

RESEARCH ARTICLE

A new unified theory of electromagnetic and gravitational interactions

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In this paper we present a new unified theory of electromagnetic and gravitational interactions. By considering a four-dimensional spacetime as a hypersurface embedded in a five-dimensional bulk spacetime, we derive the complete set of field equations in the four-dimensional spacetime from the five-dimensional Einstein field equation. Besides the Einstein field equation in the four-dimensional spacetime, an electromagnetic field equation is obtained: $\nabla_a F^{ab} - \xi R^b_a A^a = -4\pi J^b$ with $\xi = -2$, where F^{ab} is the antisymmetric electromagnetic field tensor defined by the potential vector A^a , R_{ab} is the Ricci curvature tensor of the hypersurface, and J^a is the electric current density vector. The electromagnetic field equation differs from the Einstein–Maxwell equation by a curvature-coupled term $\xi R^b_a A^a$, whose presence addresses the problem of incompatibility of the Einstein–Maxwell equation with a universe containing a uniformly distributed net charge, as discussed in a previous paper by the author [L.-X. Li, *Gen. Relativ. Gravit.* 48, 28 (2016)]. Hence, the new unified theory is physically different from Kaluza–Klein theory and its variants in which the Einstein–Maxwell equation is derived. In the four-dimensional Einstein field equation derived in the new theory, the source term includes the stress-energy tensor of electromagnetic fields as well as the stress-energy tensor of other unidentified matter. Under certain conditions the unidentified matter can be interpreted as a cosmological constant in the four-dimensional spacetime. We argue that, the electromagnetic field equation and hence the unified theory presented in this paper can be tested in an environment with a high mass density, e.g., inside a neutron star or a white dwarf, and in the early epoch of the universe.

Keywords general relativity, Maxwell’s equations, unified field theory, Kaluza–Klein theory, brane world theory

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1 Introduction

Since Einstein [1, 2] discovered the theory of general relativity, many people (including Einstein himself) have attempted to develop a theory that unifies all of the fundamental interactions in nature (see Ref. [3] for a review). The success of general relativity in interpreting gravity as the geometry of spacetime made Einstein enthusiastic about geometrizing electromagnetic fields and combining the electromagnetic interaction with the gravitational interaction (the only long-range forces known to exist in nature) in a unified geometric frame. To accomplish this goal, Einstein chose to extend the metric tensor of a four-dimensional spacetime to accommodate the electromagnetic field, e.g., by introducing a complex, Hermitian metric tensor [4] or a real but asymmetric metric tensor [5]. Einstein was not alone in pursuing the unified theory for gravitational and electromagnetic interactions. Many others, including Weyl [6], Eddington [7], and Schrödinger [8], have worked along similar or distinct ways in the frame of a four-dimensional spacetime [3], and some of their work predates that of Einstein. None of these efforts have turned out to be successful, since the derived theory cannot describe the physical reality.

After Nordström [9], Kaluza [10] explored the possibility of unifying gravity and electromagnetism in a five-dimensional spacetime: Starting from a vacuum Einstein field equation in a five-dimensional spacetime, he was able to derive the Maxwell equation and the Einstein field equation in a four-dimensional spacetime with the stress-energy tensor of an electromagnetic field and an unidentified scalar field as the source. Of the total 15 independent components of the symmetric five-dimensional metric tensor, 10 were interpreted as the components of a four-dimensional spacetime metric, four were interpreted as proportional to the components of an electromagnetic field potential vector in the four-dimensional spacetime, and one was interpreted as an unidentified scalar field.

In his original work Kaluza adopted the so-called cylinder condition, in which one assumes that all metric components do not depend on the fifth dimension. Klein [11, 12] proposed a quantum interpretation for Kaluza's theory. He introduced the hypothesis that the fifth spacetime dimension is compactified with a very small circumference (e.g., on the order of the Planck length). Then, all components of the metric were expanded in a Fourier

series and the n th Fourier mode was interpreted as the n th excited state according to quantum theory. Because of the extreme smallness of the size of the fifth dimension, all excited states (i.e., all $n \geq 1$ states) would correspond to extremely high energy (e.g., the Planck energy) states so would not be accessible to experiments even in high-energy physics. Then, only the ground state with $n = 0$ was accessible and Kaluza's cylinder condition was interpreted. For comprehensive reviews on Kaluza–Klein theory and its variants, see Refs. [13, 14].

Although Kaluza–Klein theory has not been accepted as the ultimate theory for the unification of electromagnetism and gravity, the idea of extra dimensions has become popular in modern theories attempting to unify all of the fundamental interactions in nature (the electromagnetic, weak, strong, and gravitational interactions), including supergravity theory, superstring theory, and brane world theory. In both supergravity theory and superstring theory it is assumed that the spacetime contains four macroscopic dimensions (one dimension of time and three dimensions of space) and a number of compactified extra space dimensions (seven in supergravity theory and six in superstring theory) that are hypothetically of Planck length scales so that we cannot see the extra dimensions under laboratory conditions [14].

Since 1998, researchers started to realize that large extra dimensions (i.e., extra dimensions of scales much larger than the Planck length) are possible, provided that the standard model particles are confined in the macroscopic four-dimensional spacetime (a four-dimensional membrane) and only the gravitational interaction can propagate in extra dimensions [15, 16]. The theory of large extra dimensions was proposed to explain the very weakness of gravity relative to the other three kinds of interactions and to address the hierarchy problem in theoretical physics. A more interesting scenario — called brane gravity or brane world — was proposed in 1999 by Randall and Sundrum [17, 18]: The four-dimensional spacetime in which we live is in fact a four-dimensional membrane (a hypersurface with a surface stress-energy density) embedded in a five-dimensional anti-de Sitter space. As in the large extra dimension model, it is hypothesized that the standard model particles are confined in the four-dimensional brane so that we do not see the fifth dimension. However, in the brane world model, the fifth dimension need not be compactified. It is the curvature of the five-dimensional bulk spacetime that makes gravity on the four-dimensional brane weak and appear four-dimensional on scales larger than the curvature radius of the bulk spacetime.

In this paper, we propose a new unified theory of electromagnetic and gravitational interactions. Similar to the brane world model, we assume that the four-dimensional spacetime in which we live is a hypersurface

embedded in a five-dimensional bulk spacetime. However, we do not assume that the standard model particles are confined in the four-dimensional spacetime *a priori*. We also do not attempt to address the hierarchy problem as in brane world theory. Instead, similar to what is done in Kaluza–Klein theory, we derive the electromagnetic field equation from the vacuum Einstein field equation in the five-dimensional bulk spacetime. The approach can be outlined as follows: Through projection, the metric tensor in the five-dimensional bulk spacetime naturally induces a metric tensor on the four-dimensional spacetime hypersurface embedded in the bulk space. The five-dimensional metric tensor has in total 15 independent components. The four-dimensional metric tensor has 10 independent components when it is expressed in a coordinate system in the four-dimensional spacetime. Of the remaining five independent components of the five-dimensional metric tensor, as in Kaluza–Klein theory, four are interpreted as proportional to the components of an electromagnetic potential vector in the four-dimensional spacetime and the remaining one is a scalar function. Then, on the four-dimensional spacetime hypersurface, we derive an electromagnetic field equation of the form

$$\nabla_a F^{ab} - \xi R^b{}_a A^a = -4\pi J^b \quad (1)$$

with $\xi = -2$, where F^{ab} is the usual antisymmetric electromagnetic field tensor defined by the potential vector A^a , R_{ab} is the Ricci curvature tensor of the four-dimensional spacetime, and J^a is the electric current density vector.

Technically, the approach adopted in this paper is similar to that adopted in the Hamiltonian formulation of general relativity (see, e.g., [19]), where a scalar “lapse” function N and a “shift” vector N^a tangent to the hypersurface are introduced. However, in the Hamiltonian formulation of general relativity, a spacetime is foliated along a spacelike direction; i.e., the normal to the hypersurface is a timelike vector. For the problem studied in this paper, a five-dimensional spacetime is foliated along a timelike direction, i.e., the normal to the hypersurface is a spacelike vector. In the theory presented in this paper, the “shift” vector N^a is interpreted as proportional to the electromagnetic potential vector A^a on the hypersurface: $N^a = 2NA^a$ in Planck units. Then, when N is constant on the hypersurface (but can vary with the fifth dimension), the electromagnetic field Eq. (1) is derived from the five-dimensional vacuum Einstein field equation, as will be detailed in the paper.

Generally, for an n -dimensional spacetime as a hypersurface embedded in an $(n + 1)$ -dimensional spacetime, the n -dimensional spacetime has two curvatures: a Riemann curvature, determined by the metric tensor (also called the first fundamental form in differential geometry)

on the hypersurface, and an extrinsic curvature (also called the second fundamental form), determined by the normal to the hypersurface and the metric tensor in the $(n + 1)$ -dimensional spacetime. The Riemann curvature tensor of the hypersurface depends on the intrinsic geometry of the hypersurface (the metric and the derivative operator associated with it). The extrinsic curvature tensor depends on the way the hypersurface is embedded in the bulk space [20, 21]. In the theory presented in this paper, electromagnetic fields are contained in the extrinsic curvature of the four-dimensional spacetime. As in standard general relativity, gravitational fields are represented by the Riemann curvature. The electromagnetism in the four-dimensional spacetime arises from the gravity in a five-dimensional spacetime containing the four-dimensional spacetime. So, unlike in brane world theory, in our theory electromagnetic fields are not assumed to be confined in a four-dimensional membrane *a priori*, and the hypersurface representing the four-dimensional spacetime need not be a discontinuous surface layer with a surface stress-energy tensor.

The electromagnetic field Eq. (1) is the most important result of this paper. It differs from the Einstein–Maxwell equation [Eq. (20) in Section 2] by the presence of a term with A^a coupled to the Ricci tensor R_{ab} and thus is a new electromagnetic field equation. The electromagnetic field equation of the form (1) with an undetermined dimensionless factor ξ was proposed by Li [22] to address the incompatibility of the Einstein–Maxwell equation with a homogeneous and isotropic universe: If a homogeneous and isotropic universe contains a uniformly distributed net charge, $F_{ab} = 0$ by the symmetry of the spacetime but $J^b \neq 0$; then Eq. (1) is violated if the $\xi R^b{}_a A^a$ term is not present. In the theory presented in this paper, from a five-dimensional Einstein field equation we can derive an electromagnetic field equation that has exactly the form of Eq. (1), but with a fixed $\xi = -2$.

The fact that an electromagnetic field equation in the form of Eq. (1) is derived also indicates that the theory presented in this paper is distinctly different from Kaluza–Klein theory. In fact, in Kaluza–Klein theory, the four-dimensional spacetime metric appearing in the decomposition of a five-dimensional metric is not defined on the spacetime hypersurface mentioned above. The electromagnetic potential vector in Kaluza–Klein theory is also not a vector tangent to the hypersurface. They are in fact defined on a different hypersurface and are not related to the tensor variables used in this paper by diffeomorphisms, as will be proved in the paper. Hence, the theory presented in this paper is physically distinguishable from Kaluza–Klein theory.

To some degree, the present work was motivated by an attempt to “unify” Kaluza–Klein theory and brane world theory, i.e., to derive an electromagnetic field equation

on a spacetime brane. Although the theory presented in this paper cannot be considered as a unification or merging of Kaluza–Klein and brane world theories, it borrows essential ingredients from each of them: derivation of the electromagnetic field equation from the five-dimensional Einstein field equation as in Kaluza–Klein theory, and derivation of the gravitation field equation in a four-dimensional spacetime by direct projection of the Einstein field equation in a five-dimensional bulk spacetime onto a hypersurface as in brane world theory.

The paper is organized as follows. In Section 2, we revisit the Maxwell equation in a flat Minkowski spacetime and its generalization to a curved spacetime. We show that the Maxwell equation can also be expressed in terms of a symmetric tensor instead of the antisymmetric electromagnetic field tensor. When the Maxwell equation expressed in the symmetric tensor is extended to a curved spacetime, the field Eq. (1) is naturally obtained, with $\xi = -2$. In Sections 3–6 we describe the theory proposed in this paper in detail, including the $4 + 1$ decomposition of the five-dimensional gravity described by the vacuum Einstein field equation, the action and the Lagrangian density expressed in terms of the scalar curvature and the extrinsic curvature of the four-dimensional spacetime as a hypersurface, derivation of electromagnetic fields and the electromagnetic field equation, and the Einstein field equation in the four-dimensional spacetime with the stress-energy tensor including electromagnetic fields and other matter as the source. In Section 7, the relation and difference between Kaluza–Klein theory and the theory presented in this paper are discussed. In Section 8, we analyze and discuss gauge and diffeomorphic transformations of electromagnetic fields and gravity.

In Section 9, we discuss the remaining terms in the total Lagrangian density, i.e., the terms in addition to those representing four-dimensional gravity and electromagnetism. We show that, under certain conditions, they may represent a cosmological constant. In Section 10, we present a discussion of the electromagnetic field Eq. (1). In particular, we check under what conditions and in what kind of environments the field Eq. (1) can be tested with experiments and observations. In Section 11 we summarize the results obtained in this paper and draw our conclusions.

The paper contains three Appendixes. In Appendix A, which is included as a supplement to Section 7, we describe the geometric interpretation of the metric decomposition in Kaluza–Klein theory and derive its relation to the variables used in our theory. In Appendix B, we derive the field equations in four-dimensional spacetime by projection of the five-dimensional Einstein field equation and discuss their relationships to the equations derived from the Lagrangian formulation in Sections 5 and 6. In Appendix C, we present a pseudo-Hamiltonian formula-

tion of the new theory and show that the field equations derived from the pseudo-Hamiltonian formulation confirm the results obtained in Sections 5 and 6.

Throughout the paper we use Planck units with $G = c = \hbar = 1$, where G is the gravitational constant, c is the speed of light, and \hbar is the reduced Planck constant. However, in a few places the units are restored to get the magnitudes and dimensions of physical quantities.

2 Electromagnetic field equations and general relativity

In the paper we adopt the abstract index notation for tensors as used in Ref. [19]. That is, a tensor will be denoted by a letter followed by lowercase Latin indices, e.g., v^a , T_{ab} , etc. The component of a tensor in any basis is denoted by a letter followed by Greek letter indices (occasionally by lowercase Latin letters from i and onward and capital Latin letters, as will be manifested in the text), e.g., v^μ , $T_{\mu\nu}$, etc. The summation convention for tensor components is also adopted: An index appearing in both subscripts and superscripts is summed over all dimensions represented by the index.

2.1 Revisiting Maxwell's equations in a flat spacetime

In terms of the electric field \mathbf{E} and the magnetic field \mathbf{B} , in a global inertial frame in a flat spacetime, Maxwell's equations can be written as

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad (2)$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{J}, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (5)$$

where ρ_e is the charge density and \mathbf{J} is the current density vector.

In the theory of special relativity, the electric field \mathbf{E} and the magnetic field \mathbf{B} are combined to form an antisymmetric electromagnetic field tensor F_{ab} , with components in Cartesian coordinates

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \quad (6)$$

The charge density and the current density vector are combined to form a four-dimensional current density vector J^a , with components (ρ_e, J_x, J_y, J_z) .

With the antisymmetric field tensor F_{ab} , the inhomogeneous Maxwell Eqs. (2) and (3) are equivalent to the

equation

$$\partial_a F^{ab} = -4\pi J^b, \tag{7}$$

where ∂_a is the ordinary derivative operator of the global inertial coordinates. The homogeneous Maxwell Eqs. (4) and (5) are equivalent to the equation

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0. \tag{8}$$

Note that ∂_a is associated with the Minkowski metric tensor η_{ab} (i.e., $\partial_a \eta_{bc} = 0$).

The action of ∂_b on the Maxwell Eq. (7) leads to the equation for charge conservation,

$$\partial_a J^a = 0 = \frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}. \tag{9}$$

By the converse of the Poincaré lemma, Eq. (8) indicates that there must exist a four-dimensional potential vector A^a so that (see, e.g., [19])

$$F_{ab} = \partial_a A_b - \partial_b A_a. \tag{10}$$

If A^a is taken to be the fundamental variable, Eq. (8) is automatically satisfied and hence trivial. Then Eq. (7) is the only equation determining the evolution of electromagnetic fields.

It is well known that Maxwell’s equations are invariant under the gauge transformation. That is, under the gauge transformation $A_a \rightarrow A_a + \partial_a \chi$, where χ is any scalar function, F_{ab} is unchanged and hence Maxwell’s equations are unchanged.

By adopting A^a as the fundamental variable, the Maxwell Eq. (7) can also be expressed with a symmetric tensor H_{ab} instead of the antisymmetric tensor F_{ab} , where H_{ab} is defined by [22]

$$H_{ab} \equiv \partial_a A_b + \partial_b A_a. \tag{11}$$

By the definitions of F_{ab} and H_{ab} , we have

$$F_{ab} = H_{ab} - 2\partial_b A_a. \tag{12}$$

Hence,

$$\partial_a F^{ab} = \partial_a H^{ab} - \partial^b H, \tag{13}$$

where $H \equiv \eta_{ab} H^{ab} = 2\partial_a A^a$. The Maxwell Eq. (7) can then be written as

$$\partial_a H^{ab} - \partial^b H = -4\pi J^b. \tag{14}$$

Although in a flat spacetime Eqs. (7) and (14) are equivalent, in the next section we will see that they lead to different equations for the electromagnetic field in a curved spacetime. In particular, later we will see that expressing the electromagnetic field equation in terms of a symmetric tensor makes it easier to derive electromagnetism from five-dimensional gravity.

2.2 Electromagnetic field equations in a curved spacetime

In the theory of general relativity, a spacetime is defined by a manifold \mathcal{M} with a symmetric metric tensor g_{ab} on it. With a derivative operator ∇_a associated with the metric (i.e., $\nabla_a g_{bc} = 0$), the Riemann curvature of the spacetime is defined and hence so is the Ricci tensor R_{ab} . By Einstein’s field equations, R_{ab} is related to the stress-energy tensor of matter, T_{ab} , by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}, \tag{15}$$

where the Ricci scalar $R \equiv g^{ab}R_{ab}$. By the Bianchi identity $\nabla_a G^{ab} = 0$, the Einstein field equation leads to the equation for the conservation of stress-energy:

$$\nabla_a T^{ab} = 0. \tag{16}$$

An equation of physics in a flat spacetime is transplanted into a general curved spacetime usually by the “minimal substitution rule” (not applied to the equation of gravity, of course), i.e., by replacing the Minkowski metric tensor η_{ab} appearing in the equation by a general spacetime metric tensor g_{ab} and replacing the derivative operator ∂_a associated with η_{ab} by the derivative operator ∇_a associated with g_{ab} [19]. This is essentially reflection of the equivalence principle [23, 24]: At any point in a curved spacetime, it is possible to choose a local inertial frame so that within a sufficiently small region around that point the laws of nature take the forms as in a flat Minkowski spacetime. The critical point is that the region must be sufficiently small (i.e., much smaller than the radius of the spacetime curvature) so that any term of matter coupled to the spacetime curvature can be ignored. So, with the minimal substitution rule, it is not possible to recover terms of matter coupled to the spacetime curvature, if those terms exist in the laws.

Applying the minimal substitution rule to Eqs. (10) and (11), we get the definitions of F_{ab} and H_{ab} in a curved spacetime:

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \tag{17}$$

and

$$H_{ab} = \nabla_a A_b + \nabla_b A_a. \tag{18}$$

The definition of F_{ab} in Eq. (17) leads to

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0, \tag{19}$$

which is identical to the equation obtained by application of the minimal substitution rule to Eq. (8).

Application of the minimal substitution rule to Eq. (7) leads to an inhomogeneous electromagnetic field equation in a general curved spacetime:

$$\nabla_a F^{ab} = -4\pi J^b, \tag{20}$$

which implies conservation of charge in a curved spacetime:

$$\nabla_a J^a = 0, \quad (21)$$

since $\nabla_a \nabla_b F^{ab} = 0$. Equation (20) was originally proposed by Einstein [5, 25]. Since then it was widely accepted as the standard generalization of the Maxwell equation in a curved spacetime and is sometimes called the Einstein–Maxwell equation.

However, if we apply the minimal substitution rule to Eq. (14), we get the following electromagnetic field equation in a curved spacetime:

$$\nabla_a H^{ab} - \nabla^b H = -4\pi J^b, \quad (22)$$

where

$$H = g^{ab} H_{ab} = 2\nabla_a A^a. \quad (23)$$

By the identity

$$\nabla_a H^{ab} - \nabla^b H = \nabla_a F^{ab} + 2R^b{}_a A^a, \quad (24)$$

we find that Eq. (22) is equivalent to

$$\nabla_a F^{ab} + 2R^b{}_a A^a = -4\pi J^b. \quad (25)$$

Equation (25) differs from Eq. (20) by a curvature-coupled term $2R^b{}_a A^a$ on the left-hand side, but it is identical to Eq. (1) with $\xi = -2$.

The action of ∇_b on Eq. (25) leads to $\nabla_a J_{\text{eff}}^a = 0$, where

$$J_{\text{eff}}^a \equiv J^a + \frac{1}{2\pi} R^a{}_b A^b \quad (26)$$

can be interpreted as an effective current density vector. The usual equation of charge conservation, $\nabla_a J^a = 0$, is preserved if and only if

$$\nabla^a (R_{ab} A^b) = 0, \quad (27)$$

which is equivalent to

$$R_{ab} H^{ab} + A^b \nabla_b R = 0, \quad (28)$$

after application of the Bianchi identity.

Both Eqs. (20) and (25) are generally covariant and return to the Maxwell Eq. (7) in a flat spacetime. We have seen that, starting from the same equation in the flat spacetime, we can get different corresponding equations in a curved spacetime with the minimal substitution rule. This is easily understood since the Maxwell Eq. (7) contains second-order derivatives of A^a and the order of derivatives of a vector matters in a curved spacetime. That is, $\partial_a \partial^b A^a$ becomes $\nabla_a \nabla^b A^a$ in a curved spacetime, but $\partial^b \partial_a A^a$ becomes $\nabla^b \nabla_a A^a = \nabla_a \nabla^b A^a - R^b{}_a A^a$, although $\partial_a \partial^b A^a = \partial^b \partial_a A^a$. So, the difference between

Eqs. (20) and (25) shows the ambiguity in deriving an equation of physics in a curved spacetime from the corresponding equation in a flat spacetime. Although it is not possible to decide which equation is correct *a priori*, in Ref. [22] it has been shown that Eq. (20) is not compatible with a universe with a uniformly distributed charge, but Eq. (25) is.

Equations (20) and (25) are not the only possible equations obtained by extension of the Maxwell equation to a curved spacetime, even if the minimal substitution rule is forced. Since $\partial_a \partial^b A^a = (1 + \xi) \partial_a \partial^b A^a - \xi \partial^b \partial_a A^a$, where ξ is any number, $\partial_a F^{ab} = \partial_a \partial^a A^b - \partial_a \partial^b A^a$ can be substituted by $\nabla_a \nabla^a A^b - (1 + \xi) \nabla_a \nabla^b A^a + \xi \nabla^b \nabla_a A^a = \nabla_a F^{ab} - \xi R^b{}_a A^a$ in a curved spacetime, according to the minimal substitution rule. Then Eq. (1) is obtained. Since ξ is arbitrary (and can even be a function), there are an infinite number of possible equations in a curved spacetime corresponding to the Maxwell equation in a flat spacetime. It is worth pointing out that these equations do not violate the equivalence principle, since, in a region of size much smaller than the spacetime curvature radius, the curvature-coupled term in the equation is negligibly small for an electromagnetic field with a coherent space scale smaller than or comparable to the size of the region.

The electromagnetic field Eq. (22) can be recast in a neater form

$$\nabla_a \Theta^{ab} = -4\pi J^b, \quad (29)$$

where

$$\Theta_{ab} = \Theta_{ba} \equiv H_{ab} - H g_{ab}. \quad (30)$$

Then, conservation of charge demands that

$$\nabla_a \nabla_b \Theta^{ab} = 0. \quad (31)$$

Because of the presence of the curvature-coupled term, in a spacetime with a nonvanishing R_{ab} the field Eq. (25) [identically, Eqs. (22) and (29)] is not invariant under gauge transformations. This issue will be discussed in detail in Section 8.

2.3 The stress-energy tensor of electromagnetic fields

The Einstein field equation and the electromagnetic field equations discussed in previous subsections hold in a spacetime of any integer number dimensions, although the representation of electromagnetic fields in terms of the vectors \mathbf{E} and \mathbf{B} is applicable only to a four-dimensional spacetime. Here we present the stress-energy tensor of electromagnetic fields in an n -dimensional spacetime, where n is any integer ≥ 4 (then we have $g^{ab} g_{ab} = n$).

It can be checked that the electromagnetic field Eq. (29) [identical to Eqs. (22) and (25)] can be derived from an action

$$S_{EM} = \int \mathcal{L}_{EM}(g_{ab}, A_a) e \quad (32)$$

by variation with respect to A_a , where e is a fixed volume element and

$$\mathcal{L}_{EM}(g_{ab}, A_a) = \sqrt{-g} \left(-\frac{1}{4} F_{ab} F^{ab} + R_{ab} A^a A^b + 4\pi A_a J^a \right) \quad (33)$$

$$= \sqrt{-g} \left[-\frac{1}{4} (H_{ab} H^{ab} - H^2) + 4\pi A_a J^a \right] \quad (34)$$

$$= \sqrt{-g} \left[-\frac{1}{4} \left(\Theta_{ab} \Theta^{ab} - \frac{1}{n-1} \Theta^2 \right) + 4\pi A_a J^a \right] \quad (35)$$

is the Lagrangian density of electromagnetic fields. Here, g is the determinant of the component matrix of g_{ab} in the coordinate system associated with e , and

$$\Theta \equiv g^{ab} \Theta_{ab} = -(n-1)H. \quad (36)$$

The stress-energy tensor of electromagnetic fields is derived from the action by variation with respect to g^{ab} (after ignoring the source term $4\pi A_a J^a$ in the Lagrangian density) [19]:

$$T_{EM,ab} = -\frac{1}{2\pi\sqrt{-g}} \frac{\delta}{\delta g^{ab}} S_{EM}(J^a = 0). \quad (37)$$

Then, for the action defined by Eqs. (32)–(35), we get

$$\begin{aligned} T_{EM,ab} = & \frac{1}{4\pi} \left(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right) \\ & - \frac{1}{4\pi} \left\{ \nabla^c \nabla_c (A_a A_b) - 2\nabla^c \nabla_{(a} (A_b) A_c) \right. \\ & \left. + 4A^c R_{c(a} A_{b)} + g_{ab} [\nabla_c \nabla_d (A^c A^d) - R_{cd} A^c A^d] \right\}; \end{aligned} \quad (38)$$

or, equivalently,

$$\begin{aligned} T_{EM,ab} = & \frac{1}{4\pi} \left(\Theta_{ac} \Theta_b{}^c - \frac{1}{n-1} \Theta \Theta_{ab} \right) \\ & - \frac{1}{16\pi} \left(\Theta_{cd} \Theta^{cd} - \frac{1}{n-1} \Theta^2 \right) g_{ab} \\ & - \frac{1}{4\pi} \nabla^c (2A_{(a} \Theta_{b)c} - A_c \Theta_{ab}). \end{aligned} \quad (39)$$

In Eqs. (38) and (39), the parentheses in the indexes of a tensor denote symmetrization of the tensor.

The term in the first line on the right-hand side of Eq. (38) is just the usual stress-energy tensor of electromagnetic fields described by the Einstein–Maxwell Eq. (20). The terms in the second line are derived from the curvature term in Eq. (35), $R_{ab} A^a A^b$, which was discussed in Ref. [22] in detail. The divergence of $T_{EM,ab}$, i.e., $\nabla^a T_{EM,ab}$, was also calculated and discussed in Ref. [22] (see also Section 10 of this paper).

3 4+1 decomposition of five-dimensional gravity

By considering the 4+1 decomposition of the Einstein field equation in a five-dimensional (5D) spacetime, Kaluza [10] and Klein [11, 12] were able to derive the four-dimensional Einstein–Maxwell equation and the four-dimensional Einstein field equation from five-dimensional gravity. In other words, electromagnetic fields and gravity were unified into five-dimensional gravity in Kaluza–Klein theory, at least formally.

The strategy of Kaluza and Klein was to, without loss of generality, take a five-dimensional coordinate system $\{x^0, x^1, x^2, x^3, x^4 = w\}$ and express a five-dimensional spacetime metric tensor \tilde{g}_{AB} ($A, B = 0, 1, 2, 3, 4$) in the form

$$\tilde{g}_{AB} = \begin{pmatrix} {}_k g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}, \quad (40)$$

where $\mu, \nu = 0, 1, 2, 3$. Kaluza and Klein interpreted ${}_k g_{\mu\nu}$ as the spacetime metric in the four-dimensional spacetime defined by the coordinates $\{x^0, x^1, x^2, x^3\}$, A_μ as the electromagnetic field potential vector in the four-dimensional spacetime, and ϕ as an unidentified scalar field. By the assumption that ${}_k g_{\mu\nu}$ and A_μ do not depend on the w coordinate and $\phi = \text{constant}$, Kaluza and Klein have shown that the vacuum five-dimensional Einstein field equation is equivalent to the combination of a four-dimensional Einstein–Maxwell equation and a four-dimensional Einstein field equation with the stress-energy tensor of electromagnetic fields as the source. This is the basis of Kaluza–Klein theory.

In this paper, we consider a different 4+1 decomposition of a five-dimensional spacetime, which is constructed by direct projection of the five-dimensional metric onto a timelike hypersurface embedded in the five-dimensional spacetime. The scheme we take is much like that in the Hamiltonian formulation of general relativity and canonical quantization of gravity (see, e.g., [19, 24, 26]), except that in the latter case the hypersurface is spacelike, i.e., has a timelike normal but here the hypersurface is timelike (i.e., has a spacelike normal). Assuming that the bulk spacetime is described by the vacuum five-dimensional Einstein field equation, we investigate the field equations induced on the spacetime hypersurface by projection. We will find that, in addition to a four-dimensional Einstein field equation, an electromagnetic field equation in the form of Eq. (25) is derived. The electromagnetic field equation is equivalent to the Eq. (25) but different from the Einstein–Maxwell Eq. (20). As will be explained in detail in Section 7 and Appendix A, our procedure is distinctly different from that of Kaluza and Klein, and, as a result, the derived field equations are also different.

For generality, we consider an $(n + 1)$ -dimensional spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ that is (locally at least) covered by a coordinate system $\{x^0, \dots, x^{n-1}, x^n = w\}$ and a hypersurface \mathcal{M} in it defined by $w = \text{constant}$ [27]. We denote the unit normal to \mathcal{M} by n^a , which is a spacelike vector and hence satisfies the condition

$$n^a n_a = 1. \quad (41)$$

Then, the tangent vector of the w -coordinate line, $w^a = (\partial/\partial w)^a$, can be decomposed as

$$w^a = N n^a + N^a, \quad (42)$$

where N is a scalar function and the vector N^a is tangent to \mathcal{M} , i.e.,

$$n^a N_a = 0. \quad (43)$$

In the Hamiltonian formulation of general relativity, N is called the lapse function and N^a is called the shift vector.

Given \tilde{g}_{ab} and n^a , an n -dimensional metric tensor g_{ab} is naturally induced on \mathcal{M} :

$$g_{ab} \equiv \tilde{g}_{ab} - n_a n_b, \quad (44)$$

which satisfies

$$g_{ab} n^b = 0. \quad (45)$$

Then, (\mathcal{M}, g_{ab}) forms an n -dimensional spacetime. By definition, N^a is a vector field on \mathcal{M} , i.e., $N^a \in \mathcal{T}(\mathcal{M})$ [28]. The inverse metric tensor on \mathcal{M} is

$$g^{ab} = \tilde{g}^{ac} \tilde{g}^{bd} g_{cd} = \tilde{g}^{ab} - n^a n^b. \quad (46)$$

It can be checked that

$$g_a^b \equiv \tilde{g}^{bc} g_{ac} = g_{ac} g^{bc} = \tilde{\delta}_a^b - n_a n^b, \quad (47)$$

where $\tilde{\delta}_a^b$ is the identity operator on $\tilde{\mathcal{M}}$. The tensor $g_a^b = \delta_a^b$ defined above is the identity operator on \mathcal{M} ; i.e., for any vector $v^a \in \mathcal{T}(\mathcal{M})$, we have $g_a^b v^a = v^b$.

Note that, for any tensor $\in \mathcal{T}(\tilde{\mathcal{M}})$, the index can be raised and lowered by \tilde{g}^{ab} and \tilde{g}_{ab} . For any tensor $\in \mathcal{T}(\mathcal{M})$, the index can be raised and lowered by \tilde{g}^{ab} and \tilde{g}_{ab} or, equivalently, by g^{ab} and g_{ab} . The metric tensor g_{ab} can be used as a projection operator to project a tensor on $\tilde{\mathcal{M}}$ onto \mathcal{M} .

With the metric tensor g_{ab} defined above, the matrix representation of the $(n + 1)$ -dimensional metric tensor \tilde{g}_{ab} in the coordinate system $\{x^0, \dots, x^{n-1}, x^n = w\}$ can be written as

$$\tilde{g}_{AB} = \begin{pmatrix} g_{\mu\nu} & N_\mu \\ N_\nu & N^2 + N_\rho N^\rho \end{pmatrix}, \quad (48)$$

where $\mu, \nu, \rho = 0, 1, \dots, n - 1$ and $A, B = 0, 1, \dots, n$. By comparison with Eq. (40), we can see the difference between our decomposition of the metric \tilde{g}_{AB} and that of Kaluza and Klein. The metric tensor $g_{\mu\nu}$ is defined on \mathcal{M} by projection of the bulk metric \tilde{g}_{AB} , but ${}_k g_{\mu\nu}$ in Eq. (40) is not.

Given the spacetime metric tensor \tilde{g}_{ab} , we can define a derivative operator associated with \tilde{g}_{ab} , which is denoted by $\tilde{\nabla}_a$ (i.e., $\tilde{\nabla}_a \tilde{g}_{bc} = 0$). Since \mathcal{M} is considered as a hypersurface embedded in an $(n + 1)$ -dimensional spacetime $\tilde{\mathcal{M}}$, we can define the extrinsic curvature tensor of \mathcal{M} by (see, e.g., [19])

$$K_{ab} \equiv g_a^c \tilde{\nabla}_c n_b = \frac{1}{2} \tilde{\mathcal{L}}_n g_{ab} = K_{ba}, \quad (49)$$

where $\tilde{\mathcal{L}}_n$ denotes the Lie derivative with respect to the vector n^a . The extrinsic curvature K_{ab} defines how \mathcal{M} is embedded in $\tilde{\mathcal{M}}$ and is tangent to \mathcal{M} , i.e., $n^a K_{ab} = 0$.

By application of Frobenius's theorem [19], it can be derived that

$$\tilde{\nabla}_a n_b = K_{ab} + n_a a_b, \quad (50)$$

where

$$a^b \equiv n^a \tilde{\nabla}_a n^b, \quad a^b n_b = 0. \quad (51)$$

The acceleration vector $a^a \in \mathcal{T}(\mathcal{M})$ describes the curvature of a curve orthogonally intersecting the hypersurface \mathcal{M} [24]. If the curve is a geodesic, we have $a^a = 0$.

The derivative operator associated with g_{ab} on \mathcal{M} , denoted by ∇_a (i.e., $\nabla_a g_{bc} = 0$), is defined by [19]

$$\nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_l} \equiv g^{a_1}_{d_1} \dots g_{b_l}^{e_l} g_c^f \tilde{\nabla}_f T^{d_1 \dots d_k}_{e_1 \dots e_l}, \quad (52)$$

for any tensor $T^{a_1 \dots a_k}_{b_1 \dots b_l} \in \mathcal{T}(\mathcal{M})$.

With the derivative operator $\tilde{\nabla}_a$, the Riemann curvature tensor \tilde{R}_{abc}^d on $\tilde{\mathcal{M}}$ is defined, and then the Ricci tensor \tilde{R}_{ab} and the Ricci scalar \tilde{R} are defined. With the derivative operator ∇_a , the Riemann curvature tensor R_{abc}^d on \mathcal{M} is defined and then the Ricci tensor R_{ab} and the Ricci scalar R are defined. It can be derived that \tilde{R} and R are related by (e.g., [19, 24])

$$\tilde{R} = R - K_{ab} K^{ab} + K^2 - 2\tilde{\nabla}_a v^a, \quad (53)$$

where

$$K \equiv g^{ab} K_{ab} = \tilde{\nabla}_a n^a \quad (54)$$

and

$$v^a \equiv n^a \tilde{\nabla}_c n^c - n^c \tilde{\nabla}_c n^a = K n^a - a^a. \quad (55)$$

The Einstein-Hilbert action of gravity in the $(n + 1)$ -dimensional spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ is

$$\begin{aligned} \tilde{S}_G &= \int \sqrt{-\tilde{g}} \tilde{R} \tilde{e} \\ &= \int \sqrt{-\tilde{g}} (R - K_{ab}K^{ab} + K^2) \tilde{e}, \end{aligned} \quad (56)$$

where \tilde{e} is a fixed volume element associated with the coordinate system on $\tilde{\mathcal{M}}$, and \tilde{g} is the determinant of the component matrix of \tilde{g}_{ab} . In Eq. (56) we have substituted Eq. (53) and ignored the term $2\tilde{\nabla}_a v^a$ since it does not contribute to the action integral with suitable boundary conditions. The field equation derived from $\delta\tilde{S}_G/\delta\tilde{g}^{ab} = 0$ is just the vacuum Einstein field equation in the $(n+1)$ -dimensional spacetime:

$$\tilde{G}_{ab} \equiv \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} = 0, \quad (57)$$

which is equivalent to

$$\tilde{R}_{ab} = 0. \quad (58)$$

Substituting Eq. (42) into Eq. (49), we can express K_{ab} in terms of N , N_a , and g_{ab} as

$$K_{ab} = \frac{1}{2}N^{-1}(\dot{g}_{ab} - M_{ab}), \quad (59)$$

where

$$\dot{g}_{ab} \equiv \frac{\partial}{\partial w} g_{ab} \equiv g_a^c g_b^d \tilde{L}_w g_{cd} \quad (60)$$

and

$$M_{ab} \equiv \nabla_a N_b + \nabla_b N_a. \quad (61)$$

Both \dot{g}_{ab} and M_{ab} are symmetric tensors tangent to \mathcal{M} . The trace of K_{ab} is

$$K = \frac{1}{2}N^{-1}(g^{ab}\dot{g}_{ab} - M), \quad (62)$$

where

$$M \equiv g^{ab}M_{ab} = 2\nabla_a N^a. \quad (63)$$

Substituting Eqs. (59), (62), and $\sqrt{-\tilde{g}} = N\sqrt{-g}$ into Eq. (56), we get

$$\tilde{S}_G = \int \tilde{\mathcal{L}}_G(N, N_a, g_{ab})\tilde{e}, \quad (64)$$

where the Lagrangian density $\tilde{\mathcal{L}}_G$ is defined by

$$\begin{aligned} \tilde{\mathcal{L}}_G(N, N_a, g_{ab}) &= \sqrt{-g}N \left[R - \frac{1}{4}N^{-2}(M_{ab}M^{ab} - M^2) \right] - \frac{1}{4}\sqrt{-g}N^{-1} \\ &\quad \times (g^{ac}g^{bd} - g^{ab}g^{cd})(\dot{g}_{ab}\dot{g}_{cd} - 2\dot{g}_{ab}M_{cd}). \end{aligned} \quad (65)$$

In the following sections we will investigate the field equations derived from the action defined by Eqs. (64) and (65) by variation with respect to N , N_a , and g_{ab} . We will show that electromagnetic fields are contained in M_{ab} and M . The variation of \tilde{S}_G with respect to N , N_a , and g_{ab} leads to a scalar constraint equation, the electromagnetic field equation, and the Einstein field equation on \mathcal{M} , respectively.

4 Inspection of the Lagrangian

The Lagrangian density in Eq. (65) can be separated into several parts as follows:

$$\tilde{\mathcal{L}}_G = \mathcal{L}_G + \mathcal{L}_{EM} + \mathcal{L}_{Other}, \quad (66)$$

where

$$\mathcal{L}_G \equiv \sqrt{-g}NR, \quad (67)$$

$$\mathcal{L}_{EM} \equiv -\frac{1}{4}\sqrt{-g}N^{-1}(M_{ab}M^{ab} - M^2), \quad (68)$$

and

$$\begin{aligned} \mathcal{L}_{Other} &\equiv -\frac{1}{4}\sqrt{-g}N^{-1}(g^{ac}g^{bd} - g^{ab}g^{cd}) \\ &\quad \times (\dot{g}_{ab}\dot{g}_{cd} - 2\dot{g}_{ab}M_{cd}). \end{aligned} \quad (69)$$

The function \mathcal{L}_G in Eq. (67) will be interpreted as the Lagrangian density of gravity on \mathcal{M} , since it is proportional to the n -dimensional Ricci scalar R . The function \mathcal{L}_{EM} in Eq. (68) will be interpreted as the Lagrangian density of electromagnetic fields on \mathcal{M} , which will be explained in detail below. The function \mathcal{L}_{Other} in Eq. (69) contains all other terms in the total Lagrangian density, so will be interpreted as the Lagrangian density of other matter fields on \mathcal{M} and their interaction with the electromagnetic field, although the nature of the matter cannot be determined.

Recall that the total Lagrangian density of gravity and electromagnetic fields in an n -dimensional spacetime is [see Eq. (35) and Ref. [19]]

$$\mathcal{L} = \sqrt{-g} [l_P^{-n+2}R - (H_{ab}H^{ab} - H^2)], \quad (70)$$

where l_P is the Planck length in the n -dimensional spacetime, which is related to the n -dimensional gravitational constant by $l_P = G_n^{1/(n-2)}$.

If we assume that $\nabla_a N = 0$ and let

$$N_a = 2Nl_P^{n/2-1}A_a, \quad (71)$$

then the \mathcal{L}_{EM} in Eq. (68) can be written as

$$\mathcal{L}_{EM} = -\sqrt{-g}Nl_P^{n-2}(H_{ab}H^{ab} - H^2). \quad (72)$$

Then we have

$$\mathcal{L}_G + \mathcal{L}_{EM} = \sqrt{-g}Nl_P^{n-2} [l_P^{-n+2}R - (H_{ab}H^{ab} - H^2)]. \quad (73)$$

Comparing Eqs. (70) and (73), we see that $\mathcal{L}_G + \mathcal{L}_{EM}$ is identical to the Lagrangian density of electromagnetic fields and gravity, up to a constant multiplier in the total Lagrangian. This fact indicates that \mathcal{L}_{EM} can be interpreted as the Lagrangian density of electromagnetic fields on \mathcal{M} . The Lagrangian \mathcal{L}_{Other} in Eq. (69) contains a term proportional to M_{cd} and hence proportional to H_{cd} , which can be interpreted as representing the interaction of electromagnetic fields with other matter. Variation of the total action with respect to A_a will then give rise to an electromagnetic field equation in the form of Eq. (25), at least in the case of $\nabla_a N = 0$ (see the next section). Hence, we see that, in a spacetime as a hypersurface embedded in a higher dimensional spacetime, electromagnetism is contained in the extrinsic curvature tensor of the hypersurface.

Because the total Lagrangian can be defined only up to a constant multiplier, the gravitational constant on \mathcal{M} cannot be determined from $\mathcal{L}_G + \mathcal{L}_{EM}$ in Eq. (73). However, this does not affect the field equations on \mathcal{M} derived from the action. With \mathcal{L}_G , \mathcal{L}_{EM} , and \mathcal{L}_{Other} defined above, variation of the total action with respect to g^{ab} (the details of which will be given in Section 6) leads to the Einstein field equation on \mathcal{M} :

$$G_{ab} = 8\pi l_P^{n-2} (T_{EM,ab} + T_{Other,ab}), \quad (74)$$

where $T_{EM,ab}$ is the stress-energy tensor of electromagnetic fields given by Eq. (39) [equivalent to Eq. (38)] and

$$T_{Other,ab} = -\frac{1}{8\pi N l_P^{n-2} \sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int \mathcal{L}_{Other} \tilde{e} \quad (75)$$

is the stress-energy tensor of other matter fields, including their interaction with electromagnetic fields.

Finally, we have some comments on the condition $\nabla_a N = 0$. By the definition of n^a , we have

$$n_a = N dw_a = N \nabla_a w. \quad (76)$$

Hence, we have

$$\tilde{\nabla}_b n_a = N \tilde{\nabla}_a \tilde{\nabla}_b w + \tilde{\nabla}_a w \tilde{\nabla}_b N. \quad (77)$$

Since $n_a n^a = 1$, we get

$$0 = n^a \tilde{\nabla}_b n_a = N n^a \tilde{\nabla}_a \tilde{\nabla}_b w + n^a \tilde{\nabla}_a w \tilde{\nabla}_b N. \quad (78)$$

From Eq. (76) we have $\nabla_a w = N^{-1} n_a$. Substituting it into Eq. (78), we get

$$\begin{aligned} 0 &= n^a \tilde{\nabla}_a n_b + \frac{1}{N} \left(\tilde{\nabla}_b N - n_b n^a \tilde{\nabla}_a N \right) \\ &= n^a \tilde{\nabla}_a n_b + \frac{1}{N} \nabla_b N. \end{aligned} \quad (79)$$

Hence, by the definition of a_a [Eq. (51)], we get

$$a_a = -\nabla_a \ln N. \quad (80)$$

Therefore, the condition $\nabla_a N = 0$ (i.e., N is constant on \mathcal{M}) is equivalent to the condition $a_a = 0$; i.e., \mathcal{M} is orthogonal to a congruence of spacelike geodesics. This condition must also be fulfilled if we require that \mathcal{L}_G in Eq. (67) be equal to R multiplied by a constant on \mathcal{M} .

5 Derivation of the electromagnetic field equation

The Lagrangian density $\tilde{\mathcal{L}}_G$ in Eq. (65), which contains three independent variables N , N_a , and g_{ab} , can be divided into several parts according to Eqs. (66)–(69). As discussed in Section 4, \mathcal{L}_G is interpreted as the Lagrangian density of gravity on \mathcal{M} , \mathcal{L}_{EM} is interpreted as the Lagrangian density of electromagnetic fields, and \mathcal{L}_{Other} is interpreted as the Lagrangian density of other matter fields and their interaction with electromagnetic fields. The field equations on \mathcal{M} are derived by variation of the action \tilde{S}_G with respect to N , N_a , and g_{ab} , respectively.

The Lagrangian \mathcal{L}_{Other} can be further separated into two parts as follows:

$$\mathcal{L}_{Other} = \mathcal{L}_m + \mathcal{L}_{int}, \quad (81)$$

where

$$\mathcal{L}_m \equiv -\frac{1}{4} \sqrt{-g} N^{-1} (g^{ac} g^{bd} - g^{ab} g^{cd}) \dot{g}_{ab} \dot{g}_{cd} \quad (82)$$

and

$$\mathcal{L}_{int} \equiv \frac{1}{2} \sqrt{-g} N^{-1} (g^{ac} g^{bd} - g^{ab} g^{cd}) \dot{g}_{ab} M_{cd}. \quad (83)$$

The Lagrangian \mathcal{L}_m does not contain N_a and is interpreted as the Lagrangian density of matter fields. The tensor \dot{g}_{ab} is interpreted as representing a matter field, although we do not know the nature of the matter (i.e., it could be normal matter, dark matter, or dark energy whose presence in nature has been confirmed by astronomical and cosmological observations). The Lagrangian \mathcal{L}_{int} contains both \dot{g}_{ab} and N_a ; hence it is interpreted as representing the interaction between the electromagnetic field and the matter field [29].

Then, the total Lagrangian density can be written as

$$\tilde{\mathcal{L}}_G = \mathcal{L}_G + \mathcal{L}_{EM} + \mathcal{L}_m + \mathcal{L}_{int}. \quad (84)$$

Accordingly, the total action can be written as

$$\tilde{S}_G = S_G + S_{EM} + S_m + S_{int}, \quad (85)$$

where S_G , S_{EM} , S_m , and S_{int} are integrals of the corresponding Lagrangian densities.

To simplify the derived equations, we define variables

$$\begin{cases} \Psi_{ab} \equiv \frac{1}{2}N^{-1}(M_{ab} - Mg_{ab}), \\ \Psi \equiv g^{ab}\Psi_{ab} = -\frac{1}{2}(n-1)N^{-1}M, \end{cases} \quad (86)$$

and

$$\begin{cases} \Phi_{ab} \equiv -\frac{1}{2}N^{-1}(g_a{}^c g_b{}^d - g_{ab}g^{cd})\dot{g}_{cd}, \\ \Phi \equiv g^{ab}\Phi_{ab} = \frac{1}{2}(n-1)N^{-1}g^{ab}\dot{g}_{ab}. \end{cases} \quad (87)$$

The variable Ψ_{ab} is used to replace M_{ab} and hence represents the electromagnetic field, and Φ_{ab} is used to replace \dot{g}_{ab} and hence represents the presumed matter field.

Then, \mathcal{L}_{EM} , \mathcal{L}_m , and \mathcal{L}_{int} , can be rewritten as

$$\mathcal{L}_{EM} = -\sqrt{-g}N \left(\Psi_{ab}\Psi^{ab} - \frac{1}{n-1}\Psi^2 \right), \quad (88)$$

$$\mathcal{L}_m = -\sqrt{-g}N \left(\Phi_{ab}\Phi^{ab} - \frac{1}{n-1}\Phi^2 \right), \quad (89)$$

and

$$\mathcal{L}_{int} = -2\sqrt{-g}N \left(\Psi_{ab}\Phi^{ab} - \frac{1}{n-1}\Psi\Phi \right), \quad (90)$$

respectively.

The variable N appears in the Lagrangian density only as a multiplication parameter. Variation of the Lagrangian density $\tilde{\mathcal{L}}_G$ with respect to N leads to [30]

$$\delta\tilde{\mathcal{L}}_G = \sqrt{-g} \left(R + \Pi_{ab}\Pi^{ab} - \frac{1}{n-1}\Pi^2 \right) \delta N, \quad (91)$$

where

$$\Pi_{ab} \equiv \Psi_{ab} + \Phi_{ab}, \quad \Pi \equiv g^{ab}\Pi_{ab} = \Psi + \Phi. \quad (92)$$

Hence, $\delta\tilde{\mathcal{S}}_G/\delta N = 0$ leads to a scalar constraint equation

$$R + \Pi_{ab}\Pi^{ab} - \frac{1}{n-1}\Pi^2 = 0. \quad (93)$$

By Eq. (59), we have

$$\Pi_{ab} = -K_{ab} + Kg_{ab}. \quad (94)$$

Hence, Eq. (93) is equivalent to

$$R + K_{ab}K^{ab} - K^2 = 0, \quad (95)$$

which is in fact just $\tilde{G}_{ab}n^a n^b = 0$ (see Appendix B).

The electromagnetic field equation is obtained by variation of the action with respect to N_a , which is contained in M_{ab} and Ψ_{ab} . By Eqs. (61) and (63), we get

$$\delta M_{ab} = \nabla_a \delta N_b + \nabla_b \delta N_a, \quad \delta M = 2\nabla^c \delta N_c. \quad (96)$$

Then, by Eq. (86), we get

$$\delta\Psi_{ab} = \frac{1}{2}N^{-1}(\nabla_a \delta N_b + \nabla_b \delta N_a - 2g_{ab}\nabla^c \delta N_c) \quad (97)$$

and

$$\delta\Psi = g^{ab}\delta\Psi_{ab} = -(n-1)N^{-1}\nabla^c \delta N_c. \quad (98)$$

Hence,

$$\begin{cases} \delta(N\Psi_{ab}\Psi^{ab}) = 2\Psi^{ab}\nabla_a \delta N_b - 2\Psi\nabla^c \delta N_c, \\ \delta\left(\frac{N}{n-1}\Psi^2\right) = -2\Psi\nabla^c \delta N_c, \end{cases} \quad (99)$$

and

$$\begin{cases} \delta(N\Psi_{ab}\Phi^{ab}) = \Phi^{ab}\nabla_a \delta N_b - \Phi\nabla^c \delta N_c, \\ \delta\left(\frac{N}{n-1}\Psi\Phi\right) = -\Phi\nabla^c \delta N_c. \end{cases} \quad (100)$$

Therefore, we get

$$\begin{aligned} \delta\mathcal{L}_{EM} &= -2\sqrt{-g}\Psi^{ab}\nabla_a \delta N_b \\ &= \sqrt{-g}\nabla_a [\dots]^a + 2\sqrt{-g}(\nabla_a \Psi^{ab})\delta N_b \end{aligned} \quad (101)$$

and

$$\begin{aligned} \delta\mathcal{L}_{int} &= -2\sqrt{-g}\Phi^{ab}\nabla_a \delta N_b \\ &= \sqrt{-g}\nabla_a [\dots]^a + 2\sqrt{-g}(\nabla_a \Phi^{ab})\delta N_b. \end{aligned} \quad (102)$$

In Eqs. (101) and (102), the terms denoted with $[\dots]$ are not written out since $\nabla_a [\dots]^a$ does not contribute to the action integral. Since \mathcal{L}_G and \mathcal{L}_m do not contain N_a , we have $\delta\mathcal{L}_G = \delta\mathcal{L}_m = 0$. Then, in terms of Ψ_{ab} and Φ_{ab} , $\delta\tilde{\mathcal{L}}_G$ can be written as

$$\delta\tilde{\mathcal{L}}_G \doteq 2\sqrt{-g}(\nabla_a \Pi^{ab})\delta N_b, \quad (103)$$

where \doteq means ‘‘equal up to a divergence term that does not contribute to the action integral.’’

Hence, $\delta\tilde{\mathcal{S}}_G/\delta N_a = 0$ leads to a vector equation

$$\nabla_a \Pi^{ab} = 0, \quad (104)$$

or, equivalently,

$$\nabla_a \Psi^{ab} = -\nabla_a \Phi^{ab}. \quad (105)$$

Since Ψ_{ab} is interpreted as the electromagnetic field, the right-hand side of Eq. (105) can be interpreted as the charge current density vector. If we define a current vector J^b by

$$J^b \equiv \frac{1}{4\pi}\nabla_a \Phi^{ab}, \quad (106)$$

the vector Eq. (105) can be written as

$$\nabla_a \Psi^{ab} = -4\pi J^b. \quad (107)$$

This equation will be interpreted as the electromagnetic field equation according to the reasons given below.

According to Eq. (71), when $\nabla_a N = 0$ is satisfied we have $M_{ab} = 2Nl_P^{n/2-1}H_{ab}$ and $M = 2Nl_P^{n/2-1}H$, where H_{ab} is defined by Eq. (18). Hence, when $\nabla_a N = 0$, by Eq. (86) we have

$$\Psi_{ab} = l_P^{n/2-1}\Theta_{ab}, \quad (108)$$

where Θ_{ab} is defined by Eq. (30). Therefore, if we adopt Planck units by setting $l_P = 1$, Ψ_{ab} is identical to Θ_{ab} , and Eq. (107) is identical to the electromagnetic field Eq. (29), which is equivalent to Eq. (25).

By Eq. (94), Eq. (105) is equivalent to

$$\nabla_a K^{ab} - \nabla^b K = 0, \quad (109)$$

which is in fact just $g^{bc}\tilde{R}_{cd}n^d = 0$ (which is equivalent to $g^{bc}\tilde{G}_{cd}n^d = 0$; see Appendix B).

6 Derivation of the gravitational field equation

The gravitational field equation on \mathcal{M} is obtained by variation of \tilde{S}_G with respect to g^{ab} . For variation with respect to g^{ab} , N and N_a are treated as invariant quantities, but

$$\delta N^a = N_b \delta g^{ab}, \quad \delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}, \quad (110)$$

and

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}. \quad (111)$$

The variation of S_G with respect to g^{ab} is simple to evaluate. The result is

$$\frac{1}{\sqrt{-g}}\frac{\delta S_G}{\delta g^{ab}} = NG_{ab} - \nabla_a \nabla_b N + g_{ab} \nabla_c \nabla^c N. \quad (112)$$

We denote the results of variation of S_{EM} , S_m , and S_{int} with respect to g^{ab} by $\kappa T_{EM,ab}$, $\kappa T_{m,ab}$, and $\kappa T_{int,ab}$, respectively, where $\kappa = 8\pi$ in Planck units. That is, we define

$$\kappa T_{EM,ab} \equiv -\frac{1}{N\sqrt{-g}}\frac{\delta S_{EM}}{\delta g^{ab}}, \quad (113)$$

$$\kappa T_{m,ab} \equiv -\frac{1}{N\sqrt{-g}}\frac{\delta S_m}{\delta g^{ab}}, \quad (114)$$

and

$$\kappa T_{int,ab} \equiv -\frac{1}{N\sqrt{-g}}\frac{\delta S_{int}}{\delta g^{ab}}. \quad (115)$$

Then, the gravitational field equation on \mathcal{M} is

$$G_{ab} = \kappa(T_{EM,ab} + T_{m,ab} + T_{int,ab}) + \frac{1}{N}(\nabla_a \nabla_b N - g_{ab} \nabla_c \nabla^c N). \quad (116)$$

Since $\nabla_a G^{ab} = 0$, the divergence of the right-hand side of Eq. (116) vanishes. Hence, the right-hand side can be interpreted as the stress-energy tensor of matter. Then, Eq. (116) is just the Einstein field equation in an n -dimensional spacetime,

$$G_{ab} = \kappa T_{ab}, \quad (117)$$

if we define the total stress-energy tensor

$$T_{ab} \equiv T_{EM,ab} + T_{m,ab} + T_{int,ab} + \frac{1}{\kappa N}(\nabla_a \nabla_b N - g_{ab} \nabla_c \nabla^c N). \quad (118)$$

When $\nabla_a N = 0$, the terms in the parentheses of Eq. (118) vanish.

6.1 Derivation of $T_{EM,ab}$

By the definition of M_{ab} , for variation with respect to g^{ab} we have

$$\begin{cases} \delta M_{ab} = -2N_c \delta \Gamma^c_{ab}, \\ \delta M = -2N_c g^{de} \delta \Gamma^c_{de} + M_{cd} \delta g^{cd}, \end{cases} \quad (119)$$

where Γ^c_{ab} is the Christoffel symbol. Then, by the definition of Ψ_{ab} , we have

$$\begin{aligned} \delta \Psi_{ab} &= N^{-1}(-N_c \delta \Gamma^c_{ab} + g_{ab} N_c g^{de} \delta \Gamma^c_{de}) \\ &\quad - \frac{1}{2} N^{-1}(g_{ab} M_{cd} - g_{ac} g_{bd} M) \delta g^{cd} \end{aligned} \quad (120)$$

and

$$\delta \Psi = N^{-1}(n-1)\left(N_c g^{de} \delta \Gamma^c_{de} - \frac{1}{2} M_{cd} \delta g^{cd}\right). \quad (121)$$

From Eqs. (120) and (121) we get

$$\begin{aligned} &\delta\left(\Psi_{ab}\Psi^{ab} - \frac{1}{n-1}\Psi^2\right) \\ &= -2N^{-1}N_c\Psi^{ab}\delta\Gamma^c_{ab} + 2\Psi_{ac}\Psi_b{}^c\delta g^{ab} \\ &\quad + N^{-1}M\Psi_{ab}\delta g^{ab}. \end{aligned} \quad (122)$$

The variation of Γ^c_{ab} is given by [19]

$$\delta\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}). \quad (123)$$

Then, by integration by parts and substitution of Eq. (86) for M , we get

$$\begin{aligned} &\delta\left[N\left(\Psi_{ab}\Psi^{ab} - \frac{1}{n-1}\Psi^2\right)\right] \\ &= -\nabla^c(N_a\Psi_{bc} + N_b\Psi_{ac} - N_c\Psi_{ab})\delta g^{ab} \\ &\quad + 2N\left(\Psi_{ac}\Psi_b{}^c - \frac{1}{n-1}\Psi\Psi_{ab}\right)\delta g^{ab}. \end{aligned} \quad (124)$$

Hence, by Eqs. (111) and (113) and the expression for \mathcal{L}_{EM} in Eq. (88), we get the stress-energy tensor of electromagnetic fields:

$$T_{EM,ab} = \frac{2}{\kappa} \left(\Psi_{ac} \Psi_b{}^c - \frac{1}{n-1} \Psi \Psi_{ab} \right) - \frac{1}{2\kappa} \left(\Psi_{cd} \Psi^{cd} - \frac{1}{n-1} \Psi^2 \right) g_{ab} - \frac{1}{\kappa N} \nabla^c (2N_{(a} \Psi_{b)c} - N_c \Psi_{ab}), \quad (125)$$

which agrees with Eq. (39) when $\nabla_a N = 0$. The trace of $T_{EM,ab}$ is

$$T_{EM} = \frac{2}{\kappa} \left(1 - \frac{1}{4}n \right) \left(\Psi_{cd} \Psi^{cd} - \frac{1}{n-1} \Psi^2 \right) - \frac{1}{\kappa N} \nabla^c (2N^a \Psi_{ac} - N_c \Psi). \quad (126)$$

When $n = 4$, we have

$$T_{EM} = -\frac{1}{\kappa N} \nabla^c (2N^a \Psi_{ac} - N_c \Psi). \quad (127)$$

6.2 Derivation of $T_{m,ab}$

By the definition of Φ_{ab} , we have

$$\delta \Phi^{ab} = -\frac{1}{2} N^{-1} (g^{ac} g^{bd} - g^{ab} g^{cd}) \delta g_{cd} - \frac{1}{2} N^{-1} \dot{g}_{cd} \times (g^{ac} \delta g^{bd} + g^{bd} \delta g^{ac} - g^{ab} \delta g^{cd} - g^{cd} \delta g^{ab}). \quad (128)$$

Hence,

$$\delta (\Phi_{ab} \Phi^{ab}) = -N^{-1} (\Phi^{ab} - \Phi g^{ab}) \delta g_{ab} - 2\Phi_{ac} \Phi_b{}^c \delta g^{ab} - N^{-1} \Phi_{ab} \dot{g}_{cd} (2g^{ac} \delta g^{bd} - g^{ab} \delta g^{cd} - g^{cd} \delta g^{ab}). \quad (129)$$

By Eq. (87) we have

$$\dot{g}_{ab} = -2N \left(\Phi_{ab} - \frac{1}{n-1} \Phi g_{ab} \right). \quad (130)$$

Substituting Eq. (130) into Eq. (129), we get

$$\delta (\Phi_{ab} \Phi^{ab}) = -N^{-1} (\Phi^{ab} - \Phi g^{ab}) \delta g_{ab} + 2\Phi_{ac} \Phi_b{}^c \delta g^{ab} - \frac{2}{n-1} (n\Phi \Phi_{ab} - \Phi^2 g_{ab}) \delta g^{ab}. \quad (131)$$

By Eqs. (130) and (128), we get

$$\delta \Phi^2 = (n-1) (N^{-1} \Phi g^{ab} \delta g_{ab} - 2\Phi \Phi_{ab} \delta g^{ab}) + 2\Phi^2 g_{ab} \delta g^{ab}. \quad (132)$$

Hence,

$$\delta \left(\Phi_{ab} \Phi^{ab} - \frac{1}{n-1} \Phi^2 \right) = -N^{-1} \Phi^{ab} \delta g_{ab} + 2 \left(\Phi_{ac} \Phi_b{}^c - \frac{1}{n-1} \Phi \Phi_{ab} \right) \delta g^{ab}. \quad (133)$$

Since

$$\delta \dot{g}_{ab} = \delta \frac{\partial}{\partial w} g_{ab} = \frac{\partial}{\partial w} \delta g_{ab}, \quad (134)$$

by Eqs. (89) and (133) we get

$$\delta \mathcal{L}_m \doteq -\frac{\partial}{\partial w} (\sqrt{-g} \Phi^{ab}) \delta g_{ab} - 2\sqrt{-g} N \left(\Phi_{ac} \Phi_b{}^c - \frac{1}{n-1} \Phi \Phi_{ab} \right) \delta g^{ab} - N \left(\Phi_{ab} \Phi^{ab} - \frac{1}{n-1} \Phi^2 \right) \delta \sqrt{-g}. \quad (135)$$

Then, by Eqs. (110) and (111), we get

$$\delta \mathcal{L}_m \doteq g_{ac} g_{bd} \frac{\partial}{\partial w} (\sqrt{-g} \Phi^{cd}) \delta g^{ab} - 2\sqrt{-g} N \left(\Phi_{ac} \Phi_b{}^c - \frac{1}{n-1} \Phi \Phi_{ab} \right) \delta g^{ab} + \frac{1}{2} \sqrt{-g} N \left(\Phi_{cd} \Phi^{cd} - \frac{1}{n-1} \Phi^2 \right) g_{ab} \delta g^{ab}. \quad (136)$$

Then, by Eq. (114), we get the stress-energy tensor of the matter field:

$$T_{m,ab} = -g_{ac} g_{bd} \frac{1}{\kappa N \sqrt{-g}} \frac{\partial}{\partial w} (\sqrt{-g} \Phi^{cd}) + \frac{2}{\kappa} \left(\Phi_{ac} \Phi_b{}^c - \frac{1}{n-1} \Phi \Phi_{ab} \right) - \frac{1}{2\kappa} \left(\Phi_{cd} \Phi^{cd} - \frac{1}{n-1} \Phi^2 \right) g_{ab}. \quad (137)$$

The trace of $T_{m,ab}$ is

$$T_m = -g_{cd} \frac{1}{\kappa N \sqrt{-g}} \frac{\partial}{\partial w} (\sqrt{-g} \Phi^{cd}) + \frac{2}{\kappa} \left(1 - \frac{1}{4}n \right) \left(\Phi_{cd} \Phi^{cd} - \frac{1}{n-1} \Phi^2 \right). \quad (138)$$

When $n = 4$, we have

$$T_m = -g_{cd} \frac{1}{\kappa N \sqrt{-g}} \frac{\partial}{\partial w} (\sqrt{-g} \Phi^{cd}). \quad (139)$$

6.3 Derivation of $T_{int,ab}$

By Eqs. (128) and (130), we get

$$\Psi_{ab} \delta \Phi^{ab} = -\frac{1}{2} N^{-1} (\Psi^{ab} - \Psi g^{ab}) \delta g_{ab} - \Psi \Phi_{ab} \delta g^{ab} + \left[2\Phi_{c(a} \Psi_{b)c} - \frac{1}{n-1} \Phi (\Psi_{ab} - \Psi g_{ab}) \right] \delta g^{ab} \quad (140)$$

and

$$\Psi \delta \Phi = \frac{1}{2} (n-1) (N^{-1} \Psi g^{ab} \delta g_{ab} - 2\Psi \Phi_{ab} \delta g^{ab}) + \Phi \Psi g_{ab} \delta g^{ab}. \quad (141)$$

Hence,

$$\begin{aligned} & \Psi_{ab}\delta\Phi^{ab} - \frac{1}{n-1}\Psi\delta\Phi \\ &= -\frac{1}{2}N^{-1}\Psi^{ab}\delta\dot{g}_{ab} + \left[2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}\Phi\Psi_{ab}\right]\delta g^{ab}. \end{aligned} \quad (142)$$

By Eqs. (120) and (121), we get

$$\begin{aligned} & \Phi^{ab}\delta\Psi_{ab} - \frac{1}{n-1}\Phi\delta\Psi \\ &= -N^{-1}N_c\Phi^{ab}\delta\Gamma^c_{ab} + \frac{1}{2}N^{-1}M\Phi_{ab}\delta g^{ab}. \end{aligned} \quad (143)$$

Combination of Eqs. (142) and (143) leads to

$$\begin{aligned} & \delta\left(\Psi_{ab}\Phi^{ab} - \frac{1}{n-1}\Psi\Phi\right) \\ &= \left[2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}(\Phi\Psi_{ab} + \Psi\Phi_{ab})\right]\delta g^{ab} \\ & \quad - \frac{1}{2}N^{-1}\Psi^{ab}\delta\dot{g}_{ab} - N^{-1}N_c\Phi^{ab}\delta\Gamma^c_{ab}, \end{aligned} \quad (144)$$

where Eq. (86) has been used to substitute for M .

Then, by Eq. (90), we have

$$\begin{aligned} \delta\mathcal{L}_{\text{int}} &= \sqrt{-g}\Psi^{ab}\delta\dot{g}_{ab} + 2\sqrt{-g}N_c\Phi^{ab}\delta\Gamma^c_{ab} \\ & \quad - 2\sqrt{-g}N\left[2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}(\Phi\Psi_{ab} + \Psi\Phi_{ab})\right]\delta g^{ab} \\ & \quad - 2N\left(\Psi_{cd}\Phi^{cd} - \frac{1}{n-1}\Psi\Phi\right)\delta\sqrt{-g}. \end{aligned} \quad (145)$$

By Eqs. (110) and (134), we get

$$\begin{aligned} \sqrt{-g}\Psi^{ab}\delta\dot{g}_{ab} &= \frac{\partial}{\partial w}(\sqrt{-g}\Psi^{ab}\delta g_{ab}) - \frac{\partial}{\partial w}(\sqrt{-g}\Psi^{ab})\delta g_{ab} \\ & \quad \doteq g_{ac}g_{bd}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd})\delta g^{ab}. \end{aligned} \quad (146)$$

By Eq. (123), we get

$$2\sqrt{-g}N_c\Phi^{ab}\delta\Gamma^c_{ab} \doteq \sqrt{-g}\nabla^c[2N_{(a}\Phi_{b)c} - N_c\Phi_{ab}]\delta g^{ab}. \quad (147)$$

Hence, we have

$$\begin{aligned} \delta\mathcal{L}_{\text{int}} & \doteq g_{ac}g_{bd}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd})\delta g^{ab} \\ & \quad + \sqrt{-g}\nabla^c[2N_{(a}\Phi_{b)c} - N_c\Phi_{ab}]\delta g^{ab} \\ & \quad - 2\sqrt{-g}N\left[2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}(\Phi\Psi_{ab} + \Psi\Phi_{ab})\right]\delta g^{ab} \\ & \quad + \sqrt{-g}N\left(\Psi_{cd}\Phi^{cd} - \frac{1}{n-1}\Psi\Phi\right)g_{ab}\delta g^{ab}, \end{aligned} \quad (148)$$

where Eq. (111) has been used.

Finally, by Eq. (15), we get the stress-energy tensor of the interaction:

$$\begin{aligned} T_{\text{int},ab} &= -g_{ac}g_{bd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd}) \\ & \quad + \frac{2}{\kappa}\left[2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}(\Phi\Psi_{ab} + \Psi\Phi_{ab})\right] \\ & \quad - \frac{1}{\kappa}\left(\Phi_{cd}\Psi^{cd} - \frac{1}{n-1}\Phi\Psi\right)g_{ab} \\ & \quad - \frac{1}{\kappa N}\nabla^c[2N_{(a}\Phi_{b)c} - N_c\Phi_{ab}]. \end{aligned} \quad (149)$$

The trace of $T_{\text{int},ab}$ is

$$\begin{aligned} T_{\text{int}} &= -g_{cd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd}) \\ & \quad + \frac{4}{\kappa}\left(1 - \frac{1}{4}n\right)\left(\Phi_{cd}\Psi^{cd} - \frac{1}{n-1}\Phi\Psi\right) \\ & \quad - \frac{1}{\kappa N}\nabla^c(2N^a\Phi_{ac} - N_c\Phi). \end{aligned} \quad (150)$$

When $n = 4$, we have

$$\begin{aligned} T_{\text{int}} &= -g_{cd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd}) \\ & \quad - \frac{1}{\kappa N}\nabla^c(2N^a\Phi_{ac} - N_c\Phi). \end{aligned} \quad (151)$$

6.4 The total stress-energy tensor

Substituting Eqs. (125), (137), and (149) into Eq. (118), we get the total stress-energy tensor

$$\begin{aligned} T_{ab} &= -g_{ac}g_{bd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Pi^{cd}) \\ & \quad + \frac{2}{\kappa}\left(\Pi_{ac}\Pi^c_b - \frac{1}{n-1}\Pi\Pi_{ab}\right) \\ & \quad - \frac{1}{2\kappa}\left(\Pi_{cd}\Pi^{cd} - \frac{1}{n-1}\Pi^2\right)g_{ab} \\ & \quad + \frac{1}{\kappa N}(\nabla_a\nabla_b N - g_{ab}\nabla_c\nabla^c N) \\ & \quad - \frac{1}{\kappa N}\nabla^c(2N_{(a}\Pi_{b)c} - N_c\Pi_{ab}). \end{aligned} \quad (152)$$

The trace of T_{ab} is

$$\begin{aligned} T &= -g_{cd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Pi^{cd}) - \frac{n-1}{\kappa N}\nabla_c\nabla^c N \\ & \quad + \frac{2}{\kappa}\left(1 - \frac{1}{4}n\right)\left(\Pi_{cd}\Pi^{cd} - \frac{1}{n-1}\Pi^2\right) \\ & \quad - \frac{1}{\kappa N}\nabla^c(2N^a\Pi_{ac} - N_c\Pi). \end{aligned} \quad (153)$$

By identities

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial w}\sqrt{-g} = \frac{1}{2}g^{ab}\dot{g}_{ab} = \frac{1}{n-1}N\dot{\Phi} \quad (154)$$

and

$$\frac{\partial}{\partial w} \Pi^{ab} = N g^a_c g^b_d \tilde{\mathcal{L}}_n \Pi^{cd} + N^c \nabla_c \Pi^{ab} - 2 \Pi^{c(a} \nabla_c N^{b)}, \tag{155}$$

we get

$$\begin{aligned} \kappa T_{ab} = & -g_{ac} g_{bd} \tilde{\mathcal{L}}_n \Pi^{cd} + 2 \Pi_{ac} \Pi^c_b - \frac{3}{n-1} \Pi \Pi_{ab} \\ & - \frac{1}{2} \left(\Pi_{cd} \Pi^{cd} - \frac{1}{n-1} \Pi^2 \right) g_{ab} - \frac{2}{N} N_{(a} \nabla^{c} \Pi_{b)c} \\ & + \frac{1}{N} (\nabla_a \nabla_b N - g_{ab} \nabla_c \nabla^c N). \end{aligned} \tag{156}$$

By the vector field Eq. (104), the term $N_{(a} \nabla^{c} \Pi_{b)c}$ in Eq. (156) vanishes. Hence we have

$$\begin{aligned} \kappa T_{ab} = & -g_{ac} g_{bd} \tilde{\mathcal{L}}_n \Pi^{cd} + 2 \Pi_{ac} \Pi^c_b - \frac{3}{n-1} \Pi \Pi_{ab} \\ & - \frac{1}{2} \left(\Pi_{cd} \Pi^{cd} - \frac{1}{n-1} \Pi^2 \right) g_{ab} \\ & + \frac{1}{N} (\nabla_a \nabla_b N - g_{ab} \nabla_c \nabla^c N). \end{aligned} \tag{157}$$

Substituting Eq. (94) into Eq. (157) and making use of the relation

$$\begin{aligned} g_{ac} g_{bd} \tilde{\mathcal{L}}_n \Pi^{cd} = & - (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} + 4 K_{ac} K_b^c \\ & - 2 K K_{ab} - 2 K_{cd} K^{cd} g_{ab}, \end{aligned} \tag{158}$$

we get

$$\begin{aligned} \kappa T_{ab} = & -2 K_{ac} K_b^c + K K_{ab} + \frac{1}{2} (3 K_{cd} K^{cd} - K^2) g_{ab} \\ & + (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} \\ & + \frac{1}{N} (\nabla_a \nabla_b N - g_{ab} \nabla_c \nabla^c N). \end{aligned} \tag{159}$$

By Eq. (80), we have

$$\nabla_a a_b - a_a a_b = -\frac{1}{N} \nabla_a \nabla_b N \tag{160}$$

and

$$\nabla_c a^c - a_c a^c = -\frac{1}{N} \nabla_c \nabla^c N. \tag{161}$$

Hence, Eq. (159) is equivalent to

$$\begin{aligned} \kappa T_{ab} = & -2 K_{ac} K_b^c + K K_{ab} + \frac{1}{2} (3 K_{cd} K^{cd} - K^2) g_{ab} \\ & + (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} \\ & - \nabla_a a_b + a_a a_b + (\nabla_c a^c - a_c a^c) g_{ab}. \end{aligned} \tag{162}$$

The expression of κT_{ab} in Eq. (162) agrees with the right-hand side of Eq. (B39) in Appendix B. Hence, the n -dimensional Einstein field Eq. (117) derived from the Lagrangian formulation is equal to the full projection of the

$(n+1)$ -dimensional vacuum Einstein field equation onto \mathcal{M} , up to addition of a term proportional to $g_b^c \tilde{R}_{cd} n^d$.

Equation (162) can be further simplified. By Eq. (162), the trace of κT_{ab} is

$$\begin{aligned} \kappa T = & \left(\frac{3}{2} n - 2 \right) K_{cd} K^{cd} - \left(\frac{n}{2} - 1 \right) K^2 \\ & - (n-1) (g^{cd} \tilde{\mathcal{L}}_n K_{cd} - \nabla_c a^c + a_c a^c). \end{aligned} \tag{163}$$

The trace of Eq. (117) gives rise to

$$\kappa T = -\left(\frac{n}{2} - 1 \right) R. \tag{164}$$

Substituting Eq. (95) into Eq. (164), we get

$$\kappa T = \left(\frac{n}{2} - 1 \right) (K_{cd} K^{cd} - K^2). \tag{165}$$

Then, eliminating κT from Eqs. (163) and (165), we get

$$g^{cd} \tilde{\mathcal{L}}_n K_{cd} - K_{cd} K^{cd} - \nabla_c a^c + a_c a^c = 0. \tag{166}$$

Substituting Eq. (166) into Eq. (162), we get

$$\begin{aligned} \kappa T_{ab} = & -2 K_{ac} K_b^c + K K_{ab} + \frac{1}{2} (K_{cd} K^{cd} - K^2) g_{ab} \\ & + g_a^c g_b^d \tilde{\mathcal{L}}_n K_{cd} - \nabla_a a_b + a_a a_b. \end{aligned} \tag{167}$$

Equation (166) is another scalar constraint equation on \mathcal{M} , which can be used to replace Eq. (95). In fact, with T_{ab} given by Eq. (167), the n -dimensional Einstein field Eq. (117) implies Eq. (95). By Eq. (B21), the scalar Eq. (166) corresponds to $\tilde{R}_{ab} n^a n^b = 0$.

It can be checked that the scalar Eq. (166) is equivalent to

$$\begin{aligned} & g_{cd} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial w} (\sqrt{-g} \Pi^{cd}) \\ & = (3-n) N \left(\Pi_{cd} \Pi^{cd} - \frac{1}{n-1} \Pi^2 \right) \\ & - (n-1) \nabla_c \nabla^c N - \nabla^c (2 N^a \Pi_{ac} - N_c \Pi), \end{aligned} \tag{168}$$

i.e.,

$$\begin{aligned} g_{cd} \tilde{\mathcal{L}}_n \Pi^{cd} = & -(n-3) \Pi_{cd} \Pi^{cd} + \frac{n-4}{n-1} \Pi^2 \\ & - (n-1) \frac{1}{N} \nabla_c \nabla^c N, \end{aligned} \tag{169}$$

after substitution of Eq. (104).

With the supplement of the scalar constraint Eq. (169) [or, equivalently, Eq. (166)], the electromagnetic field Eq. (107) [or, equivalently, Eq. (109)] and the Einstein field Eq. (117) with the stress-energy tensor T_{ab} given by Eq. (157) [or, equivalently, Eq. (167)] form a complete system of field equations on \mathcal{M} .

7 Relation to Kaluza–Klein Theory

Since both the theory presented in this paper and Kaluza–Klein theory are constructed from Einstein’s theory of gravity in a five-dimensional bulk spacetime, and in both theories electromagnetic field equations are derived from the five-dimensional Einstein field equation, the relation and difference between the two theories should be clarified.

The geometric interpretation of Kaluza–Klein decomposition of a five-dimensional metric is described in Appendix A. It is shown that the metric tensor constructed from ${}_k g_{\mu\nu}$, i.e., ${}_k g_{ab} = {}_k g_{\mu\nu} dx_a^\mu dx_b^\nu$ where $\mu, \nu = 0, 1, 2, 3$, is a tensor $\in \mathcal{T}(\mathcal{M}_k)$, where \mathcal{M}_k is a hypersurface orthogonal to the integral curves of $w^a = (\partial/\partial w)^a$. Similarly, $A_a = A_\mu dx_a^\mu$ is a vector $\in \mathcal{T}(\mathcal{M}_k)$. Since when $N^a \neq 0$ the hypersurface \mathcal{M} defined by $w = \text{constant}$ is not orthogonal to w^a , \mathcal{M}_k and \mathcal{M} are two hypersurfaces intersecting at a three-dimensional manifold Σ (Fig. 1).

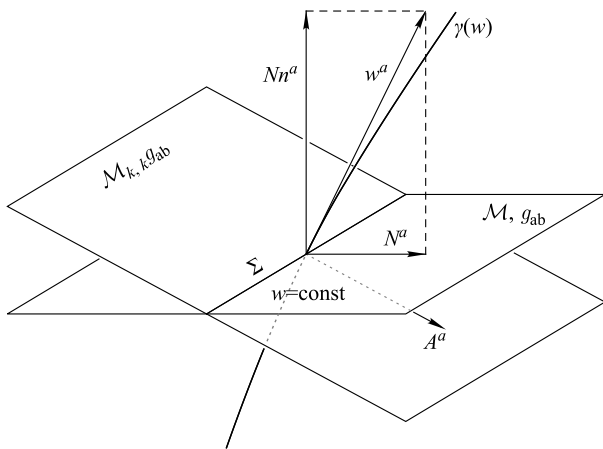


Fig. 1 Geometric relation between Kaluza–Klein theory and the new unified theory presented in this paper. In coordinate system $\{x^0, x^1, x^2, x^3, w\}$ in a five-dimensional spacetime, a hypersurface \mathcal{M} is defined by $w = \text{constant}$. If w -coordinate lines, $\gamma(w)$, are hypersurface orthogonal, there exists a hypersurface \mathcal{M}_k orthogonal to the vector w^a tangent to $\gamma(w)$. The \mathcal{M}_k and \mathcal{M} intersect at a three-dimensional manifold Σ . Projection of the five-dimensional metric tensor \tilde{g}_{ab} onto \mathcal{M} gives rise to a four-dimensional metric tensor g_{ab} on \mathcal{M} . Projection of \tilde{g}_{ab} onto \mathcal{M}_k gives rise to a four-dimensional metric tensor ${}_k g_{ab}$ on \mathcal{M}_k . Projection of w^a onto \mathcal{M} gives rise to a four-dimensional vector field N^a on \mathcal{M} . Projection of $-N^{-1}n^a$ (normal to \mathcal{M}) onto \mathcal{M}_k gives rise to a four-dimensional vector field A^a on \mathcal{M}_k [Eq. (A30)]. The theory presented in this paper is defined by $(\mathcal{M}, g_{ab}, N^a, N)$, where $N = \tilde{g}_{ab}w^a w^b$. Kaluza–Klein theory is defined by $(\mathcal{M}_k, {}_k g_{ab}, A^a, \phi)$, where $\phi = (\tilde{g}_{ab}w^a w^b)^{1/2}$. As discussed in Section 7, the two theories are not related by diffeomorphisms hence they represent different physics in a four-dimensional spacetime.

By Eq. (40), in the Kaluza–Klein representation the line element in the five-dimensional spacetime is

$$d\tilde{s}^2 = {}_k g_{\mu\nu} dx^\mu dx^\nu + \phi^2 (A_\mu dx^\mu + dw)(A_\nu dx^\nu + dw), \quad (170)$$

from which we derive the line element on the hypersurface \mathcal{M} defined by $w = \text{constant}$:

$$ds^2 = ({}_k g_{\mu\nu} + \phi^2 A_\mu A_\nu) dx^\mu dx^\nu. \quad (171)$$

Hence, in Kaluza–Klein theory, the metric tensor on \mathcal{M} is ${}_k g_{\mu\nu} + \phi^2 A_\mu A_\nu$, not ${}_k g_{\mu\nu}$. This indicates that Kaluza–Klein theory is not a unified theory of electromagnetism and gravity in the four-dimensional spacetime spanned by the coordinates $\{x^0, x^1, x^2, x^3\}$.

By Eq. (A33), ${}_k g_{ab}$ is a projection operator onto \mathcal{M}_k . In contrast, g_{ab} is a projection operator onto \mathcal{M} . They are related by Eq. (A36). By Eq. (A30), A^a is related to the normal to the hypersurface \mathcal{M} , i.e., defined by the projection of $-N^{-1}n^a$ onto \mathcal{M}_k . In contrast, N^a is the projection of w^a onto \mathcal{M} . A^a and N^a are related by Eq. (A19).

Since the variables ${}_k g_{ab}$ and A^a are defined on \mathcal{M}_k , Kaluza–Klein theory is inherently defined on the hypersurface \mathcal{M}_k . In contrast, the theory presented in this paper is defined on \mathcal{M} . The existence of \mathcal{M}_k orthogonal to the vector field w^a requires that w^a be hypersurface orthogonal; i.e., w^a must satisfy the condition $w_{[a} \nabla_b w_{c]} = 0$ [19]. Hence, the new unified theory described in this paper is more general than Kaluza–Klein theory, since it does not require that w^a be hypersurface orthogonal.

The unit normal to \mathcal{M} is the vector n^a defined by Eq. (42), i.e.,

$$n^a = \frac{1}{N}(w^a - N^a). \quad (172)$$

By Eq. (A13), the unit normal to \mathcal{M}_k is

$${}_k n^a = \hat{w}^a = \frac{1}{\phi} w^a. \quad (173)$$

Assume that the hypersurface \mathcal{M}_k is defined by $w = f(x^\mu)$, i.e., by $F(x^\mu, w) = f(x^\mu) - w = 0$. Then we have

$$\begin{aligned} {}_k n^a \propto \tilde{\nabla}^a F &= \tilde{g}^{ab} \left(\frac{\partial f}{\partial x^\mu} dx_b^\mu - dw_b \right) \\ &= \left({}_k g^{\mu\nu} \frac{\partial f}{\partial x^\mu} + A^\nu \right) \left(\frac{\partial}{\partial x^\nu} \right)^a \\ &\quad - \left(A^\mu \frac{\partial f}{\partial x^\mu} + \frac{1}{\phi^2} + A_\rho A^\rho \right) \left(\frac{\partial}{\partial w} \right)^a, \end{aligned} \quad (174)$$

where Eq. (A4) has been applied. Therefore, we get the following equation that defines the function $f(x^\mu)$:

$$\frac{\partial f}{\partial x^\mu} = -A_\mu. \quad (175)$$

From Eqs. (172) and (173) we get

$$\tilde{g}_{ab}n^a{}_kn^b = \frac{N}{\sqrt{N^2 + N_c N^c}} = \frac{1}{\sqrt{1 + \phi^2 A_c A^c}}. \quad (176)$$

When N^a and A^a are timelike, we have $\tilde{g}_{ab}n^a{}_kn^b > 1$ and the inclination between the two hypersurfaces \mathcal{M} and \mathcal{M}_k can be interpreted as relative motion between them, with a relative velocity $\beta = \phi\sqrt{-A_c A^c} = N^{-1}\sqrt{-N_c N^c}$. When N^a and A^a are spacelike, we have $\tilde{g}_{ab}n^a{}_kn^b < 1$ and the inclination between \mathcal{M} and \mathcal{M}_k can be interpreted as relative spatial rotation, with the rotation angle $\alpha = \arccos(\tilde{g}_{ab}n^a{}_kn^b)$. When N^a and A^a are null, we have $\tilde{g}_{ab}n^a{}_kn^b = 1$, with both N^a and A^a being tangent to Σ (Proposition 12 in Appendix A). \mathcal{M} and \mathcal{M}_k are still two different hypersurfaces.

The fundamental variables on \mathcal{M} are g_{ab} , N^a , and N . The fundamental variables on \mathcal{M}_k are ${}_k g_{ab}$, A^a , and ϕ . From the results in Appendix A, we can derive the relations between the two groups of variables, which are given by

$$g_{ab} = {}_k g_{ab} + \frac{1}{1 + \phi^2 A_c A^c} \times (A_c A^c w_a w_b + A_a w_b + A_b w_a - \phi^2 A_a A_b) \quad (177)$$

and

$$\begin{cases} N^a = \frac{\phi^2}{1 + \phi^2 A_c A^c} (A^a + A_c A^c w^a), \\ N = \frac{\phi}{\sqrt{1 + \phi^2 A_c A^c}}. \end{cases} \quad (178)$$

Clearly, the transformations described by Eqs. (177) and (178) are not diffeomorphic transformations. Diffeomorphisms map tensors by linear transformations (see, e.g., [19, 21]), but the transformations in (177) and (178) are nonlinear.

Therefore, the new theory presented in this paper is physically different from Kaluza–Klein theory, although both aim at unifying the electromagnetic and gravitational interactions in the framework of general relativity in a five-dimensional spacetime. The two theories are defined on two different hypersurfaces in a five-dimensional spacetime and are not related by diffeomorphisms. In general relativity, theories defined by tensor fields on manifolds are physically identical only if they are related by diffeomorphic transformations (see Ref. [19] and the next section of this paper). Since both theories are derived from Einstein’s theory of gravity in a five-dimensional spacetime, mathematically they are related by the transformations in Eqs. (177) and (178). However, the two theories are physically distinguishable. This is similar to the case of conformal transformations: Two spacetimes related by conformal transformations are usually not identical in physics, unless the conformal factor

is a constant. For example, the Robertson–Walker spacetime metric of a universe is related to the flat Minkowski spacetime metric by a conformal transformation, but the physics in a curved universe is different from that in a flat spacetime.

The physical difference between the two theories is manifested by the fact that in Kaluza–Klein theory the standard Maxwell equation is derived, but in our theory the derived electromagnetic field equation contains a term coupled to the spacetime curvature. The curvature-coupled term cannot be eliminated by diffeomorphic transformations. Only in a Ricci-flat four-dimensional spacetime are the two field equations identical. In addition, Kaluza–Klein theory contains an unidentified scalar field, and the total five physical degrees of freedom of gravitons in the five-dimensional bulk spacetime are shared by gravitons (two), photons (two), and the scalar field (one) in the four-dimensional spacetime. In our theory, the scalar field does not exist. The total five physical degrees of freedom of gravitons in the bulk spacetime are shared by gravitons (two) and photons (three) in the four-dimensional spacetime. In our theory, photons have three degrees of freedom since the curvature-coupled term in the field equation breaks the gauge symmetry and causes photons to acquire an effective mass.

In the two theories, the derived gravitational field equations in a four-dimensional spacetime are also different, since the stress-energy tensors on the right-hand side of the field equation are different. This can be directly verified by comparison of the stress-energy tensor derived in this paper [Eqs. (118), (125), (137), and (149)] with that in Kaluza–Klein theory [13, 14].

Finally, our theory and the procedure adopted in this paper are more general than Kaluza–Klein theory and the procedure used in it. As already mentioned above, our theory does not require that the vector field w^a be hypersurface orthogonal, but Kaluza–Klein theory does. In addition, the notation of “normal” or “orthogonality” requires the existence of a prespecified metric tensor, but in both theories the metric tensor needs to be solved from field equations. This is also the reason why in the Hamiltonian formulation of general relativity a Gaussian normal coordinate system cannot be used to simplify the problem.

8 Diffeomorphism and gauge symmetry

It is well known that the Maxwell theory of electromagnetic fields is invariant under the gauge transformation

$$A_a \rightarrow A_a + \nabla_a \chi, \quad (179)$$

where χ is any scalar function. Under this gauge transformation, the antisymmetric electromagnetic field ten-

for F_{ab} defined by Eq. (17) is unchanged, and hence the Einstein–Maxwell Eq. (20) is unchanged. In a Ricci-flat spacetime, the electromagnetic field Eq. (1) reduces to the Einstein–Maxwell Eq. (20) and so is invariant under the gauge transformation. However, when the spacetime is not Ricci-flat, the electromagnetic field Eq. (1) is not invariant under the gauge transformation. So, in the theory presented in this paper, it appears that the presence of Ricci curvature leads to gauge symmetry breaking to the electromagnetic field equation.

The above observation reminds us the Proca equation, which is a generalization of the Maxwell equation in which a photon mass term has been introduced. The presence of a photon mass term makes the Proca equation not invariant under the gauge transformation. However, it is possible to restore the gauge symmetry by introducing a complex scalar field interacting with the electromagnetic field [31, 32]. In a high-energy state, the scalar field has a true vacuum at a zero value and photons remain massless. In a low-energy state, the scalar field has a true vacuum at a nonzero value through spontaneous symmetry breaking, and photons acquire a mass through the Higgs-like mechanism. Hence, a theory with a broken gauge symmetry can be fitted into an underlying and more fundamental theory with gauge symmetry. This is a well known fact in quantum field theory.

For the theory presented in this paper we have a similar situation. Despite the fact that the derived electromagnetic field equation in a four-dimensional spacetime is not invariant under the gauge transformation defined by Eq. (179), the underlying theory — Einstein’s theory of gravity in a five-dimensional bulk spacetime from which the electromagnetic field equation is derived — is invariant under the gauge transformation defined by diffeomorphisms in the bulk spacetime. The apparent breakdown of gauge invariance for the electromagnetic field Eq. (1) with $\xi = -2$ — then the field equation can be derived from the Einstein field equation in the bulk spacetime — originates from slicing of the bulk spacetime with timelike hypersurfaces.

One may wonder how the gauge transformation in Eq. (179) is related to diffeomorphisms in a spacetime. In this section, we show that the gauge transformation (179) can be interpreted as a diffeomorphic transformation in the background spacetime and hence can be regarded as arising from diffeomorphisms of spacetime. Since the electromagnetic field Eq. (1) is invariant under any diffeomorphism, it is invariant under the gauge transformation (179) provided that the corresponding diffeomorphism is applied to all variables appearing in the field equation.

As a geometric theory, general relativity is known to be invariant under diffeomorphic transformations. In fact, all field equations expressed in tensors (including

scalar functions and vectors as special cases) defined on a manifold are invariant under diffeomorphism. Hence, diffeomorphisms comprise the gauge freedom of any theory formulated in terms of tensor fields, including general relativity itself [19–21]. Assume that $(\mathcal{M}, g_{ab}, \psi)$ defines a spacetime (\mathcal{M}, g_{ab}) and matter fields ψ on it, where ψ can be scalar, vector, and tensor fields. Under a diffeomorphism $\phi : \mathcal{M} \rightarrow \mathcal{M}$, with the map ϕ and its inverse ϕ^{-1} any tensor field T is transformed to $T' = \phi_* T$. Then, $(\mathcal{M}, g_{ab}, \psi)$ is transformed to $(\mathcal{M}, \phi_* g_{ab}, \phi_* \psi)$. According to general relativity, $(\mathcal{M}, g_{ab}, \psi)$ and $(\mathcal{M}, \phi_* g_{ab}, \phi_* \psi)$ belong to the same equivalent class under diffeomorphisms and represent the same physics.

Before demonstrating that the gauge transformation (179) can be generated by a diffeomorphism on the manifold where electromagnetic fields are defined, we present some lemmas that ensure that any theory defined on a manifold in terms of tensors is invariant under diffeomorphic transformations.

Lemma 1. On a manifold \mathcal{M} , any action defined by

$$S = \int \mathcal{L} \epsilon \quad (180)$$

is invariant under diffeomorphisms, where the Lagrangian density \mathcal{L} is a scalar function and ϵ is a volume element.

Lemma 1 is a direct consequence of the fact that the integral of any n -form on an n -dimensional manifold is invariant under diffeomorphisms [20]. It can also be proved as follows. Let $\epsilon = \sqrt{-g} e$, where e is a fixed volume on \mathcal{M} . Under an infinitesimal diffeomorphism generated by a vector field $v^a = (\partial/\partial\tau)^a$, we have

$$\delta g_{ab} = -(\mathcal{L}_v g_{ab})\delta\tau = -2\nabla_{(a} v_{b)}\delta\tau, \quad (181)$$

where \mathcal{L} is the Lie derivative operator on \mathcal{M} and ∇_a is the derivative operator associated with the metric tensor g_{ab} . Then, by Eq. (111) we get $\delta\sqrt{-g} = -\sqrt{-g}(\nabla_a v^a)\delta\tau$. On the other hand, we have $\delta\mathcal{L} = -(\mathcal{L}_v \mathcal{L})\delta\tau = -(v^a \nabla_a \mathcal{L})\delta\tau$. Hence, we get

$$\delta(\sqrt{-g} \mathcal{L}) = -\sqrt{-g} \nabla_a (\mathcal{L} v^a \delta\tau). \quad (182)$$

The term $\nabla_a (\mathcal{L} v^a \delta\tau)$ in Eq. (182) is a boundary term that has no contribution to the action integral by Gauss’s theorem, if we set $v^a = 0$ on the boundary. Hence, we get $\delta S = 0$ under the variation generated by a diffeomorphism.

Lemma 2. For any tensor field $T^{a_1 \dots a_k}_{b_1 \dots b_l}$ on \mathcal{M} ,

$$\phi_* (\nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_l}) = \nabla'_c (\phi_* T^{a_1 \dots a_k}_{b_1 \dots b_l}), \quad (183)$$

where ∇_a is the derivative operator associated with g_{ab} and ∇'_a is the derivative operator associated with $\phi_* g_{ab}$ (see, e.g., [33]).

Proof. Since $\phi_*(\nabla T) = \phi_*(\nabla(\phi_*^{-1}\phi_*T)) \equiv f(\phi_*T)$, $\phi_*(\nabla T)$ is a map of ϕ_*T . It can be checked that $f(\phi_*T)$ satisfies the definition for a derivative operator on \mathcal{M} , because ϕ is a linear map preserving tensor type and the relations in the tensor algebra, and ∇ is a derivative operator. For example, the Leibnitz rule can be verified as follows: $f(\phi_*T_1 \otimes \phi_*T_2) = f(\phi_*(T_1 \otimes T_2)) = \phi_*(\nabla(T_1 \otimes T_2)) = \phi_*(\nabla T_1 \otimes T_2 + T_1 \otimes \nabla T_2) = \phi_*(\nabla T_1) \otimes \phi_*T_2 + \phi_*T_1 \otimes \phi_*(\nabla T_2) = f(\phi_*T_1) \otimes \phi_*T_2 + \phi_*T_1 \otimes f(\phi_*T_2)$. Hence, we can write $\phi_*(\nabla T) = \nabla'(\phi_*T)$, where ∇' is a derivative operator. Since $\nabla g = 0$ everywhere, we get $\nabla'(\phi_*g) = \phi_*(\nabla g) = 0$. *End of Proof.*

Therefore, the action of diffeomorphisms preserves both algebraic and derivative relations in tensors, if the derivative operator is associated with the metric tensor. As an example, diffeomorphic transformation of a Riemann curvature tensor defined by a metric and the derivative operator associated with it is equivalent to the Riemann curvature tensor defined by the diffeomorphic transformation of the metric tensor and the derivative operator associated with it. That is, $\phi_*(R_{abc}{}^d) = R'_{abc}{}^d$, where $R'_{abc}{}^d$ is defined by $\phi_*(g_{ab})$ and $\nabla'_a, \nabla'_a(\phi_*g_{bc}) = 0$. Hence we have the following lemma:

Lemma 3. In a spacetime, physical laws expressed by tensors and their derivatives defined by the derivative operator associated with the metric tensor are invariant under diffeomorphisms.

The above discussion and results apply to any spacetime of any dimension, including both the $(n + 1)$ -dimensional $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ and the n -dimensional (\mathcal{M}, g_{ab}) as a hypersurface embedded in $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$. The Lie derivative operator \mathcal{L} on \mathcal{M} is related to the Lie derivative operator $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{M}}$ by

$$\mathcal{L}_v T^{a_1 \dots a_k}_{b_1 \dots b_l} = g^{a_1}_{c_1} \dots g_{b_l}^{d_l} \tilde{\mathcal{L}}_v T^{c_1 \dots c_k}_{d_1 \dots d_l}, \quad (184)$$

for any v^a and $T^{a_1 \dots a_k}_{b_1 \dots b_l} \in \mathcal{T}(\mathcal{M})$.

Since the theory presented in this paper is based on an $(n + 1)$ -dimensional Einstein field equation on $\tilde{\mathcal{M}}$, Lemmas 1–3 guarantee that all the field equations derived in previous sections are invariant under diffeomorphisms on $\tilde{\mathcal{M}}$. In particular, the derived electromagnetic field Eq. (107) is invariant under the gauge transformation defined by all diffeomorphisms on $\tilde{\mathcal{M}}$.

Here we consider diffeomorphisms restricted to the n -dimensional \mathcal{M} , which are a subset of the diffeomorphisms on $\tilde{\mathcal{M}}$. Let ϕ_τ be a one-parameter group of diffeomorphisms, which is generated by a vector field $v^a = (\partial/\partial\tau)^a$ on \mathcal{M} . Then, by Eq. (181), under an infinitesimal transformation the metric tensor g_{ab} is transformed to $g'_{ab} = g_{ab} - 2\nabla_{(a}v_{b)}\delta\tau$. Any vector $N_a \in \mathcal{T}(\mathcal{M})$ is transformed to $N'_a = N_a - (\mathcal{L}_v N_a)\delta\tau$, with

$$\mathcal{L}_v N_a = \nabla_a(v^c N_c) + v^c \mathcal{F}_{ca}, \quad (185)$$

where the antisymmetric tensor

$$\mathcal{F}_{ca} \equiv \nabla_c N_a - \nabla_a N_c. \quad (186)$$

Lemma 4. For any vector field $N_a \in \mathcal{T}(\mathcal{M})$ and any smooth function χ on \mathcal{M} , there exists a vector field $v^a \in \mathcal{T}(\mathcal{M})$ so that

$$\mathcal{L}_v N_a = \nabla_a \chi. \quad (187)$$

Proof. When \mathcal{F}_{ca} is nondegenerate, i.e., $\det \mathcal{F}_{ca} \neq 0$, there exists an inverse antisymmetric tensor $\hat{\mathcal{F}}^{ab}$ defined by $\mathcal{F}_{ca} \hat{\mathcal{F}}^{ab} = \delta_c^b$. Then, for any smooth function χ' , the linear algebraic equation $v^c \mathcal{F}_{ca} = \nabla_a \chi'$ has a unique solution $v^a = \hat{\mathcal{F}}^{ca} \nabla_c \chi'$. Substituting it into Eq. (185), we get Eq. (187) with

$$\chi = \chi' - N_c \hat{\mathcal{F}}^{ca} \nabla_a \chi'. \quad (188)$$

Given any function $\chi(x^0, \dots, x^{n-1})$ on \mathcal{M} , Eq. (188) can be solved for $\chi'(x^0, \dots, x^{n-1})$ as follows. Let $x^\mu = x^\mu(s)$ ($\mu = 0, \dots, n-1$) define the integral curves of the vector field $k^a \equiv -N_c \hat{\mathcal{F}}^{ca}$. Then, $k^\mu = dx^\mu/ds$. By $d\chi'/ds = (\partial_\mu \chi') dx^\mu/ds$, Eq. (188) is equivalent to the first-order ordinary differential equation

$$e^{-s} \frac{d}{ds} (e^s \chi'(s)) = \chi(x^0(s), \dots, x^{n-1}(s)), \quad (189)$$

whose solution is given by the integral

$$\chi'(s) = e^{-s} \int e^s \chi(x^0(s), \dots, x^{n-1}(s)) ds. \quad (190)$$

When \mathcal{F}_{ca} is degenerate, i.e., $\det \mathcal{F}_{ca} = 0$, the linear algebraic equation $v^c \mathcal{F}_{ca} = 0$ of v^c has an infinite number of solutions. Let v^c_1 be a solution. Then, $v^c = \chi' v^c_1$ must also be a solution, where χ' is any function. Hence, $v^c \mathcal{F}_{ca} = 0$, and Eq. (185) becomes Eq. (187) with $\chi = \chi' v^c_1 N_c$. *End of Proof.*

Theorem 1. For any vector field $N_a \in \mathcal{T}(\mathcal{M})$ and any smooth function χ on \mathcal{M} , there exists a diffeomorphism ϕ so that

$$\phi_* N_a = N_a + \nabla_a \chi. \quad (191)$$

Proof. By Lemma 4, for any function χ_1 there exists a vector field $v^a = (\partial/\partial\tau)^a$ so that $N'_a = N_a - \delta\tau \nabla_a \chi_1$ under an infinitesimal diffeomorphism generated by v^a . Then, at any point p on \mathcal{M} , we have (e.g., [20])

$$\begin{aligned} \phi_{\tau*} N_a|_p &= N_a|_p - \int_0^\tau \phi_{\tau'*} \left(\mathcal{L}_v N_a|_{\phi_{-\tau'(p)}} \right) d\tau' \\ &= N_a|_p - \int_0^\tau \phi_{\tau'*} \left(\nabla_a \chi_1|_{\phi_{-\tau'(p)}} \right) d\tau' \\ &= N_a|_p - \int_0^\tau \nabla_a|_p \phi_{\tau'*} \left(\chi_1|_{\phi_{-\tau'(p)}} \right) d\tau' \\ &= N_a|_p + \nabla_a|_p \chi_p, \end{aligned} \quad (192)$$

where

$$\chi_p \equiv - \int_0^\tau \phi_{\tau'} * \left(\chi_1|_{\phi_{-\tau'}(p)} \right) d\tau'. \quad (193)$$

Equation (193) indicates that $\chi_1 = -\phi_*^{-1}(d\chi/d\tau)$. *End of Proof.*

Theorem 2. For any closed two-form \mathcal{F}_{ab} on \mathcal{M} , there exists a symmetry transformation. That is, there exists a vector field v^a on \mathcal{M} so that

$$\mathcal{L}_v \mathcal{F}_{ab} = 0. \quad (194)$$

Proof. By the converse of the Poincaré lemma, any closed two-form \mathcal{F}_{ab} can be expressed as in Eq. (186), i.e., $\mathbf{F} = d\mathbf{N}$. Since the exterior derivative operator d commutes with the Lie derivative operator [20], we have $\mathcal{L}_v \mathbf{F} = d\mathcal{L}_v \mathbf{N}$. By Lemma 4, there exists a vector field v^a so that $\mathcal{L}_v \mathbf{N} = d\chi$ [Eq. (187)]. Then, by the Poincaré lemma, we get $\mathcal{L}_v \mathbf{F} = d^2\chi = 0$. *End of Proof.*

Theorems 1 and 2 indicate that in a spacetime there always exists a diffeomorphism that gives rise to the gauge transformation in Eq. (179) for electromagnetic fields, where χ is any smooth function. The electromagnetic field tensor F_{ab} is invariant under the diffeomorphism. Since the derivative operator in the definition of F_{ab} , i.e., Eq. (17), can be any derivative operator, we can take it to be the ∇_a before the diffeomorphism. If we use g^{ab} before the diffeomorphism to raise the indexes of A_a , $\nabla_a \chi$, and F_{ab} , then we get that $A^a \rightarrow A^a + \nabla^a \chi$, and F^{ab} is also invariant under the gauge transformation. Then, if in the field equation we take the metric tensor to be g_{ab} and g^{ab} before the diffeomorphism, the derivative operator to be the ∇_a associated with g_{ab} , we find that the Einstein–Maxwell Eq. (20) is invariant under the gauge transformation. Thus, we have successfully fitted the electromagnetic gauge symmetry into diffeomorphisms in the background spacetime.

Of course, the electromagnetic field Eq. (1) is not invariant under the gauge transformation if we stay on the original spacetime background, unless $R_{ab} = 0$. However, if we apply the diffeomorphic transformation to all tensor variables appearing in the equation, including the electromagnetic field vector and tensor, the current density vector, and the metric tensor and quantities derived from it, then Eq. (1) must be invariant according to Lemma 3.

9 The cosmological constant

We have interpreted \dot{g}_{ab} as the representation of a matter field on \mathcal{M} , but we have not specified the nature of the matter. It may represent dark matter that has been observed to exist in galaxies and clusters of galaxies and interact with ordinary matter only through the gravitational interaction [34] or dark energy (equivalent to a

cosmological constant in some models) that is uniformly distributed in the universe and responsible for the observed accelerating expansion of the universe [35, 36]. It is also possible that \dot{g}_{ab} represents some other kind of unknown matter. In this section we investigate whether \dot{g}_{ab} can behave as a cosmological constant in certain situations [37].

We consider a special case in which, in the neighborhood of the hypersurface \mathcal{M} defined by $w = 0$, the metric tensors g_{ab} and g^{ab} can be approximated by

$$\begin{cases} g_{ab}(w) = g_{ab}(1 + \lambda w), \\ g^{ab}(w) = g^{ab}(1 - \lambda w), \end{cases} \quad (195)$$

where $g_{ab} = g_{ab}(w = 0)$, $g^{ab} = g^{ab}(w = 0)$, λ is a constant, and $\lambda w \ll 1$. Then, we have

$$\dot{g}_{ab} = \lambda g_{ab}, \quad \dot{g}^{ab} = -\lambda g^{ab}, \quad (196)$$

and

$$\dot{g}_a{}^b = \ddot{g}_{ab} = \ddot{g}^{ab} = 0. \quad (197)$$

Then, by the definition of Φ_{ab} , we have

$$\begin{cases} \Phi_{ab} = \frac{1}{2}(n-1)N^{-1}\lambda g_{ab}, \\ \Phi = \frac{1}{2}n(n-1)N^{-1}\lambda, \end{cases} \quad (198)$$

and

$$\begin{aligned} \dot{\Phi}^{ab} &= -\frac{1}{2}N^{-1}(\dot{g}^{ac}g^{bd} + g^{ac}\dot{g}^{bd} - \dot{g}^{ab}g^{cd} - g^{ab}\dot{g}^{cd})\dot{g}_{cd} \\ &\quad -\frac{1}{2}N^{-2}\dot{N}(g^{ac}g^{bd} - g^{ab}g^{cd})\dot{g}_{cd} \\ &= -(n-1)\lambda\left(\lambda + \frac{1}{2}N^{-1}\dot{N}\right)N^{-1}g^{ab}, \end{aligned} \quad (199)$$

where $\dot{N} = \partial N/\partial w$.

By Eq. (154), we get

$$\begin{aligned} &g_{ac}g_{bd}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Phi^{cd}) \\ &= \frac{1}{n-1}N\Phi\Phi_{ab} + g_{ac}g_{bd}\dot{\Phi}^{cd} \\ &= (n-1)\lambda N^{-1}\left[\left(\frac{n}{4}-1\right)\lambda - \frac{1}{2}N^{-1}\dot{N}\right]g_{ab}. \end{aligned} \quad (200)$$

By Eq. (198), we have

$$\Phi_{cd}\Phi^{cd} - \frac{1}{n-1}\Phi^2 = -\frac{1}{4}n(n-1)N^{-2}\lambda^2 \quad (201)$$

and

$$\Phi_{ac}\Phi_b{}^c - \frac{1}{n-1}\Phi\Phi_{ab} = -\frac{1}{4}(n-1)N^{-2}\lambda^2 g_{ab}. \quad (202)$$

Substituting Eqs. (200)–(202) into Eq. (137), we get

$$T_{m,ab} = -\frac{1}{2\kappa}(n-1)\lambda N^{-2} \left[\left(\frac{n}{4} - 1\right) \lambda - N^{-1}\dot{N} \right] g_{ab}. \quad (203)$$

Letting $n = 4$, we have

$$T_{m,ab} = \frac{3}{2\kappa}\lambda N^{-3}\dot{N}g_{ab}, \quad (204)$$

which corresponds to a cosmological constant

$$\Lambda = -\frac{3}{2}\lambda N^{-3}\dot{N} \quad (205)$$

in the four-dimensional spacetime (\mathcal{M}, g_{ab}) .

By Eq. (198), we have

$$\Phi_{cd}\Psi^{cd} - \frac{1}{n-1}\Phi\Psi = -\frac{1}{2}N^{-1}\lambda\Psi \quad (206)$$

and

$$\begin{aligned} & 2\Phi_{c(a}\Psi_{b)}^c - \frac{1}{n-1}(\Phi\Psi_{ab} + \Psi\Phi_{ab}) \\ &= \left(\frac{n}{2} - 1\right)N^{-1}\lambda\Psi_{ab} - \frac{1}{2}N^{-1}\lambda\Psi g_{ab}. \end{aligned} \quad (207)$$

By Eqs. (154) and (198), we derive that

$$\begin{aligned} & g_{ac}g_{bd}\frac{1}{\kappa N\sqrt{-g}}\frac{\partial}{\partial w}(\sqrt{-g}\Psi^{cd}) \\ &= \frac{1}{\kappa N}g_{ac}g_{bd}\dot{\Psi}^{cd} + \frac{1}{2\kappa}nN^{-1}\lambda\Psi_{ab}. \end{aligned} \quad (208)$$

By Eq. (198) and the convention $\nabla_a N = 0$, we have $\nabla_a\Phi_{bc} = 0$. Hence we get

$$\begin{aligned} & \frac{1}{\kappa N}\nabla^c [2N_{(a}\Phi_{b)c} - N_c\Phi_{ab}] \\ &= \frac{1}{\kappa}N^{-1}\lambda \left[(n-1)\Psi_{ab} - \frac{1}{2}\Psi g_{ab} \right]. \end{aligned} \quad (209)$$

Substituting Eqs. (206)–(209) into Eq. (149), we get

$$T_{\text{int},ab} = -\frac{1}{\kappa N} \left[\left(\frac{n}{2} + 1\right) \lambda\Psi_{ab} + g_{ac}g_{bd}\dot{\Psi}^{cd} \right]. \quad (210)$$

Setting $n = 4$, we get

$$\begin{aligned} T_{\text{int},ab} &= -\frac{1}{\kappa N} \left(3\lambda\Psi_{ab} + g_{ac}g_{bd}\dot{\Psi}^{cd} \right) \\ &= -\frac{1}{\kappa N} \left(\lambda\Psi_{ab} + \dot{\Psi}_{ab} \right). \end{aligned} \quad (211)$$

By Eqs. (106) and (198), we have $4\pi J^b = \nabla_a\Phi^{ab} = 0$ (since $\nabla_a N = 0$). Then, the electromagnetic field Eq. (107) becomes a source-free equation,

$$\nabla_a\Psi^{ab} = 0. \quad (212)$$

Equations (204) and (205) indicate that, in appropriate conditions, the matter represented by \dot{g}_{ab} behaves

like a cosmological constant Λ on \mathcal{M} . However, by Eq. (205), $\Lambda \neq 0$ only if $\dot{N} = \partial N/\partial w \neq 0$, and the sign of Λ is determined by the sign of \dot{N} . To get a positive cosmological constant on \mathcal{M} , we need to have $\dot{N} < 0$. Let us assume that $\dot{N} \sim -\lambda N$, by Eq. (205) we have $\Lambda \sim N^{-2}\lambda^2$, and the mass density corresponding to Λ is $\rho_\Lambda = \Lambda/\kappa \sim \kappa^{-1}N^{-2}\lambda^2$. Hence,

$$\frac{\rho_\Lambda}{\rho_P} \sim \kappa^{-1}N^{-2}\lambda^2 l_P^2, \quad (213)$$

where $\rho_P = l_P^{-2} = 5.2 \times 10^{93} \text{ g cm}^{-3}$ is the Planck mass density. Observations from the Wilkinson Microwave Anisotropy Probe (WMPA) and Planck satellites have indicated that there may exist a positive cosmological constant in the universe with an equivalent mass density $\approx 0.7\rho_{\text{crit}}$, where ρ_{crit} is the critical mass density [35, 36], which leads to $\rho_\Lambda/\rho_P \approx 10^{-123}$. Then, if we set the dimensionless function $N \sim 1$ and $\kappa = 8\pi$, Eq. (213) leads to

$$\lambda^{-1} \sim 6.3 \times 10^{60} l_P \sim 10^{28} \text{ cm}, \quad (214)$$

which is of the same order as the present Hubble distance.

However, if λ^{-1} is much smaller than the Hubble distance, Eq. (213) indicates a cosmological constant that is much larger than that we have observed, being on the order of

$$\frac{\rho_\Lambda}{\rho_P} \sim 10^{-65} \left(\frac{\lambda^{-1}}{1 \text{ mm}} \right)^{-2}. \quad (215)$$

Such an unrealistically large cosmological constant may be canceled by a native cosmological constant in the five-dimensional bulk spacetime, as has been assumed in brane world theory [38].

The equations derived in Sections 5 and 6 can be extended to the case in which there is a cosmological constant $\tilde{\Lambda}$ in the $(n+1)$ -spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$, i.e., when the vacuum Einstein field equation on $\tilde{\mathcal{M}}$ is $\tilde{G}_{ab} + \tilde{\Lambda}\tilde{g}_{ab} = 0$, corresponding to the following action of gravity on $\tilde{\mathcal{M}}$:

$$\begin{aligned} \tilde{S}_G &= \int \sqrt{-\tilde{g}}(\tilde{R} - 2\tilde{\Lambda})\tilde{e} \\ &= \int \sqrt{-\tilde{g}} \left(R - K_{ab}K^{ab} + K^2 - 2\tilde{\Lambda} \right) \tilde{e}. \end{aligned} \quad (216)$$

The presence of $\tilde{\Lambda}$ in the action leads only to modification of \mathcal{L}_G in Eq. (67), and the modified \mathcal{L}_G is

$$\mathcal{L}_G \equiv \sqrt{-g}N(R - 2\tilde{\Lambda}). \quad (217)$$

The Lagrangians \mathcal{L}_{EM} , \mathcal{L}_m , and \mathcal{L}_{int} defined in Eqs. (88)–(90) are not changed.

It is easy to derive that, when $\tilde{\Lambda}$ is present, Eqs. (93) and (95) become

$$R + \Pi_{ab}\Pi^{ab} - \frac{1}{n-1}\Pi^2 = 2\tilde{\Lambda} \quad (218)$$

and

$$R + K_{ab}K^{ab} - K^2 = 2\tilde{\Lambda}, \quad (219)$$

Equations (107) and (109), which are interpreted as the electromagnetic field equations, are not changed since, in the Lagrangian density, $\tilde{\Lambda}$ is not coupled to N_a . The gravitational field equation on \mathcal{M} , i.e., Eq. (117), becomes

$$G_{ab} + \tilde{\Lambda}g_{ab} = \kappa T_{ab}, \quad (220)$$

where the stress-energy tensor T_{ab} is unchanged [being still given by Eqs. (118), (125), (137), and (149)].

Correspondingly, Eq. (166) becomes

$$g^{cd}\tilde{\mathcal{L}}_n K_{cd} - K_{cd}K^{cd} - \nabla_c a^c + a_c a^c + \frac{2}{n-1}\tilde{\Lambda} = 0, \quad (221)$$

Eq. (168) becomes

$$\begin{aligned} & g_{cd} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial w} (\sqrt{-g} \Pi^{cd}) \\ &= (3-n)N \left(\Pi_{cd} \Pi^{cd} - \frac{1}{n-1} \Pi^2 \right) - (n-1) \nabla_c \nabla^c N \\ & \quad - \nabla^c (2N^a \Pi_{ac} - N_c \Pi) - 2N\tilde{\Lambda}, \end{aligned} \quad (222)$$

and Eq. (169) becomes

$$\begin{aligned} g_{cd} \tilde{\mathcal{L}}_n \Pi^{cd} &= -(n-3) \Pi_{cd} \Pi^{cd} + \frac{n-4}{n-1} \Pi^2 \\ & \quad - (n-1) \frac{1}{N} \nabla_c \nabla^c N - 2\tilde{\Lambda}. \end{aligned} \quad (223)$$

The effective cosmological constant on \mathcal{M} is $\Lambda_{\text{eff}} = \tilde{\Lambda} + \Lambda$, with Λ being determined by Eq. (205). With some unknown fine-tuning mechanism (as in brane world theory), Λ_{eff} may become zero or small enough to be compatible with the observations in cosmology.

10 Discussion: The new electromagnetic field equation

This section focuses on a discussion of the new electromagnetic field Eq. (1), which is the most important equation derived in this paper and marks the physical difference between the new unified theory and Kaluza–Klein theory. Although $\xi = -2$ is preferred since then the equation can be derived from the Einstein field equation in a five-dimensional spacetime, here for generality we treat ξ as an undetermined number of order unity (being either positive or negative). Solutions to (1) will not be provided here, except for a very simple case involving a Killing vector field as a solution. Consequently, we will not attempt to explore phenomenological signals explicitly and quantitatively predicted by solutions to the field

Eq. (1). Instead, we will only present a general discussion and make some comments relevant to experimental tests of the new field equation in a four-dimensional spacetime.

In terms of A^a , the electromagnetic field Eq. (1) can be written as

$$\nabla^a \nabla_a A^b - \nabla^b \nabla_a A^a - (\xi + 1) R^b{}_a A^a = -4\pi J^b. \quad (224)$$

We know that a Killing vector ψ^a in a spacetime satisfies Eqs. [19]

$$\nabla_a \psi^a = 0, \quad \nabla^a \nabla_a \psi^b + R^b{}_a \psi^a = 0. \quad (225)$$

Comparison of Eqs. (224) and (225) leads to the result that *when $\xi = -2$, any Killing vector field in a spacetime (\mathcal{M}, g_{ab}) is a solution of the source-free electromagnetic field Eq. (1)*. This is in contrast to the case of the Maxwell Eq. (20), where a Killing vector field solves the source-free equation only if the spacetime is Ricci-flat [19, 39].

The above result can be more easily derived from Eq. (22), which is equivalent to the field Eq. (1) with $\xi = -2$. If we set $A^a = \psi^a$, by the Killing equation $\nabla_a \psi_b + \nabla_b \psi_a = 0$ we get $H_{ab} = 0$ and $H = 0$. Hence a Killing vector ψ^a solves the electromagnetic field Eq. (22) when $J^a = 0$.

The result can be somewhat generalized. Let ψ^a be a conformal Killing vector field, i.e., ψ^a satisfies [19]

$$\nabla_a \psi_b + \nabla_b \psi_a = \alpha g_{ab}, \quad (226)$$

where α is any function. Setting $A^a = \psi^a$, we get $H_{ab} = \alpha g_{ab}$ and $H = n\alpha$, where $n = \dim \mathcal{M}$. Substituting these expressions into the electromagnetic field Eq. (22), we find that Eq. (22) is solved when $J^a = 0$ if and only if $\nabla_a \alpha = 0$. Hence, *any conformal Killing vector field with a constant α solves the source-free electromagnetic field Eq. (1) with $\xi = -2$* .

The above conclusion applies to a spacetime of any dimensions. However, in the remaining part of this section we assume that $n = 4$ since the focus will be on experimental tests of the electromagnetic field Eq. (1) in a four-dimensional spacetime.

The stress-energy tensor of electromagnetic fields described by Eq. (1) was derived in Ref. [22]; it becomes Eq. (38) when $\xi = -2$. For any value of ξ , the divergence of the stress-energy tensor of electromagnetic fields is evaluated to be [22]

$$\nabla^a T_{\text{EM},ab} = -F_{ba} J^a + A_b \nabla_a J^a, \quad (227)$$

where the electromagnetic field equation (1) has been applied. When the electric current is conserved (i.e., $\nabla_a J^a = 0$), Eq. (227) becomes

$$\nabla^a T_{\text{EM},ab} = -F_{ba} J^a. \quad (228)$$

If we denote the total stress-energy tensor by $T_{ab} = T_{EM,ab} + T_{Other,ab}$, where $T_{Other,ab}$ represents the stress-energy tensor of other matter fields, by the conservation equation $\nabla^a T_{ab} = 0$ we get

$$\nabla^a T_{Other,ab} = F_{ba} J^a, \tag{229}$$

which is just the Lorentz force law.

Hence, the curvature-coupled term in the electromagnetic field equation does not affect the Lorentz force law. The force of a charged particle or an electric current in an electromagnetic field is determined by the antisymmetric tensor F_{ab} (i.e., by the electric field \mathbf{E} and the magnetic field \mathbf{B}) solving the electromagnetic field Eq. (1). Although the potential vector A^a explicitly appears in the electromagnetic field equation through the curvature-coupled term, it interacts with charges and currents only through the antisymmetric tensor F_{ab} .

When the spacetime is Ricci-flat ($R_{ab} = 0$, e.g., outside of a black hole or a star in a vacuum environment [40]), Eq. (1) is equivalent to the Einstein–Maxwell Eq. (20). Hence, in a flat spacetime or in a curved but Ricci-flat spacetime, the new electromagnetic field equation is identical to the Einstein–Maxwell equation. As a result, the curvature-coupled term in the new equation has no effect on experiments under laboratory conditions.

In a spacetime with $R_{ab} \neq 0$, the curvature-coupled term can have an effect on the electromagnetic field solution (e.g., inside a star or in a cosmological environment). To estimate the effect, we write

$$|\nabla_a F^{ab}| \sim \frac{|A^b|}{l_e^2}, \quad |\xi R^b{}_a A^a| \sim \frac{|A^b|}{r_c^2}, \tag{230}$$

where l_e is the spacetime scale on which the electromagnetic field varies (e.g., the wavelength of an electromagnetic wave and the size of the source) and r_c is the spacetime curvature radius defined by the Ricci tensor, i.e., $r_c \equiv (R_{ab} R^{ab})^{-1/4}$. So, we have

$$\frac{|\xi R^b{}_a A^a|}{|\nabla_a F^{ab}|} \sim \frac{l_e^2}{r_c^2}. \tag{231}$$

When $r_c \gg l_e$, we expect that the term $\xi R^b{}_a A^a$ is not important. By the Einstein field equation, we can estimate the order of r_c by

$$r_c \sim \frac{l_P}{\sqrt{8\pi}} \left(\frac{\rho}{\rho_P} \right)^{-1/2}, \tag{232}$$

where ρ is the mass density at the place where the electromagnetic field is present.

The density and curvature radius of some objects are listed in Table 1. For comparison with the curvature radius, the sizes (radii or heights) of the selected objects

Table 1 Density ρ , radius (height) r , and the curvature radius r_c of some objects. The r_c is estimated by Eq. (232), except for the universe where both r and r_c are taken to be the Hubble distance d_H .^a

Object	ρ (g · cm ⁻³)	r (cm)	r_c (cm)
Atmosphere ^b	0.001225	$\sim 10^6$	6.6×10^{14}
Earth ^c	5.5	6.4×10^8	1.0×10^{13}
Jupiter ^c	1.3	7.0×10^9	2.0×10^{13}
Sun ^c	1.4	7.0×10^{10}	2.0×10^{13}
White Dwarf ^d	10^6	7×10^8	2×10^{10}
Neutron Star ^e	5×10^{14}	10^6	1.0×10^6
Universe ^f	$2 \times 10^{-29} h^2$	$9 \times 10^{27} h^{-1}$	$9 \times 10^{27} h^{-1}$

^aIn the case of the universe, the r_c estimated by Eq. (232) is of the same order as d_H .

^bThe density is measured at sea level and 15 °C. The r refers to the approximate height above sea level.

^cThe density ρ and radius r are averaged values.

^dThe ρ is the averaged density, and r is the radius of a typical white dwarf.

^eThe ρ is the core density, and r is the radius of a typical neutron star.

^fThe ρ is taken to be equal to ρ_{crit} , where h is the Hubble constant in units of $100 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$.

are also listed. It can be imagined that the curvature-coupled term is important if the size of an object is larger than or at least comparable to the curvature radius given by Eq. (232). Thus, according to the results in Table 1, we can expect that the curvature-coupled term in the new electromagnetic field Eq. (1) will give rise to detectable effects for white dwarfs, neutron stars, and the universe. It is noteworthy that Turner and Widrow [41] have shown that an electromagnetic field equation of the form of Eq. (1) with a negative ξ can lead to fast growth of a primordial magnetic field during the inflation epoch through a mechanism akin to “superadiabatic amplification”, which is not possible for the Maxwell equation without a curvature-coupled term. This can be regarded as evidence that the curvature-coupled term can lead to detectable effects in the early universe.

In Table 1 we also list the atmosphere, which is relevant for the experiments of electromagnetism under laboratory conditions, and determination of the magnetic field of the earth. For the density of the atmosphere, the curvature radius is $\sim 10^6$ times the radius of the earth and the correction from $\xi R^b{}_a A^a$ should be of the order of $< 10^{-12}$ in a fraction of $\nabla_a F^{ab}$. According to [42], the best constraint from laboratory tests on the reduced Compton wavelength associated with the rest mass of photons is $\lambda \gtrsim 2 \times 10^9 \text{ cm}$. Although the limit is distant from r_c for the atmosphere by five orders of magnitudes, testing the effect of $\xi R^b{}_a A^a$ under laboratory conditions may become possible in the future with more advanced techniques.

For a vacuum spacetime with a cosmological constant Λ (e.g., a de Sitter spacetime or an anti-de Sitter space-

time), by the Einstein field equation we have $R_{ab} = \Lambda g_{ab}$ and $R_{ab}A^b = \Lambda A_a$. Then, the electromagnetic field equation (1) becomes

$$\nabla_a F^{ab} - \xi \Lambda A^b = -4\pi J^b. \quad (233)$$

In this case, the curvature-coupled term $\xi R_{ab}A^b$ is equivalent to a photon mass term $m^2 A_a$, with

$$m^2 = \xi \Lambda. \quad (234)$$

When $\xi \Lambda < 0$, we have $m^2 < 0$; i.e., the photon has an imaginary mass. Then a photon will travel with a speed faster than c . For instance, for a de Sitter spacetime ($\Lambda > 0$) with an electromagnetic field equation derived from five-dimensional gravity we have $\xi = -2$ and $m^2 = -2\Lambda < 0$. We can then expect that photons can travel superluminally in the de Sitter spacetime. For a Λ value comparable to the observed one in the current universe, m^2 is sufficiently small and hence the strength of the signal off the light cone should be very weak. However, in the very early universe, e.g., during the inflationary phase, Λ is not small, so the signal off the light cone may be appreciable.

11 Summary and conclusions

A new unified theory of electromagnetic and gravitational interactions is presented in this paper. A four-dimensional spacetime is assumed to be a hypersurface embedded in a five-dimensional bulk spacetime. Then, the field equations in the four-dimensional spacetime are determined by the projection of the Einstein field equation in the five-dimensional spacetime onto the hypersurface and the contraction with the normal to the hypersurface. Three independent equations are obtained. They form a complete set of field equations in the four-dimensional spacetime, including determination of the metric tensor off the hypersurface.

The first is a scalar constraint equation given by Eq. (93) [or, equivalently, Eq. (95)], which relates the scalar curvature to the extrinsic curvature tensor of the hypersurface. The second is a vector constraint equation given by Eq. (104) [or, equivalently, Eq. (109)], which can be interpreted as the electromagnetic field equation in a four-dimensional spacetime [Eq. (107)]. The third is a tensor equation, i.e., Eq. (117) with the stress-energy tensor T_{ab} given by Eqs. (118), (125), (137), and (149), which can be interpreted as the Einstein field equation with the stress-energy tensor of electromagnetic fields and other matter as the source. The constraint Eq. (93) [or, equivalently, Eq. (95)] can also be replaced by Eq. (168) [or, equivalently, Eq. (166)], since the latter is derived from the combination of Eqs. (93), (104), and (117).

The most important result of the new unified theory is that a new electromagnetic field equation in a four-dimensional spacetime is derived from the five-dimensional vacuum Einstein field equation. The new electromagnetic field equation is given by Eq. (107), which is equivalent to Eq. (1) with $\xi = -2$ with the assumption $\nabla_a N = 0$ (i.e., N is constant in the four-dimensional spacetime). The new field equation differs from the Einstein–Maxwell Eq. (20) by a curvature-coupled term $\xi R^b{}_a A^a$, which vanishes in a Ricci-flat spacetime but can be important in an environment with a high mass density. Although practical solutions to the new electromagnetic field equation are not studied, we have shown that a conformal Killing vector field with a constant α (or a Killing vector field when $\alpha = 0$) solves the source-free field equation with $\xi = -2$. We have also argued that the effect of the curvature-coupled term can be detectable in electromagnetic processes inside a neutron star or a white dwarf and in the early epoch of the universe (see Table 1 and the relevant discussion in the text).

The electromagnetic field Eq. (1) with an undetermined ξ was originally proposed by Li [22] to address the incompatibility problem in application of the Einstein–Maxwell equation to a universe with a uniformly distributed net charge. The fact that it can be derived from the five-dimensional Einstein field equation with a determined $\xi = -2$ supports the proposal of Eq. (1) as a solution to the inconsistency problem. Another support for Eq. (1) comes from the fact that the Maxwell equation in a flat spacetime can be expressed in terms of a symmetric tensor instead of an antisymmetric tensor. When the Maxwell equation expressed in a symmetric tensor is extended to a curved spacetime via the minimal substitution rule, the field Eq. (1) with $\xi = -2$ is naturally obtained. Therefore, we believe that, the electromagnetic field Eq. (1) and the unified theory of gravity and electromagnetism proposed in this paper deserve further study, although whether they describe the real physical world can only be ultimately determined by experiments and observations.

Geometrization of electromagnetic fields and unification of electromagnetic and gravitational interactions had been Einstein’s ultimate goal for his life, which had cost the entire latter half of his life to find the solution. The theory proposed and studied in this paper provides a new possible solution to the big problem. In the theory, electromagnetic fields are contained in the extrinsic curvature tensor of a four-dimensional spacetime as a hypersurface in a high-dimensional spacetime. The idea of interpreting a four-dimensional spacetime as a hypersurface embedded in a five-dimensional bulk spacetime has also been explored in brane world theory. However, unlike in brane world theory, where electromagnetic

fields are assumed to be confined in a four-dimensional membrane *a priori*, here electromagnetic fields and the electromagnetic field equation are *derived* from the five-dimensional Einstein field equation and present themselves on the four-dimensional hypersurface. The theory is also different from Kaluza–Klein theory, since in Kaluza–Klein theory the Einstein–Maxwell equation was derived.

Besides electromagnetic fields, the extrinsic curvature of a four-dimensional spacetime hypersurface also contains a term proportional to the derivative of the four-dimensional spacetime metric with respect to the fifth dimension (\dot{g}_{ab}), i.e., a term depending on the evolution of the metric off the hypersurface. We have attempted to interpret it as representing some unidentified matter in the four-dimensional spacetime. The stress-energy tensors of electromagnetic fields and the unidentified matter and their interaction are derived. We have shown that, under some conditions, the stress-energy tensor of the unidentified matter may behave like a cosmological constant.

It is well known that the five-dimensional Kaluza–Klein theory can be generalized to a higher dimensional theory to include the weak and strong interactions (see [13] and references therein). It would be interesting to extend the theory presented in this paper to the case of a four-dimensional spacetime embedded in an $n > 5$ dimensional bulk spacetime and investigate whether non-Abelian gauge interactions can be derived from an n -dimensional Einstein field equation.

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Appendix A 5D metric in Kaluza–Klein representation

In this Appendix we study the geometry of Kaluza–Klein (KK) theory and derive the relation between Kaluza–Klein decomposition of the 5D metric and that adopted in this paper. We use the same coordinate system as in Section 3, i.e., $\{x^0, x^1, x^2, x^3, w\}$ for the 5D spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$. The coordinates $\{x^0, x^1, x^2, x^3\}$ are defined on the hypersurface manifold \mathcal{M} (defined by $w = \text{constant}$) and carried to the neighbor of \mathcal{M} in the five-dimensional $\tilde{\mathcal{M}}$ with the map generated by the coordinate lines of w .

As stated in Section 3, the strategy of Kaluza–Klein

theory is to decompose the five-dimensional metric \tilde{g}_{ab} in the form of Eq. (40), or, equivalently, as

$$\tilde{g}_{ab} = {}_k g_{\mu\nu} dx_a^\mu dx_b^\nu + \phi^2 (A_\mu dx_a^\mu + dw_a) (A_\nu dx_b^\nu + dw_b). \tag{A1}$$

Let us define

$$\tilde{g}^{AB} = \begin{pmatrix} {}_k g^{\mu\nu} & -A^\mu \\ -A^\nu & \frac{1}{\phi^2} + A_\rho A^\rho \end{pmatrix}, \tag{A2}$$

where the 4-matrix ${}_k g^{\mu\nu}$ is the inverse of ${}_k g_{\mu\nu}$, i.e., ${}_k g_{\mu\nu} {}_k g^{\nu\rho} = \delta_\mu^\rho$, and

$$A^\mu \equiv {}_k g^{\mu\nu} A_\nu. \tag{A3}$$

It can be checked that the matrix in Eq. (A2) is the inverse of the matrix in Eq. (40). Hence, we have

$$\begin{aligned} \tilde{g}^{ab} = & {}_k g^{\mu\nu} \left(\frac{\partial}{\partial x^\mu}\right)^a \left(\frac{\partial}{\partial x^\nu}\right)^b - 2A^\mu \left(\frac{\partial}{\partial x^\mu}\right)^a \left(\frac{\partial}{\partial w}\right)^b \\ & + \left(\frac{1}{\phi^2} + A_\rho A^\rho\right) \left(\frac{\partial}{\partial w}\right)^a \left(\frac{\partial}{\partial w}\right)^b. \end{aligned} \tag{A4}$$

Equation (A3), along with the fact that ${}_k g^{\mu\nu}$ and ${}_k g_{\mu\nu}$ are inverse to each other, automatically leads to

$${}_k g_{\mu\nu} A^\nu = A_\mu. \tag{A5}$$

To understand the geometric nature of the KK variables defined above, we need to determine the KK metric tensor ${}_k g_{ab}$ associated with the 4×4 matrix ${}_k g_{\mu\nu}$ and the vector A_a associated with the 4×1 matrix A_μ . Since ${}_k g_{ab}$ and A_a are tensors and vectors on $\tilde{\mathcal{M}}$, in general they can be written as

$${}_k g_{ab} = {}_k g_{\mu\nu} dx_a^\mu dx_b^\nu + 2{}_k g_{\mu 4} dx_a^\mu dw_b + {}_k g_{44} dw_a dw_b \tag{A6}$$

and

$$A_a = A_\mu dx_a^\mu + A_4 dw_a, \tag{A7}$$

although some of the tensor components may turn out to be zero.

Proposition 1. The vectors A_a and A^a are expressed in coordinate components by

$$A_a = A_\mu dx_a^\mu \tag{A8}$$

and

$$A^a = A^\mu \left(\frac{\partial}{\partial x^\mu}\right)^a - A_\rho A^\rho \left(\frac{\partial}{\partial w}\right)^a, \tag{A9}$$

where $\mu, \rho = 0, 1, 2, 3$.

Proof. By Eqs. (A4) and (A7), we have

$$A^a = \tilde{g}^{ab} A_b = (k g^{\mu\nu} A_\nu - A_4 A^\mu) \left(\frac{\partial}{\partial x^\mu} \right)^a - \left[A^\rho A_\rho - A_4 \left(\frac{1}{\phi^2} + A_\rho A^\rho \right) \right] \left(\frac{\partial}{\partial w} \right)^a, \quad (\text{A10})$$

from which we get

$$A^\mu \equiv A^a dx_a^\mu = k g^{\mu\nu} A_\nu - A_4 A^\mu, \quad (\text{A11})$$

then by Eq. (A3) we must have $A_4 = 0$. Hence we get Eq. (A8).

Setting $A_4 = 0$, by Eqs. (A10) and (A3) we get Eq. (A9). *End of Proof.*

Proposition 2.

$$A_a A^a = A_\rho A^\rho, \quad (\text{A12})$$

which is directly derived from equations (A8) and (A9).

Proposition 3. The KK variables ϕ , A_μ , and A^μ are related to N , N_μ , and N^μ by

$$\phi^2 = N^2 + N_\rho N^\rho = \tilde{g}_{ab} w^a w^b, \quad (\text{A13})$$

$$A_\mu = \frac{N_\mu}{\phi^2} = \frac{N_\mu}{N^2 + N_\rho N^\rho}, \quad (\text{A14})$$

and

$$A^\mu = \frac{N^\mu}{N^2}. \quad (\text{A15})$$

Proof. Comparison of Eqs. (48) and (40) leads to Eqs. (A13) and (A14). The inverse of the matrix in Eq. (48) is

$$\tilde{g}^{AB} = \begin{pmatrix} g^{\mu\nu} + \frac{1}{N^2} N^\mu N^\nu & -\frac{1}{N^2} N^\mu \\ -\frac{1}{N^2} N^\nu & \frac{1}{N^2} \end{pmatrix}. \quad (\text{A16})$$

Comparison of Eqs. (A16) and (A2) leads to Eq. (A15). The second equality in Eq. (A13) follows from Eq. (42). *End of Proof.*

Proposition 4.

$$A_\rho A^\rho = \frac{N_\rho N^\rho}{N^2 (N^2 + N_\rho N^\rho)}, \quad (\text{A17})$$

$$A_\rho A^\rho + \frac{1}{\phi^2} = \frac{1}{N^2}, \quad (\text{A18})$$

which are directly derived from Eqs. (A13)–(A15).

Proposition 5. The vector A^a is related to N^a by

$$A^a = \frac{1}{N^2 + N_\rho N^\rho} \left(N^a - \frac{N_\rho N^\rho}{N} n^a \right). \quad (\text{A19})$$

Proof. By Eqs. (A9), (A15), and (42), we get

$$A^a = \frac{N^\mu}{N^2} \left(\frac{\partial}{\partial x^\mu} \right)^a - A_\rho A^\rho (N n^a + N^a). \quad (\text{A20})$$

Then, since $N^a = N^\mu (\partial/\partial x^\mu)^a$, by Eq. (A17) we get

$$A^a = \frac{N^a}{N^2} - \frac{N_\rho N^\rho}{N^2 (N^2 + N_\rho N^\rho)} (N n^a + N^a), \quad (\text{A21})$$

which then leads to Eq. (A19). *End of Proof.*

Proposition 6.

$$A^a n_a = -\frac{N_\rho N^\rho}{N (N^2 + N_\rho N^\rho)} = -N A_\rho A^\rho. \quad (\text{A22})$$

The first identity follows from equation (A19) since $n_a N^a = 0$ and $n_a n^a = 1$. The second identity follows from Eq. (A17).

Since $N \neq 0$, Eq. (A22) indicates that $A^a n_a = 0$ (i.e., A^a is a vector tangent to the 4-manifold \mathcal{M}) if and only if (iff) $N_\rho N^\rho = 0$, or, equivalently, $A_\rho A^\rho = 0$. Then, by Eq. (A12), we have the following proposition:

Proposition 7. The vector A^a is tangent to \mathcal{M} iff A^a is null, i.e., iff

$$A^a A_a = 0, \quad (\text{A23})$$

or, equivalently,

$$N^a N_a = 0, \quad (\text{A24})$$

Proposition 8.

$$k g_{ab} = k g_{\mu\nu} dx_a^\mu dx_b^\nu, \quad (\text{A25})$$

where $\mu, \nu = 0, 1, 2, 3$,

$$k g^{ab} = \tilde{g}^{ac} \tilde{g}^{bd} k g_{cd} = k g^{\mu\nu} \left(\frac{\partial}{\partial x^\mu} \right)^a \left(\frac{\partial}{\partial x^\nu} \right)^b - 2 A^\mu \left(\frac{\partial}{\partial x^\mu} \right)^a \left(\frac{\partial}{\partial w} \right)^b + A_\rho A^\rho \left(\frac{\partial}{\partial w} \right)^a \left(\frac{\partial}{\partial w} \right)^b, \quad (\text{A26})$$

and

$$k g_a^c \equiv k g_{ab} k g^{bc} = \tilde{g}^{cb} k g_{ab} = dx_a^\mu \left(\frac{\partial}{\partial x^\mu} \right)^c - A_\mu dx_a^\mu \left(\frac{\partial}{\partial w} \right)^c. \quad (\text{A27})$$

Proof. By Eqs. (A6), (A9), and (A5), we get

$$A_a = k g_{ab} A^b = (A_\mu - A_\rho A^\rho k g_{\mu 4}) dx_a^\mu + (A^\rho k g_{4\rho} - A_\rho A^\rho k g_{44}) dw_a. \quad (\text{A28})$$

Comparison with Eq. (A8) leads to

$$k g_{\mu 4} = k g_{4\mu} = 0, \quad k g_{44} = 0. \quad (\text{A29})$$

Then Eq. (A25) is proved. Equations (A26) and (A27) are then derived by the application of Eq. (A4). *End of Proof.*

Proposition 9.

$${}_k g_{ab} n^b = -N A_a. \tag{A30}$$

Proof. By Eqs. (42) and (A25), we have

$$\begin{aligned} {}_k g_{ab} n^b &= {}_k g_{\mu\nu} dx_a^\mu dx_b^\nu \frac{1}{N} \left[\left(\frac{\partial}{\partial w} \right)^b - N^\rho \left(\frac{\partial}{\partial x^\rho} \right)^b \right] \\ &= -\frac{1}{N} {}_k g_{\mu\nu} N^\nu dx_a^\mu. \end{aligned} \tag{A31}$$

Then, by Eqs. (A15) and (A8), we get

$${}_k g_{ab} n^b = -N {}_k g_{\mu\nu} A^\nu dx_a^\mu = -N A_\mu dx_a^\mu. \tag{A32}$$

End of Proof.

Equation (A30) leads to the conclusion that ${}_k g_{ab}$ is not a 4-metric tensor field on the hypersurface \mathcal{M} , unless $A_a = 0$ (or, equivalently, $N^a = 0$). In fact, we have the following proposition:

Proposition 10. The tensor ${}_k g_{ab}$ is a 4-metric on a hypersurface \mathcal{M}_k orthogonal to w^a (if w^a is hypersurface orthogonal), induced from the 5-metric tensor in the bulk spacetime. A^a is a vector tangent to \mathcal{M}_k . That is, if we define $\hat{w}^a = w^a / (w_c w^c)^{1/2}$ (so that $\hat{w}_a \hat{w}^a = 1$), we have

$${}_k g_{ab} = \tilde{g}_{ab} - \hat{w}_a \hat{w}_b \tag{A33}$$

and

$$A^a \hat{w}_a = 0. \tag{A34}$$

Proof. By Eqs. (A26), (A4), and (A13), we have

$${}_k g^{ab} = \tilde{g}^{ab} - \frac{1}{\phi^2} \left(\frac{\partial}{\partial w} \right)^a \left(\frac{\partial}{\partial w} \right)^b = \tilde{g}^{ab} - \hat{w}^a \hat{w}^b, \tag{A35}$$

since $\hat{w}^a = w^a / \phi$ by Eq. (A13). By Eq. (A7), we have $A_a w^a = 0$ and hence Eq. (A34). *End of Proof.*

The metric g_{ab} on \mathcal{M} is induced from the five-dimensional metric \tilde{g}_{ab} by $g_{ab} = \tilde{g}_{ab} - n_a n_b$, as discussed in Section 3. The two four-dimensional metrics ${}_k g_{ab}$ and g_{ab} are related by

$${}_k g_{ab} = g_{ab} + n_a n_b - \hat{w}_a \hat{w}_b. \tag{A36}$$

By Eq. (A36), we have $g_{\mu\nu} = {}_k g_{\mu\nu} + \hat{w}_\mu \hat{w}_\nu$ for $n_\mu = 0$. Since $w_\mu = \tilde{g}_{\mu w} = N_\mu$ by Eq. (48), and $w_a w^a = N^2 + N_\rho N^\rho$, by Proposition 3 we have $\hat{w}_\mu \hat{w}_\nu = \phi^2 A_\mu A_\nu$. Hence, $g_{\mu\nu} = {}_k g_{\mu\nu} + \phi^2 A_\mu A_\nu$, which confirms Eq. (171).

So, A^a is a vector on \mathcal{M}_k . N^a is a vector on \mathcal{M} . \mathcal{M}_k and \mathcal{M} are orthogonal to, respectively, w^a and n^a . They intersect at a three-dimensional manifold, which we denote by Σ . Σ is a hypersurface in \mathcal{M} (and \mathcal{M}_k). The above results are illustrated in Fig. 1.

Proposition 11. When $N_a N^a \neq 0$ [which is equivalent to $A_a A^a \neq 0$ by Eq. (A17)], N^a and A^a are two independent normals to Σ .

Proof. For any $v^a \in \mathcal{T}(\Sigma)$, we have $N_a v^a = (w_a - N n_a) v^a = w_a v^a = 0$, where the last equality follows from the fact that Σ is also a hypersurface in \mathcal{M}_k and hence $v^a \in \mathcal{T}(\mathcal{M}_k)$. Then, by Eq. (A19), we have

$$A_a v^a = -\frac{N_\rho N^\rho}{N(N^2 + N_\rho N^\rho)} n_a v^a = 0, \tag{A37}$$

since $v^a \in \mathcal{T}(\mathcal{M})$. Hence, both N^a and A^a are normals to Σ . By Eq. (A19), A^a and N^a are independent iff $N_a N^a \neq 0$. *End of Proof.*

Proposition 12. When $N_a N^a = 0$ [which is equivalent to $A_a A^a = 0$ by Eq. (A17)], $N^a \in \mathcal{T}(\Sigma)$ and $A^a \in \mathcal{T}(\Sigma)$.

Proof. When $N_a N^a = 0$, by Eq. (A22) we have $A^a n_a = 0$ so $A^a \in \mathcal{T}(\mathcal{M})$. Since $A^a \in \mathcal{M}_k$ also, we must have $A^a \in \mathcal{T}(\Sigma)$. By Eq. (A19) we must also have $N^a \in \mathcal{T}(\Sigma)$. (See also Proposition 7.) *End of Proof.*

Appendix B $n+1$ decomposition of Einstein's field equations

In this Appendix we discuss direct decomposition of the $(n+1)$ -dimensional Einstein field equation and derive the equivalent equations in an n -dimensional spacetime. To make the results general, we assume that the $(n+1)$ -dimensional spacetime contains a matter field represented by an $(n+1)$ -dimensional stress-energy tensor \tilde{T}_{ab} .

The Einstein field equation on an $(n+1)$ -dimensional spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ is

$$\tilde{G}_{ab} \equiv \tilde{R}_{ab} - \frac{1}{2} \tilde{R} \tilde{g}_{ab} = \tilde{\kappa} \tilde{T}_{ab}, \tag{B1}$$

where $\tilde{\kappa}$ is the gravitational coupling constant.

Contraction of Eq. (B1) with \tilde{g}^{ab} leads to

$$\tilde{R} = -\frac{2\tilde{\kappa}}{n-1} \tilde{T}, \tag{B2}$$

where $\tilde{T} \equiv \tilde{g}^{ab} \tilde{T}_{ab}$ is the trace of the stress-energy tensor. Hence, the Einstein field Eq. (B1) can be written in an equivalent form

$$\tilde{R}_{ab} = \tilde{\kappa} \left(\tilde{T}_{ab} - \frac{1}{n-1} \tilde{T} \tilde{g}_{ab} \right). \tag{B3}$$

By Eq. (44), the $(n+1)$ -dimensional metric tensor \tilde{g}_{ab} can be decomposed into components tangent and orthogonal to \mathcal{M} by

$$\tilde{g}_{ab} = g_{ab} + n_a n_b, \tag{B4}$$

where $g_{ab} n^a = 0$ and $\tilde{g}_{ab} n^a n^b = 1$. An $(n+1)$ -dimensional tensor \tilde{T}_{ab} can be decomposed into compo-

nents tangent and orthogonal to \mathcal{M} by

$$\begin{aligned}\tilde{T}_{ab} &= \tilde{g}_a^c \tilde{g}_b^d \tilde{T}_{cd} = g_a^c g_b^d \tilde{T}_{cd} + n_a \left(g_b^d n^c \tilde{T}_{cd} \right) \\ &\quad + n_b \left(g_a^c n^d \tilde{T}_{cd} \right) + n_a n_b \left(n^c n^d \tilde{T}_{cd} \right).\end{aligned}\quad (\text{B5})$$

Applying the above decomposition mechanism to the $(n+1)$ -dimensional Einstein field Eq. (B1), we get the following three independent equations on \mathcal{M} :

$$g_a^c g_b^d \tilde{G}_{cd} = \tilde{\kappa} g_a^c g_b^d \tilde{T}_{cd}, \quad (\text{B6})$$

a tensor equation obtained by full projection of the $(n+1)$ -dimensional Einstein field equation onto \mathcal{M} ;

$$g_a^d n^c \tilde{G}_{cd} = \tilde{\kappa} g_a^d n^c \tilde{T}_{cd}, \quad (\text{B7})$$

a vector equation obtained from the $(n+1)$ -dimensional Einstein field equation with one index projected onto n^a and the other index projected onto \mathcal{M} ; and

$$\tilde{G}_{cd} n^c n^d = \tilde{\kappa} \tilde{T}_{cd} n^c n^d, \quad (\text{B8})$$

a scalar equation obtained by full projection of the $(n+1)$ -dimensional Einstein field equation onto n^a .

Alternatively, application of the decomposition mechanism to the $(n+1)$ -dimensional Einstein field Eq. (B3) leads to the following three independent equations on \mathcal{M} :

$$g_a^c g_b^d \tilde{R}_{cd} = \tilde{\kappa} \left(g_a^c g_b^d \tilde{T}_{cd} - \frac{1}{n-1} \tilde{T} g_{ab} \right), \quad (\text{B9})$$

$$g_a^d n^c \tilde{R}_{cd} = \tilde{\kappa} g_a^d n^c \tilde{T}_{cd}, \quad (\text{B10})$$

and

$$\tilde{R}_{cd} n^c n^d = \tilde{\kappa} \left(\tilde{T}_{cd} n^c n^d - \frac{1}{n-1} \tilde{T} \right). \quad (\text{B11})$$

The complete set of field equations on \mathcal{M} must contain a scalar equation [Eq. (B8) or (B11)], a vector equation [Eq. (B7) or (B10)], and a tensor equation [Eq. (B6) or (B9)]. In fact, Eqs. (B7) and (B10) are equivalent, since $g_a^d n^c \tilde{G}_{cd} = g_a^d n^c \tilde{R}_{cd}$.

B.1 The scalar equation

For $n_a n^a = 1$, the Riemann tensor on \mathcal{M} is related to that on $\tilde{\mathcal{M}}$ by [19, 24]

$$R_{abc}{}^d = g_a^e g_b^f g_c^g g^d h \tilde{R}_{efg}{}^h + (K_{ac} K_b{}^d - K_{bc} K_a{}^d). \quad (\text{B12})$$

From Eqs. (44) and (B12), we get

$$R_{ac} = g_a^b g_c^d \left(\tilde{R}_{bd} - n^e n^f \tilde{R}_{bedf} \right) - (K_a{}^b K_{bc} - K K_{ac}) \quad (\text{B13})$$

and

$$R = \tilde{R} - 2\tilde{R}_{ab} n^a n^b - (K_{ab} K^{ab} - K^2). \quad (\text{B14})$$

Equation (B14) is equivalent to

$$\tilde{G}_{ab} n^a n^b = -\frac{1}{2} (R + K_{ab} K^{ab} - K^2). \quad (\text{B15})$$

Substituting Eq. (B15) into Eq. (B8), we get the scalar equation on \mathcal{M} ,

$$R + K_{ab} K^{ab} - K^2 = -2\tilde{\kappa} \tilde{T}_{ab} n^a n^b. \quad (\text{B16})$$

When $\tilde{T}_{ab} = 0$, Eq. (B16) is equivalent to Eq. (95).

By definition of the Riemann tensor and the definition of K_{ab} , we have

$$\begin{aligned}\tilde{R}_{ab} n^a n^b &= -n^a \tilde{g}^{cd} \left(\tilde{\nabla}_a \tilde{\nabla}_c - \tilde{\nabla}_c \tilde{\nabla}_a \right) n_d \\ &= \tilde{\nabla}_a n^a \tilde{\nabla}_c n^c - \tilde{\nabla}_c n^a \tilde{\nabla}_a n^c - \tilde{\nabla}_a v^a \\ &= -K_{ab} K^{ab} + K^2 - \tilde{\nabla}_a v^a,\end{aligned}\quad (\text{B17})$$

where v^a is defined by Eq. (55). Substituting Eq. (55) into Eq. (B17), we get

$$\tilde{R}_{ab} n^a n^b = -K_{ab} K^{ab} - \tilde{\mathcal{L}}_n K + \tilde{\nabla}_a a^a, \quad (\text{B18})$$

where a^a is defined by Eq. (51). Since

$$\tilde{\mathcal{L}}_n K = g^{ab} \tilde{\mathcal{L}}_n K_{ab} - 2K_{ab} K^{ab} \quad (\text{B19})$$

and

$$\tilde{\nabla}_a a^a = \nabla_a a^a - a_a a^a, \quad (\text{B20})$$

from Eq. (B18) we get

$$\tilde{R}_{ab} n^a n^b = -g^{ab} \tilde{\mathcal{L}}_n K_{ab} + K_{ab} K^{ab} + \nabla_a a^a - a_a a^a. \quad (\text{B21})$$

Substituting Eq. (B21) into Eq. (B11), we get another scalar equation on \mathcal{M} ,

$$\begin{aligned}g^{ab} \tilde{\mathcal{L}}_n K_{ab} - K_{ab} K^{ab} - \nabla_a a^a + a_a a^a \\ = -\tilde{\kappa} \left(\tilde{T}_{ab} n^a n^b - \frac{1}{n-1} \tilde{T} \right).\end{aligned}\quad (\text{B22})$$

When $\tilde{T}_{ab} = 0$, Eq. (B22) is equivalent to Eq. (166).

By Eqs. (B15) and (B17), we get

$$\begin{aligned}\frac{1}{2} \tilde{R} &= \left(\tilde{R}_{ab} - \tilde{G}_{ab} \right) n^a n^b \\ &= \frac{1}{2} (R - K_{ab} K^{ab} + K^2) - \tilde{\nabla}_a v^a,\end{aligned}\quad (\text{B23})$$

which agrees with Eq. (53).

B.2 The vector equation

By the definition of K_{ab} [Eq. (49)], we get

$$\begin{aligned}\nabla_a K_b{}^a - \nabla_b K &= g_b^d g^{ce} \left(\tilde{\nabla}_c \tilde{\nabla}_d - \tilde{\nabla}_d \tilde{\nabla}_c \right) n_e \\ &= g_b^d g^{ce} \tilde{R}_{cde}{}^f n_f.\end{aligned}\quad (\text{B24})$$

Hence, we have

$$\nabla_a K^a_b - \nabla_b K = g_b^c \tilde{R}_{cd} n^d, \tag{B25}$$

since $\tilde{R}_{cde}{}^f n_f n^e = 0$ according to properties of the Riemann tensor. Equations (B12) and (B25) are called the Gauss–Codacci relations [19].

Substituting Eq. (B25) into Eq. (B10), we get the vector equation on \mathcal{M} ,

$$\nabla_a K^a_b - \nabla_b K = \tilde{\kappa} g_b^a \tilde{T}_{ac} n^c. \tag{B26}$$

When $\tilde{T}_{ab} = 0$, Eq. (B26) is equivalent to Eq. (109).

Since $g_b^c \tilde{G}_{cd} n^d = g_b^c \tilde{R}_{cd} n^d$, Eq. (B7) leads to the same vector Eq. (B26).

B.3 The tensor equation

From Eqs. (B13) and (B14) we get

$$\begin{aligned} G_{ac} &= g_a^b g_c^d \tilde{G}_{bd} + n^b n^d \tilde{R}_{bd} g_{ac} - g_a^b g_c^d n^e n^f \tilde{R}_{bedf} \\ &\quad - (K_a^b K_{bc} - K K_{ac}) + \frac{1}{2} (K_{bd} K^{bd} - K^2) g_{ac}. \end{aligned} \tag{B27}$$

The Riemann tensor on the $(n + 1)$ -dimensional $\tilde{\mathcal{M}}$ can be decomposed as $(n + 1 \geq 3, [19])$

$$\begin{aligned} \tilde{R}_{abcd} &= \tilde{C}_{abcd} + \frac{2}{n-1} (\tilde{g}_{a[c} \tilde{R}_{d]b} - \tilde{g}_{b[c} \tilde{R}_{d]a}) \\ &\quad - \frac{2}{n(n-1)} \tilde{R} \tilde{g}_{a[c} \tilde{g}_{d]b}, \end{aligned} \tag{B28}$$

where \tilde{C}_{abcd} is the traceless Weyl tensor on $\tilde{\mathcal{M}}$. Substituting Eq. (B28) into Eq. (B27), we get

$$\begin{aligned} G_{ab} &= \frac{n-2}{n-1} \left[g_a^c g_b^d \tilde{G}_{cd} + \left(n^c n^d \tilde{R}_{cd} - \frac{1}{2n} \tilde{R} \right) g_{ab} \right] \\ &\quad - (K_a^c K_{cb} - K K_{ab}) + \frac{1}{2} (K_{cd} K^{cd} - K^2) g_{ab} \\ &\quad - E_{ab}, \end{aligned} \tag{B29}$$

where

$$E_{ab} \equiv g_a^c g_b^d n^e n^f \tilde{C}_{cedf}, \quad g^{ab} E_{ab} = 0. \tag{B30}$$

If we substitute Eqs. (B1)–(B3) into Eq. (B29), we get a tensor equation that agrees with Eq. (8) in [38]. However, before doing the substitution, we should express E_{ab} in K_{ab} and its derivatives. This is necessary since, as we will see, the expression for E_{ab} contains \tilde{R}_{ab} and hence \tilde{T}_{ab} .

By the definition of the Riemann tensor, we have

$$\tilde{R}_{cedf} n^f = (\tilde{\nabla}_c \tilde{\nabla}_e - \tilde{\nabla}_e \tilde{\nabla}_c) n_d. \tag{B31}$$

Then, by Eq. (50), we get

$$\begin{aligned} \tilde{R}_{cedf} n^e n^f &= -K_{ed} K_c^e - K_{ed} n_c a^e + \tilde{\nabla}_c a_d - n^e \tilde{\nabla}_e K_{cd} \\ &\quad - n_c n^e \tilde{\nabla}_e a_d - a_c a_d. \end{aligned} \tag{B32}$$

Since

$$n^c \tilde{\nabla}_c K_{ab} = \tilde{\mathcal{L}}_n K_{ab} - 2K_{ac} K_b^c - K_{ac} n_b a^c - K_{bc} n_a a^c, \tag{B33}$$

from Eq. (B32) we get

$$\begin{aligned} g_a^c g_b^d n^e n^f \tilde{R}_{cedf} &= -g_a^c g_b^d \tilde{\mathcal{L}}_n K_{cd} + K_{ac} K_b^c \\ &\quad + \nabla_{(a} a_{b)} - a_a a_b, \end{aligned} \tag{B34}$$

where the indexes a and b of $\nabla_a a_b$ are symmetrized since the left-hand side of the equation is symmetric with respect to them. In fact, it can be verified that $\nabla_a a_b = \nabla_b a_a$.

Then, by Eqs. (B28), (B30), and (B34), we get

$$\begin{aligned} E_{ab} &= -g_a^c g_b^d \tilde{\mathcal{L}}_n K_{cd} + K_{ac} K_b^c + \nabla_{(a} a_{b)} - a_a a_b \\ &\quad - \frac{1}{n-1} \left(g_a^c g_b^d \tilde{R}_{cd} + n^c n^d \tilde{R}_{cd} g_{ab} \right) \\ &\quad + \frac{1}{n(n-1)} \tilde{R} g_{ab}. \end{aligned} \tag{B35}$$

Equation (B35) and the identity $g^{ab} E_{ab} = 0$ leads to Eq. (B21).

Substituting Eq. (B35) into Eq. (B29), we get

$$\begin{aligned} G_{ab} &= g_a^c g_b^d \tilde{G}_{cd} + n^c n^d \tilde{R}_{cd} g_{ab} - (2K_a^c K_{cb} - K K_{ab}) \\ &\quad + \frac{1}{2} (K_{cd} K^{cd} - K^2) g_{ab} + g_a^c g_b^d \tilde{\mathcal{L}}_n K_{cd} \\ &\quad - \nabla_{(a} a_{b)} + a_a a_b. \end{aligned} \tag{B36}$$

Substituting Eq. (B21) into Eq. (B36), we get

$$\begin{aligned} G_{ab} &= g_a^c g_b^d \tilde{G}_{cd} - (2K_a^c K_{cb} - K K_{ab}) \\ &\quad + \frac{1}{2} (3K_{cd} K^{cd} - K^2) g_{ab} \\ &\quad + (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} \\ &\quad - \nabla_{(a} a_{b)} + a_a a_b + (\nabla_c a^c - a_c a^c) g_{ab}, \end{aligned} \tag{B37}$$

from which we get the full projection of \tilde{G}_{ab} onto \mathcal{M} :

$$\begin{aligned} g_a^c g_b^d \tilde{G}_{cd} &= G_{ab} + (2K_a^c K_{cb} - K K_{ab}) \\ &\quad - \frac{1}{2} (3K_{cd} K^{cd} - K^2) g_{ab} \\ &\quad - (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} \\ &\quad + \nabla_{(a} a_{b)} - a_a a_b \\ &\quad - (\nabla_c a^c - a_c a^c) g_{ab}. \end{aligned} \tag{B38}$$

Substituting the Einstein field Eq. (B1) into Eq. (B37), we get the tensor field equation on \mathcal{M} ,

$$\begin{aligned} G_{ab} &= \tilde{\kappa} g_a^c g_b^d \tilde{T}_{cd} - (2K_a^c K_{cb} - K K_{ab}) \\ &\quad + \frac{1}{2} (3K_{cd} K^{cd} - K^2) g_{ab} \\ &\quad + (g_a^c g_b^d - g_{ab} g^{cd}) \tilde{\mathcal{L}}_n K_{cd} \\ &\quad - \nabla_{(a} a_{b)} + a_a a_b + (\nabla_c a^c - a_c a^c) g_{ab}. \end{aligned} \tag{B39}$$

The right-hand side of Eq. (B39) can be interpreted as representing the stress-energy tensor of matter on \mathcal{M} . When $\tilde{T}_{ab} = 0$, Eq. (B39) is equivalent to the n -dimensional Einstein field Eq. (117) with T_{ab} given by Eq. (167).

The tensor Eq. (B39) is obtained by full projection of the $(n+1)$ -dimensional Einstein field equation onto \mathcal{M} .

Substituting Eq. (B22) into Eq. (B39), we get another version of the tensor equation on \mathcal{M} :

$$G_{ab} = \tilde{\kappa} \left[g_a^c g_b^d \tilde{T}_{cd} + \left(\tilde{T}_{cd} n^c n^d - \frac{1}{n-1} \tilde{T} \right) g_{ab} \right] - (2K_a^c K_{cb} - K K_{ab}) + \frac{1}{2} (K_{cd} K^{cd} - K^2) g_{ab} + g_a^c g_b^d \tilde{\mathcal{L}}_n K_{cd} - \nabla_{(a} a_{b)} + a_a a_b. \quad (\text{B40})$$

Equation (B40) can also be obtained by directly substituting Eqs. (B1) and (B3) into Eq. (B36).

Appendix C The pseudo-Hamiltonian formulation

The procedure that we have used to derive the field equations on \mathcal{M} is very similar to that in the Hamiltonian formulation of general relativity: Define a hypersurface in an $(n+1)$ -dimensional spacetime by the coordinate $w = \text{constant}$ and then decompose the metric and curvature tensors in terms of N , N_a , and the metric g_{ab} on the hypersurface induced from the metric tensor in the $(n+1)$ -dimensional bulk spacetime [19, 24, 26]. However, unlike in the Hamiltonian formulation, the w coordinate used in this paper is a space coordinate instead of a time coordinate. Even so, it does not prohibit us from formulating the problem in a pseudo-Hamiltonian way. In this Appendix we derive the field equations by application of the pseudo-Hamiltonian formulation and confirm the results that we have obtained in Sections 5 and 6.

The Lagrangian density in Eq. (65) contains three independent variables: g_{ab} , N , and N_a . However, only g_{ab} appears as a “dynamical” variable, since \tilde{L}_G contains the w derivative of g_{ab} only. The momentum canonically conjugate to g_{ab} is

$$\pi^{ab} \equiv \frac{\partial \tilde{L}_G}{\partial \dot{g}_{ab}} = -\sqrt{-g} (K^{ab} - K g^{ab}) = \sqrt{-g} \Pi^{ab}, \quad (\text{C1})$$

where Π_{ab} is defined by Eq. (94).

Since \tilde{L}_G does not contain any w derivative of N and N_a , their conjugate momenta vanish identically. Hence, we get the pseudo-Hamiltonian density

$$\begin{aligned} \tilde{\mathcal{H}}_G &= \pi^{ab} \dot{g}_{ab} - \tilde{\mathcal{L}}_G \\ &= -\sqrt{-g} N R - \frac{N}{\sqrt{-g}} \left(\pi_{ab} \pi^{ab} - \frac{1}{n-1} \pi^2 \right) \\ &\quad + \pi^{ab} M_{ab}, \end{aligned} \quad (\text{C2})$$

where

$$\pi \equiv g_{ab} \pi^{ab} = \sqrt{-g} (n-1) K. \quad (\text{C3})$$

The Hamiltonian is defined by

$$\tilde{H}_G \equiv \int_{\mathcal{M}} \tilde{\mathcal{H}}_G (g_{ab}, \pi^{ab}, N, N_a) e. \quad (\text{C4})$$

Variation of \tilde{H}_G with respect to N and N_a leads to two constraint equations:

$$0 = \frac{\delta \tilde{H}_G}{\delta N}, \quad 0 = \frac{\delta \tilde{H}_G}{\delta N_a}. \quad (\text{C5})$$

Variation of \tilde{H}_G with respect to g_{ab} and π^{ab} leads to two “dynamical” equations:

$$\dot{g}_{ab} = \frac{\delta \tilde{H}_G}{\delta \pi^{ab}}, \quad \dot{\pi}^{ab} = -\frac{\delta \tilde{H}_G}{\delta g_{ab}}. \quad (\text{C6})$$

C.1 Variation with respect to N and N_a

Variation of \tilde{H}_G with respect to N leads to the first constraint equation

$$R - g^{-1} \left(\pi_{ab} \pi^{ab} - \frac{1}{n-1} \pi^2 \right) = 0, \quad (\text{C7})$$

which agrees with the scalar Eq. (93).

By Eq. (C2), for variation with respect to N_a we get

$$\delta \tilde{\mathcal{H}}_G = \pi^{ab} \delta M_{ab} \doteq -2\sqrt{-g} (\nabla_a \Pi^{ab}) \delta N_b. \quad (\text{C8})$$

Then, by Eq. (C5), variation of \tilde{H}_G with respect to N leads to the second constraint equation

$$\nabla_a \left(\frac{1}{\sqrt{-g}} \pi^{ab} \right) = 0, \quad (\text{C9})$$

which agrees with the vector equation (104).

C.2 Variation with respect to π^{ab} and g_{ab}

By Eq. (C2), for variation with respect to π^{ab} we get

$$\delta \tilde{\mathcal{H}}_G = -\frac{2N}{\sqrt{-g}} \left(\pi_{ab} - \frac{\pi}{n-1} g_{ab} \right) \delta \pi^{ab} + M_{ab} \delta \pi^{ab}. \quad (\text{C10})$$

Then, by Eq. (C6), we get

$$\dot{g}_{ab} = -\frac{2N}{\sqrt{-g}} \left(\pi_{ab} - \frac{1}{n-1} \pi g_{ab} \right) + M_{ab}. \quad (\text{C11})$$

This is just the same \dot{g}_{ab} derived from Eq. (59) and the definition of π^{ab} .

For variation with respect to g_{ab} , by Eq. (C2) we get

$$\delta\tilde{\mathcal{H}}_G = -N\delta(\sqrt{-g}R) + \pi^{ab}\delta M_{ab} - N\delta\left[\frac{1}{\sqrt{-g}}\left(\pi_{ab}\pi^{ab} - \frac{1}{n-1}\pi^2\right)\right]. \quad (C12)$$

Evaluation of the first term of variation leads to

$$-N\delta(\sqrt{-g}R) \doteq \sqrt{-g}(NG^{ab} - \nabla^a\nabla^b N + g^{ab}\nabla_c\nabla^c N)\delta g_{ab}. \quad (C13)$$

By the definition of M_{ab} , we have

$$\pi^{ab}\delta M_{ab} \doteq \sqrt{-g}\nabla_c\left\{\frac{1}{\sqrt{-g}}\left[2N^{(a}\pi^{b)c} - N^c\pi^{ab}\right]\right\}\delta g_{ab}. \quad (C14)$$

Since $\delta\pi^{ab} = 0$ when the variation is taken with respect to g_{ab} , we have

$$\begin{cases} \delta(\pi_{ab}\pi^{ab}) = \pi^{ab}\delta\pi_{ab} = 2\pi^a{}_c\pi^{bc}\delta g_{ab}, \\ \delta\pi^2 = 2\pi\delta\pi = 2\pi\pi^{ab}\delta g_{ab}. \end{cases} \quad (C15)$$

Hence,

$$\begin{aligned} &\delta\left[\frac{1}{\sqrt{-g}}\left(\pi_{ab}\pi^{ab} - \frac{1}{n-1}\pi^2\right)\right] \\ &= \frac{2}{\sqrt{-g}}\left(\pi^a{}_c\pi^{bc} - \frac{1}{n-1}\pi\pi^{ab}\right)\delta g_{ab} \\ &\quad - \frac{1}{2\sqrt{-g}}\left(\pi_{cd}\pi^{cd} - \frac{1}{n-1}\pi^2\right)g^{ab}\delta g_{ab}. \end{aligned} \quad (C16)$$

Substituting Eqs. (C13), (C14), and (C16) into Eq. (C12), we get

$$\begin{aligned} \delta\tilde{\mathcal{H}}_G &\doteq \sqrt{-g}(NG^{ab} - \nabla^a\nabla^b N + g^{ab}\nabla_c\nabla^c N)\delta g_{ab} \\ &\quad + \sqrt{-g}\nabla_c\left\{\frac{1}{\sqrt{-g}}\left[2N^{(a}\pi^{b)c} - N^c\pi^{ab}\right]\right\}\delta g_{ab} \\ &\quad - \frac{N}{\sqrt{-g}}\left[2\left(\pi^a{}_c\pi^{bc} - \frac{1}{n-1}\pi\pi^{ab}\right) - \frac{1}{2}\left(\pi_{cd}\pi^{cd} - \frac{1}{n-1}\pi^2\right)g^{ab}\right]\delta g_{ab}. \end{aligned} \quad (C17)$$

Therefore, by Eq. (C6), we get

$$\begin{aligned} \dot{\pi}^{ab} &= -N\sqrt{-g}G^{ab} + \frac{2N}{\sqrt{-g}}\left(\pi^a{}_c\pi^{bc} - \frac{1}{n-1}\pi\pi^{ab}\right) \\ &\quad - \frac{N}{2\sqrt{-g}}\left(\pi_{cd}\pi^{cd} - \frac{1}{n-1}\pi^2\right)g^{ab} \\ &\quad + \sqrt{-g}(\nabla^a\nabla^b N - g^{ab}\nabla_c\nabla^c N) \\ &\quad - \sqrt{-g}\nabla_c\left\{\frac{1}{\sqrt{-g}}\left[2N^{(a}\pi^{b)c} - N^c\pi^{ab}\right]\right\}. \end{aligned} \quad (C18)$$

It can be checked that Eq. (C18) is identical to the Einstein field equation on \mathcal{M} with a stress-energy tensor given by Eq. (152).

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28. In this paper tensor space on a manifold \mathcal{M} will generally be denoted by $\mathcal{T}(\mathcal{M})$, regardless of the type of tensor (scalar, vector, dual vector, or tensor of any type).
29. The Lagrangian in Eq. (65) does not contain any derivatives of N so we do not interpret N as a matter field.
30. Note that all $\Psi_{ab}\Psi^{ab}$, Ψ^2 , $\Phi_{ab}\Phi^{ab}$, Φ^2 , $\Psi_{ab}\Phi^{ab}$, and $\Psi\Phi$ are proportional to $N - 2$.
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