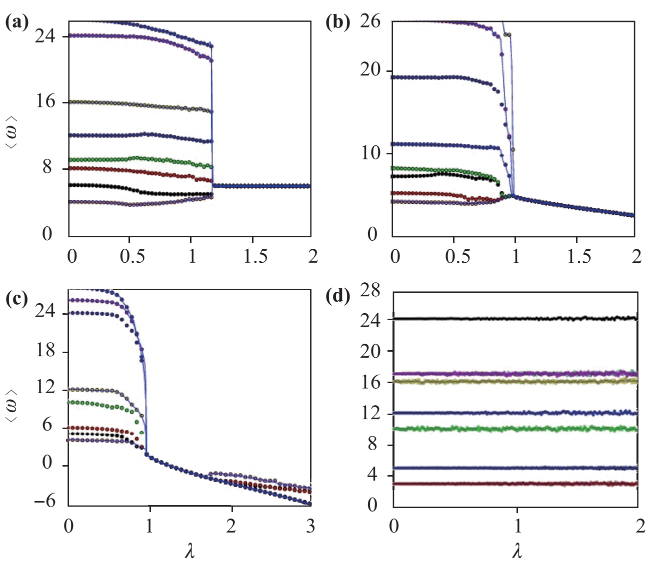

Fig. 2 The panels show the effective frequencies of different nodes with different frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.1\pi$, **(c)** $\alpha = 0.3\pi$, and **(d)** $\alpha = 0.5\pi$, where the size of the system is $N = 100$.

3 Theoretical analysis

In order to investigate the effect of frustration on the synchronization of oscillators and to obtain some analytical insights, we first analyze the two-oscillator system, which serves as the foundation for comparisons and the following discussion of many oscillators. The equations of motion for two limit-cycle oscillators with frustration α are written as

$$\dot{\theta}_1 = \omega_1 + \frac{\lambda}{2}\sin(\theta_2 - \theta_1 - \alpha), \tag{5}$$

$$\dot{\theta}_2 = \omega_2 + \frac{\lambda}{2}\sin(\theta_1 - \theta_2 - \alpha). \tag{6}$$

By introducing the phase difference ϕ, the above equations can be transformed into

$$\begin{aligned}\dot{\phi} &= \Delta\omega - \frac{\lambda}{2}[\sin(\phi + \alpha) + \sin(\phi - \alpha)] \\ &= \Delta\omega - \lambda\sin\phi\cos\alpha,\end{aligned} \tag{7}$$

and without loss of generality, we assume that $\Delta\omega \geq 0$. From Eq. (7), we obtain the condition for the synchronization between oscillator $\theta_1(t)$ and $\theta_2(t)$, i.e., $\Delta\omega \leq \lambda\cos\alpha$, where the critical coupling strength is given by

$$\lambda_c = \frac{\Delta\omega}{\cos\alpha}. \tag{8}$$

Obviously, when $\alpha = \pi/2$, the critical coupling strength $\lambda_c \to \infty$ and thus the two oscillators will never be synchronous. To quantitatively describe the influence of frustration, it is instructive to obtain explicit expressions for the order parameter $r(\alpha, \lambda)$ and the synchronous effective average frequency $\Omega(\alpha, \lambda)$, and according to the definition of Eq. (2)

$$r\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{2}\sum_{i=1}^{2}\mathrm{e}^{\mathrm{i}\theta_i} = \mathrm{e}^{\mathrm{i}\theta_1}(1 + \mathrm{e}^{\mathrm{i}\phi}), \tag{9}$$

we have $r^2 = \frac{1}{2}(1 + \cos\phi)$. For the non-synchronized state, Eq. (7) has no fixed points and the phase difference varies in the range of $[0, 2\pi]$. The probability of finding the phase difference in the interval $(\phi, \phi + \mathrm{d}\phi)$ is proportional to $\dot{\phi}$, and thus its contribution to the order parameter is

$$\begin{aligned}r_{as}^2 &= \int_0^{2\pi}\frac{1}{2}(1 + \cos\phi)P(\phi)\mathrm{d}\phi \\ &= \frac{1}{2} + \frac{1}{2}\int_0^{2\pi}\frac{\cos\phi \cdot C}{\Delta\omega - \lambda\sin\phi\cos\alpha}\mathrm{d}\phi,\end{aligned} \tag{10}$$

where C is the normalization constant. It is obvious that the second term of Eq. (10) is zero. Similarly, for the synchronized state, Eq. (7) has fixed points $\dot{\phi} = 0$, which gives

$$\Delta\omega - \lambda\sin\phi\cos\alpha = 0. \tag{11}$$

If $\alpha \in (0, \pi/2)$, $\cos\alpha > 0$, it is easy to see that the stable solution with the phase difference satisfies $\phi \in (-\pi/2, \pi/2)$. Thus, the synchronous order parameter follows

$$\begin{aligned}r_s^2 &= \frac{1}{2}(1 + \cos\phi) \\ &= \frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{\Delta\omega}{\lambda}\right)^2\frac{1}{\cos^2\alpha}}\right].\end{aligned} \tag{12}$$

By contrast, if $\alpha \in (\pi/2, \pi)$, $\cos\alpha < 0$, then the stable solution with the phase difference is $\phi \in (\pi, 3\pi/2)$, and the synchronous order parameter takes the form

$$r_s^2 = \frac{1}{2}(1+\cos\phi) = \frac{1}{2}\left[1-\sqrt{1-\left(\frac{\Delta\omega}{\lambda}\right)^2 \frac{1}{\cos^2\alpha}}\right]. \quad (13)$$

By summing Eqs. (5) and (6), the effective frequency is obtained as

$$\Omega = \frac{\omega_1+\omega_2}{2} + \frac{\lambda}{4}[\sin(\phi-\alpha) - \sin(\phi+\alpha)] = \frac{\omega_1+\omega_2}{2} - \frac{\lambda}{2}\cos\phi\sin\alpha. \quad (14)$$

According to the discussion above, we can obtain the effective average frequency as

$$\Omega = \bar{\omega} \pm \frac{\lambda}{2}\sqrt{1-\left(\frac{\Delta\omega}{\lambda}\right)^2 \frac{1}{\cos^2\alpha}}\sin\alpha, \quad (15)$$

where $\bar{\omega} = (\omega_1+\omega_2)/2$ is the average natural frequency, the minus sign "$-$" corresponds to the phase range $\alpha \in (0, \pi/2)$, and the positive sign "$+$" denotes the range $\alpha \in (\pi/2, \pi)$. According to the panels in Fig. 3 and Fig. 4, we can see that the only effect of frustration is to postpone the synchronization process. The system locks the phase, so the average frequency will stay together and the phenomenon of de-synchronization will not occur. In addition, when $\alpha < 0.5\pi$, the effective average frequency extends downward, whereas when $\alpha > 0.5\pi$, the frequency extends upward, and the two effective frequencies will never intersect in the most extreme situation when $\alpha = 0.5\pi$. It should be noted that the order parameter is a mean field concept and the size of the system should be within the thermodynamic limit $N \to \infty$. Thus, when the system is small, the order parameter may exhibit a fluctuation $\sim \sqrt{1/N}$.

To obtain analytical insights into how the frustration factor influences the synchronization process in network coupled oscillators, we reduce the problem to the analysis of a star configuration, which is a special topology structure with the main property of scale-free networks, i.e., a central node hub connected with K leaves. In the star network, the coupled form is one to one, which is similar to the two oscillators. Thus, the size of the system is $N = K+1$, the peripheral node has degree $k_i = 1$, and the hub has degree $k_h = K$. Figure 5 shows the direct numerical simulation of the order parameter in a star network with different sizes. We can see that the results obtained are similar to the scale-free network. By setting the rotated frame with the average phase $\psi(t) = \Omega t + \psi(0)$, we define the phase difference as $\phi_h = \theta_h - \psi(t)$ for the hub and $\phi_j = \theta_j - \psi(t)$ for the leaves. These definitions allow us to rewrite Eq. (3) into the set of equations given by

$$\dot{\phi}_h = (\omega_h - \Omega) + \lambda\sum_j^K \sin(\theta_j - \theta_h - \alpha), \quad (16)$$

$$\dot{\phi}_j = (\omega_j - \Omega) + \lambda\sin(\theta_h - \theta_j - \alpha). \quad (17)$$

Without any loss of generality, we set $\psi(0) = 0$ after an appropriate phase shift. Then, the order parameter

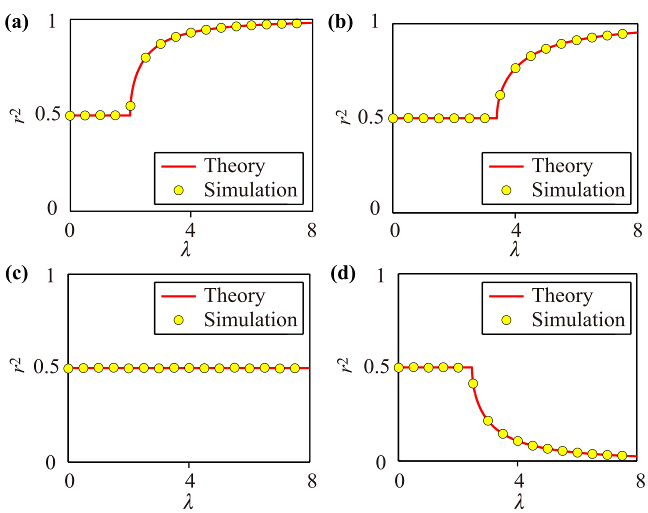

Fig. 3 The panels show the order parameters for two oscillators with different frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.3\pi$, **(c)** $\alpha = 0.5\pi$, and **(d)** $\alpha = 0.8\pi$.

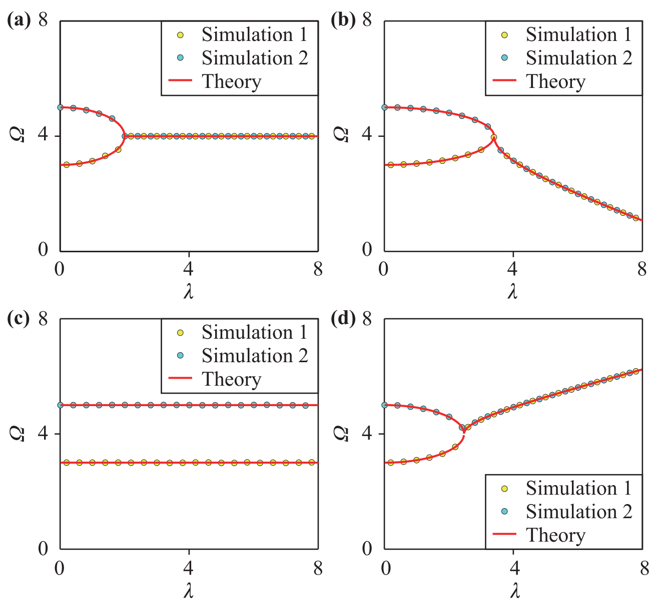

Fig. 4 The panels show the effective frequency of two oscillators with different frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.3\pi$, **(c)** $\alpha = 0.5\pi$, and **(d)** $\alpha = 0.8\pi$.

defined in Eq. (2) can be written in another form

$$re^{i\psi(t)} = \frac{e^{i\theta_h} + \sum_{j=1}^{K} e^{i(\theta_j - \alpha)}}{1+K}. \tag{18}$$

After multiplying both sides of Eq. (18) by $e^{-i\theta_h}$ and taking the imaginary part, the equations for the hub and leaves can be rewritten as

$$\dot{\phi_h} = (\omega_h - \Omega) - (1+K)r\lambda \sin\theta_h, \tag{19}$$

$$\dot{\phi_j} = (\omega - \Omega) + \lambda \sin(\phi_h - \phi_j - \alpha). \tag{20}$$

By imposing the stationary state solution $\dot{\phi_h} = 0$

$$\sin\phi_h = \frac{\omega_h - \Omega}{\lambda r(1+K)}, \tag{21}$$

and using the equations for the leaves, Eq. (20), we can evaluate the expression for $\cos(\phi_j + \alpha)$ in the locked regime

$$\begin{aligned}\cos(\phi_j+\alpha) &= \cos(\phi_j + \alpha + \phi_h - \phi_h)\\ &= \frac{\Omega - \omega}{\lambda}\sin\phi_h \pm \frac{\sqrt{(1-\sin^2\phi_h)[\lambda - (\Omega-\omega)^2]}}{\lambda}.\end{aligned} \tag{22}$$

It should be noted that when the frustration $\alpha = 0$, the same results were obtained in previous studies [12, 26, 36]. Eq. (22) is valid only for $\lambda \geq |\omega - \Omega(\lambda)|$, and it can be inferred that the necessary critical coupling strength for the onset of synchronization is $\lambda_c = |\omega - \Omega(\lambda)|$.

To understand the de-synchronization regime, it is important to know the dependence of the effective average frequency λ and Ω. When the system locks phase, all of the nodes oscillate at the same effective average frequency, i.e., $\theta_h = \Omega t$ for the hub and $\theta_i = \Omega t, (i = 1,\ldots,K)$ for the leaves. By substituting these solutions in Eq. (19) and Eq. (20), and summing all the $K+1$ equations, we obtain the average frequency Ω

$$\Omega(\lambda) = \frac{2K}{k+1}[1 - \lambda \sin\alpha \cos(\phi_j - \phi_h)]. \tag{23}$$

By imposing the synchronization condition for the leaf nodes and after some algebraic calculations, it can be inferred that the region of the stable solution (lock phase) is

$$(\phi_j - \phi_h + \alpha) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{24}$$

Finally, we obtain the function between λ and Ω

$$\begin{aligned}\Omega(\lambda) = \frac{2k}{k+1}\big\{&1 - \sqrt{\lambda^2 - [\Omega(\lambda) - \omega]^2}\sin\alpha\cos\alpha\\ &+[\Omega(\lambda) - \omega]\sin^2\alpha\big\}.\end{aligned} \tag{25}$$

When the frustration $\alpha = 0$, the result obtained above is $\Omega = 2k/(k+1)$, which is independent of the coupling strength λ and it is also a general result of synchronization [12, 26, 36]. According to the panels in Fig. 6, the black imaginary line is the synchronization boundary and the necessary condition for the system to

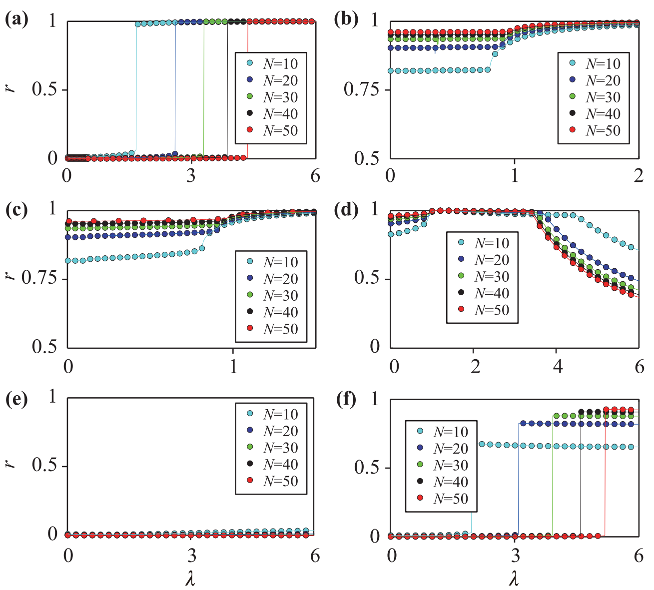

Fig. 5 The panels show the order parameter for a star network with different sizes and frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.01\pi$, **(c)** $\alpha = 0.1\pi$, **(d)** $\alpha = 0.3\pi$, **(e)** $\alpha = 0.5\pi$, and **(f)** $\alpha = 0.8\pi$.

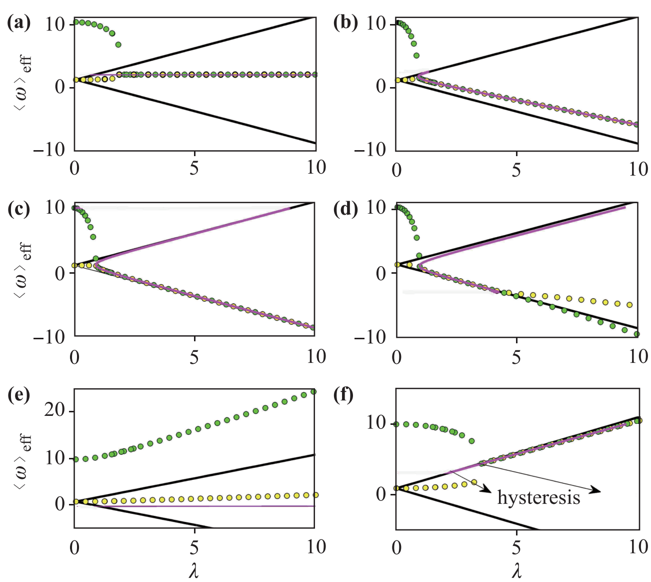

Fig. 6 The panels show the simulated effective frequency of the hub nodes (*green*) and the leaf nodes (*yellow*) with different frustration factors: **(a)** $\alpha = 0$, **(b)** $\alpha = 0.15\pi$, **(c)** $\alpha = 0.25\pi$, **(d)** $\alpha = 0.3\pi$, **(e)** $\alpha = 0.5\pi$, and **(f)** $\alpha = 0.8\pi$, where the red line represents Eq.(25).

lock phase is that the final effective average frequency must be located inside the black boundary. In Figs. 6(a) and 6(f), the empty line represents hysteresis, which is a remarkable characteristic of the first-order phase transition proposed in previous studies [12–20]. As shown in Fig. 6(b)–(d), there is an upper branch inside the boundary, which is nonphysical and unstable [36]. Analogous to the two-oscillator system, a possible effect of frustration on the effective frequency is that the effective frequency increases when $\alpha < 0.5\pi$; otherwise, the effective frequency declines when $\alpha > 0.5\pi$. It is interesting that as the frustration increases, the extent of the upper branch also increases, while the extent of the lower branch decreases and it tends to extrude the boundary. Eventually, at a specific coupling strength, λ_{ds}, the coherent effective average frequency of the hub and nodes will disperse within the restriction on the synchronization boundary. In particular, when $\alpha = 0.5\pi$, the lower branch vanishes and there is only a nonphysical branch inside the boundary, which is related to the incoherent state in Eq. (8). In order to obtain the exact expression for the critical coupling strength of the onset of synchronization, Eq. (25) can be written as

$$\lambda^2 = a\Omega^2 + 2b\Omega + c, \tag{26}$$

where $a = 1 + (2\sin^2\alpha - \frac{K+1}{K})^2/\sin^2 2\alpha, b = -1 + 2\cos^2\alpha(2\sin^2\alpha - \frac{K+1}{K})^2/\sin^2 2\alpha, c = 1 + 4\cos^4\alpha/\sin^2 2\alpha$. According to the panels in Fig. 6, it is obvious that the critical coupling strength for synchronization is the vertex of a parabola, which refers to the condition that $\mathrm{d}\lambda/\mathrm{d}\Omega = 0$. By substituting this condition into Eq. (26), the critical coupling strength for $\alpha < 0.5\pi$ is $\lambda = \sqrt{c - b^2/a}$ and the limit case is obtained for small frustration [12, 26, 36]

$$\lim_{\alpha\to 0}\sqrt{c - \frac{b^2}{c}} = \frac{K-1}{K+1}. \tag{27}$$

With no frustration, hysteresis makes the forward critical coupling strength larger than Eq. (27). However, the intrinsic difference is that the first-order phase transition will be changed and the hysteresis will vanish with little frustration. The critical coupling strength is proportional to α and its overall tendency is smaller than the non-frustration case, so the enhancement effect of frustration $\alpha < 0.5\pi$ is confirmed. In order to explain the de-synchronization phenomenon and the discontinuous phase transition with $\alpha \geq 0.5\pi$, we can quantitatively determine the de-synchronization point λ_{ds} and the backward phase transition point λ_{bs}. From Eq. (26), we obtain

$$\lambda_{ds} = \frac{b+a}{a-1} = \frac{\frac{K-1}{K}}{2\sin^2\alpha - \frac{K+1}{K}}, \qquad \alpha < 0.5\pi, \tag{28}$$

$$\lambda_{bs} = \frac{b+a}{1-a} = -\frac{\frac{K-1}{K}}{2\sin^2\alpha - \frac{K+1}{K}}, \quad \alpha \geq 0.5\pi. \tag{29}$$

$\alpha_0 = \arcsin\sqrt{(K+1)/(2K)}$ is singular in Eq. (28), and when $0 < \alpha < \alpha_0, \lambda_{ds} < 0$, this means that there is no de-synchronization point in this interval. Similarly, in Eq. (29) the singularity appears at $\pi - \alpha_0$, and when $0.5\pi \leq \alpha < \pi - \alpha_0$, $\lambda_{bs} < 0$ means that the system can never be synchronous in this angle interval.

To better understand the dynamic mechanism of de-synchronization, we consider the special topological configuration of a star network and we use a self-consistent method to deal with the dynamic equations of the star network. We assume that the hub is a host node where its oscillation is influenced by frustration and the coupling strength alone, and the peripheral nodes are driven by the hub. Thus, Eqs. (16) and (17) can be reduced to

$$\dot{\theta}_h = \Omega_h = \omega_h + K\lambda\sin\Theta, \tag{30}$$

$$\dot{\theta}_j = \omega - \lambda\sin(\theta_j - \Omega_h t + \alpha). \tag{31}$$

We assume that $\Theta = \theta_j - \theta_h - \alpha$ is a constant, which indicates that the feedback from the peripheral nodes to the hub is negligible. We then define $\theta_j' = \theta_j - \Omega_h t + \alpha$ and we obtain

$$\dot{\theta}_j' = [\omega - \Omega_h(\lambda)] - \lambda\sin\theta_j'. \tag{32}$$

From Eq. (32), it is obvious that the necessary condition for synchronization is $\lambda \geq |\omega - \Omega_h(\lambda)|$. In addition, it can be concluded that the de-synchronization point is the solution to $\lambda = |\omega - \Omega_h(\lambda)|$, which implies that

$$(K^2\sin^2\Theta - 1)\lambda^2 - 2K(1-K)\lambda\sin^2\Theta + (K-1)^2 = 0. \tag{33}$$

The roots of the equation above can be expressed as

$$\lambda_{12} = \frac{1-K}{K\sin\Theta \mp 1}, \tag{34}$$

where λ_1 is the point at which the effect of the frequency will converge and λ_2 is the point of de-synchronization that we consider. Considering the lock phase in Eq. (24), we can assume that $\theta_j - \theta_h = \pi/2 - \alpha$ on the edge of synchronization. Based on this approximation, we obtain

$$\lambda_2 = \frac{\frac{K-1}{K}}{2\sin^2\alpha - \frac{K+1}{K}}, \tag{35}$$

which is the same as Eq. (28) and thus the assumption is reasonable. According to the analysis above, the de-synchronization phenomenon can be understood well given that with some specific frustration, the increase in λ will reduce the hub's ability to drive the leaves to synchronization, and in an extreme case with $\pi/2 \le \alpha < \pi - \alpha_0$, the hub can never drive the leaf to synchronization regardless of the coupling strength.

4 Conclusion

It is helpful to understand the model proposed previously by Gómez-Gardeñes *et al.* [12] by considering frustration using an analytical treatment. Thus, in this study, we investigated the effects of frustration on the explosive synchronization of a scale-free network. It should be noted that frustration can accelerate or delay the synchronization process. Moreover, the type of phase transition may be changed. In addition, we proposed a self-consistent approach for explaining the de-synchronization regime in a star network. Therefore, the inclusion of frustration in the Kuramoto model of positive correlation with frequency and degree allows the dynamic process to change dramatically, thereby suggesting a feasible scheme for manipulating critical phenomena when engineering networks. In future research, we will focus on the case where the networked oscillators are influenced by stochastic noise. Our study may facilitate research into similar abrupt changes in dynamic structures in real-world networks.

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References

1. A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge: Cambridge University Press, 2001, pp 279–296
2. J. Buck, Synchronous rhythmic flashing of fireflies (II), *Q. Rev. Biol.* 63(3), 265 (1988)
3. B. Georges, J. Grollier, V. Cros, and A. Fert, Impact of the electrical connection of spin transfer nano-oscillators on their synchronization: An analytical study, *Appl. Phys. Lett.* 92(23), 232504 (2008)
4. I. Z. Kiss, Y. Zhai, and J. L. Hudson, Emerging coherence in a population of chemical oscillators, *Science* 296(5573), 1676 (2002)
5. B. Eckhardt, E. Ott, S. H. Strogatz, D. M. Abrams, and A. McRobie, Modeling walker synchronization on the Millennium Bridge, *Phys. Rev. E* 75(2), 021110 (2007)
6. Z. Néda, E. Ravasz, T. Vicsek, Y. Brechet, and A. L. Barabási, Physics of the rhythmic applause, *Phys. Rev. E* 61, 6987 (2000)
7. Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Berlin: Springer, 1984, pp 75–76
8. S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, *Physica D* 143(1–4), 1 (2000)
9. J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.* 77(1), 137 (2005)
10. A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Synchronization in complex networks, *Phys. Rep.* 469(3), 93 (2008)
11. F. A. Rodrigues, T. K. D. M. Peron, P. Ji, and J. Kurths, The Kuramoto model in complex networks, *Phys. Rep.* 610, 1 (2016)
12. J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, Explosive synchronization transitions in scale-free networks, *Phys. Rev. Lett.* 106(12), 128701 (2011)
13. W. Q. Liu, Y. Wu, J. H. Xiao, and M. Zhan, Effects of frequency-degree correlation on synchronization transition in scale-free networks, *Europhys. Lett.* 101(3), 38002 (2013)
14. T. K. D. M. Peron and F. A. Rodrigues, Explosive synchronization enhanced by time-delayed coupling, *Phys. Rev. E* 86(1), 016102 (2012)
15. I. Leyva, A. Navas, I. Sendina-Nadal, J. A. Almendral, J. M. Buldú, M. Zanin, D. Papo, and S. Boccaletti, Explosive transitions to synchronization in networks of phase oscillators, *Sci. Rep.* 3, 1281 (2013)
16. L. H. Zhu, L. Tian, and D. N. Shi, Criterion for the emergence of explosive synchronization transitions in networks of phase oscillators, *Phys. Rev. E* 88(4), 042921 (2013)
17. X. Zhang, X. Hu, J. Kurths, and Z. Liu, Explosive synchronization in a general complex network, *Phys. Rev. E* 88, 010802(R) (2013)
18. I. Leyva, I. Sendina-Nadal, J. A. Almendral, A. Navas, S. Olmi, and S. Boccaletti, Explosive synchronization in weighted complex networks, *Phys. Rev. E* 88(4), 042808 (2013)
19. X. Hu, S. Boccaletti, W. Huang, X. Zhang, Z. Liu, S. Guan, and C. H. Lai, Exact solution for first-order synchronization transition in a generalized Kuramoto model, *Sci. Rep.* 4, 7262 (2014)
20. C. Xu, Y. Sun, J. Gao, T. Qiu, Z. Zheng, and S. Guan, Synchronization of phase oscillators with frequency-weighted coupling, *Sci. Rep.* 6, 21926 (2016)
21. P. Li, K. Zhang, X. Xu, J. Zhang, and M. Small, Reexamination of explosive synchronization in scale-free networks: The effect of disassortativity, *Phys. Rev. E* 87(4), 042803 (2013)

22. I. Leyva, R. Sevilla-Escoboza, J. M. Buldú, I. Sendiña-Nadal, J. Gómez-Gardeñes, A. Arenas, Y. Moreno, S. Gómez, R. Jaimes-Reátegui, and S. Boccaletti, Explosive first-order transition to synchrony in networked chaotic oscillators, *Phys. Rev. Lett.* 108, 168702 (2012)
23. P. Ji, T. K. D. M. Peron, P. J. Menck, F. A. Rodrigues, and J. Kurths, Cluster Explosive Synchronization in Complex Networks, *Phys. Rev. Lett.* 110(21), 218701 (2013)
24. T. K. D. M. Peron and F. A. Rodrigues, Determination of the critical coupling of explosive synchronization transitions in scale-free networks by mean-field approximations, *Phys. Rev. E* 86(5), 056108 (2012)
25. B. C. Coutinho, A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Kuramoto model with frequency-degree correlations on complex networks, *Phys. Rev. E* 87(3), 032106 (2013)
26. Y. Zou, T. Pereira, M. Small, Z. Liu, and J. Kurths, Basin of Attraction Determines Hysteresis in Explosive Synchronization, *Phys. Rev. Lett.* 112(11), 114102 (2014)
27. P. S. Skardal and A. Arenas, Disorder induces explosive synchronization, *Phys. Rev. E* 89(6), 062811 (2014)
28. H. Bi, X. Hu, X. Zhang, Y. Zou, Z. Liu, and S. Guan, Explosive oscillation death in coupled Stuart-Landau oscillators, *Europhys. Lett.* 108(5), 50003 (2014)
29. Y. Chen, Z. Cao, S. Wang, and G. Hu, Self-organized correlations lead to explosive synchronization, *Phys. Rev. E* 92(2), 022810 (2015)
30. P. Ji, T. K. D. Peron, F. A. Rodrigues, and J. Kurths, Analysis of cluster explosive synchronization in complex networks, *Phys. Rev. E* 90(6), 062810 (2014)
31. X. Zhang, S. Boccaletti, S. Guan, and Z. Liu, Explosive Synchronization in Adaptive and Multilayer Networks, *Phys. Rev. Lett.* 114(3), 038701 (2015)
32. P. Ji, T. K. Peron, F. A. Rodrigues, and J. Kurths, Low-dimensional behavior of Kuramoto model with inertia in complex networks, Sci. Rep. 4, 4783 (2014)
33. X. Zhang, Y. Zou, S. Boccaletti, and Z. Liu, Explosive synchronization as a process of explosive percolation in dynamical phase space, *Sci. Rep.* 4, 5200 (2014)
34. R. S. Pinto and A. Saa, Explosive synchronization with partial degree-frequency correlation, *Phys. Rev. E* 91(2), 022818 (2015)
35. S. Yoon, M. Sorbaro Sindaci, A. V. Goltsev, and J. F. F. Mendes, Critical behavior of the relaxation rate, the susceptibility, and a pair correlation function in the Kuramoto model on scale-free networks, *Phys. Rev. E* 91(3), 032814 (2015)
36. C. Xu, J. Gao, Y. Sun, X. Huang, and Z. Zheng, Explosive or continuous: Incoherent state determines the route to synchronization, *Sci. Rep.* 5, 12039 (2015)
37. S. Ma, H. Bi, Y. Zou, Z. Liu, and S. Guan, Shuttle-run synchronization in mobile ad hoc networks, *Front. Phys.* 10(3), 100505 (2015)
38. S. Liu, G. Zhang, Z. He, and M. Zhan, Optimal configuration for vibration frequencies in a ring of harmonic oscillators: The nonidentical mass effect, *Front. Phys.* 10(3), 100503 (2015)
39. Y. Zhang and W. Wan, States and transitions in mixed networks, *Front. Phys.* 9(4), 523 (2014)
40. X. Huang, M. Zhan, F. Li, and Z. Zheng, Single-clustering synchronization in a ring of Kuramoto oscillators, *J. Phys. A Math. Theor.* 47(12), 125101 (2014)
41. Z. G. Zheng, Frustration effect on synchronization and chaos in coupled oscillators, *Chin. Phys.* 10, 0703 (2001)
42. H. Sakaguchi, S. Shinmoto, and Y. kuramoto, Mutual entrainment in oscillator lattices with nonvariational type interaction, *Prog. Theor. Phys.* 79, 1096 (1988)
43. E. Berg, E. Altman, and A. Auerbach, Singlet excitations in pyrochlore: A study of quantum frustration, *Phys. Rev. Lett.* 90(14), 147204 (2003)
44. O. E. Omelćhenko and M. Wolfrum, Nonuniversal transitions to synchrony in the Sakaguchi–Kuramoto model, *Phys. Rev. Lett.* 109, 164101 (2012)
45. C. Yokoi, L. Tang, and W. Chou, Ground state of the one-dimensional chiral XY model in a field, *Phys. Rev. B* 37(4), 2173 (1988)