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Research article

The combinatorics of Green's functions in planar field theories

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The aim of this exposition is to provide a detailed description of the use of combinatorial algebra in quantum field theory in the planar setting. Particular emphasis is placed on the relations between different types of planar Green's functions. The primary object is a Hopf algebra that is naturally defined on variables representing non-commuting sources, and whose coproduct splits into two half-coproducts. The latter give rise to the notion of an unshuffle bialgebra. This setting allows a description of the relation between full and connected planar Green's functions to be given by solving a simple linear fixed point equation. We also include a brief outline of the consequences of our approach in the framework of ordinary quantum field theory.

Keywords planar field theory, Green's functions, free probability, Hopf algebra, shuffle algebra, partitions

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1 Introduction

The intention for this exposition is to present a purely Hopf algebraic description of the well-known relations between full and connected Green's functions in quantum field theory (QFT) in the planar setting. Because our approach may also shed new light on the classical setting, we also include a short outline addressing ordinary QFT. That is, field theories in the non-planar setting.

We mention that the relations between connected and one-particle irreducible (1PI) planar Green's functions can be described by similar methods with some technical differences, as described in the final part of an earlier version of this article[1)]. As suggested by the referee, these ideas are worthy of a separate treatment, and will be developed systematically in a forthcoming work.

In the early 1980s, Cvitanovic *et al.* [1, 2] proposed a perturbative approach to quantum field theories in the planar setting. This was largely motivated by a desire to properly encode the behavior of the planar sector of quantum chromodynamics (QCD), based on 't Hooft's seminal 1974 paper [3]. An interesting feature of planar field theories is the manner in which the calculus of symmetry factors differs (and becomes simpler) compared to classical field theories. Planarity is reflected in the strictly non-commutative nature of the theory, which results in a rather substantial deviation from the classical description of the relations between different types of Green's functions. Cvitanovic *et al.* observed that the functional relation between the generating functionals of the full and connected planar Green's functions is encoded by a fixed point type equation, which is solved by the generating functionals. This fixed point equation replaces the common exponential map that relates the generating functionals of the full and connected Green's functions in classical theories (in this article, *classical* will refer to non-planar field theories and their associated objects, such as Green's functions, Feynman diagrams, and amplitudes).

The exponential relation between classical generating functionals is analogous to the moments-cumulants relation in classical probability theory [4]. Singer [5] realized the existence of a similar connection between planar field

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theories and Voiculescu's theory of free probability [6–8]. This link has been explored in subsequent works; see, e.g., [9, 10]. It turns out that the description given by Cvitanovic *et al.* of the relations between planar Green's functions is closely related to Speicher's combinatorial approach to the relations between moments and cumulants in free probability [11–16].

The description of the relations between planar Green's functions presented in these notes is based on our recent work on the algebraic and combinatorial structures underlying the relations between moments and cumulants in free and classical probability theory [17, 18]. In those references, it was shown that these relations can be understood algebraically in terms of a linear fixed point equation and its solution as seen in two different settings, i.e., (co)commutative and non-(co)commutative (un)shuffle Hopf algebras, which respectively correspond to classical and free probability. It turns out that in both cases the linear fixed point equation has a proper exponential solution. In the classical case, this exponential solution coincides with the standard exponential that relates classical moments and cumulants. In the non-classical setting, the relation between free cumulants and moments is also portrayed by an exponential, which is defined with respect to a non-commutative shuffle product. The difference between these two exponentials is analogous to the difference between exponential solutions of scalar- and matrix-valued non-autonomous linear differential equations. Here, we propose a similar approach to the–Hopf–algebraic understanding of the relations between full and connected Green's functions in planar QFT.

We begin by recalling the relation between full and connected Green's functions [19, pp. 4 & 28] in classical QFT. We refer the reader to Refs. [20–22] for introductions to the theory of quantum fields. Let $Z^{(n)}_{k_1\cdots k_n}$ and $W^{(n)}_{k_1\cdots k_n}$ denote the full respectively connected Green's functions of order n. The indices k_i represent all of the discrete and continuous variables that specify an external particle, e.g., momentum, spin, mass.

The generating functional for full Green's functions, $Z^{(n)}_{k_1\cdots k_n}$, is denoted by

$$Z(j) := 1 + \sum_{k>0} \frac{1}{k!} Z^{(k)}_{s_1\cdots s_k} j_{s_1}\cdots j_{s_k},$$

with j_s denoting an external source (behaving commutatively so that $j_{s_1}j_{s_2} = j_{s_2}j_{s_1}$; we remark that this property does not hold in the planar case)[2]. We follow the presentation of Refs. [1, 2], where the index of the source represents discrete as well as continuous variables, and Einstein's summation convention is employed. That is, repeated indices imply summations and integrations over the discrete and continuous variables, respectively, that characterize the actual Green's functions. The generating functional for connected Green's functions, $W^{(m)}_{k_1\cdots k_m}$, is denoted by

$$W(j) := \sum_{m>0} \frac{1}{m!} W^{(m)}_{s_1\cdots s_m} j_{s_1}\cdots j_{s_m}.$$

In both cases, the Green's functions are symmetric in the indices s_i, and are obtained by taking functional derivations, e.g., the full Green's function of order n is given by

$$\left.\frac{\partial^n}{\partial_{j_{k_1}}\cdots\partial_{j_{k_n}}}\right|_{j=0} Z(j) = Z^{(n)}_{k_1\cdots k_n}.$$

Note that on the right hand side, $Z^{(n)}_{k_1\cdots k_n}$ is the full Green's function of order n with respect to the specific set of values $(k_1,\ldots,k_n)$. We refer the reader to the aforementioned references for further details regarding functional calculus in QFT. It turns out that the above generating functionals are related through the exponential map:

$$Z(j) = \exp(W(j)). \tag{1}$$

Taking functional derivations, we find the following up to order four:

$$\begin{aligned}
Z^{(0)} &= 1,\\
Z^{(1)}_{k_1} &= W^{(1)}_{k_1},\\
Z^{(2)}_{k_1k_2} &= W^{(1)}_{k_1}W^{(1)}_{k_2} + W^{(2)}_{k_1k_2}\\
&= W^{(1)}_{k_1}Z^{(1)}_{k_2} + W^{(2)}_{k_1k_2}Z^{(0)},\\
Z^{(3)}_{k_1k_2k_3} &= W^{(1)}_{k_1}W^{(1)}_{k_2}W^{(1)}_{k_3} + W^{(1)}_{k_1}W^{(2)}_{k_2k_3}\\
&\quad + W^{(1)}_{k_2}W^{(2)}_{k_1k_3} + W^{(1)}_{k_3}W^{(2)}_{k_1k_2} + W^{(3)}_{k_1k_2k_3}\\
&= W^{(1)}_{k_1}Z^{(2)}_{k_2k_3} + W^{(2)}_{k_1k_2}Z^{(1)}_{k_3} + W^{(2)}_{k_1k_3}Z^{(1)}_{k_2}\\
&\quad + W^{(3)}_{k_1k_2k_3}Z^{(0)},\\
Z^{(4)}_{k_1k_2k_3k_4} &= W^{(1)}_{k_1}Z^{(3)}_{k_2k_3k_4} + W^{(2)}_{k_1k_2}Z^{(2)}_{k_3k_4} + W^{(2)}_{k_1k_3}Z^{(2)}_{k_2k_4}\\
&\quad + W^{(2)}_{k_1k_4}Z^{(2)}_{k_2k_3} + W^{(3)}_{k_1k_2k_3}Z^{(1)}_{k_4} + W^{(3)}_{k_1k_2k_4}Z^{(1)}_{k_3}\\
&\quad + W^{(3)}_{k_1k_3k_4}Z^{(1)}_{k_2} + W^{(4)}_{k_1k_2k_3k_4}Z^{(0)}.
\end{aligned} \tag{2}$$

Three remarks are in order. First, the exponential relation (1) between full and connected Green's functions is representative of a general principle of combinatorial nature, which applies beyond QFT. In a nutshell, the generating function of "all objects" is given as the exponential of the generating function of "irreducible objects". See Refs. [13, 14, 24] for further details. Second, the polynomial expressions giving full Green's functions in terms of

[2] The notion of sources relates to the concept of particle creation in QFT, and goes back to Schwinger [23]. Here, we consider them merely as auxiliary functions, which permit the extraction of Green's functions by taking functional derivations.

connected ones constitute a multivariate generalization of the classical Bell polynomials that relate, among others, moments and cumulants in classical probability [25]. The recursive structure featured in the order four case (4) in place of the complete expansion displays the full Green's function in terms of lower order connected ones, and will be explained in more detail below in the context of the tensor algebra approach.

The generating functionals for full and connected Green's functions of field theories in the planar setting are given by

$$\mathrm{Z}[j] := 1 + \sum_{k>0} \mathrm{Z}^{(k)}_{l_1\cdots l_k} j_{l_1}\cdots j_{l_k} \quad \text{resp.}$$
$$\mathrm{W}[j] := \sum_{m>0} \mathrm{W}^{(m)}_{l_1\cdots l_m} j_{l_1}\cdots j_{l_m}.$$

We note that no inverse factorials appear here. This is due to the fact that the external source j_l is of a strictly non-commutative nature, i.e., we have that $j_{l_1}j_{l_2} \neq j_{l_2}j_{l_1}$ for $l_1 \neq l_2$. Therefore, neither $\mathrm{Z}^{(k)}_{l_1\cdots l_k}$ nor $\mathrm{W}^{(m)}_{l_1\cdots l_m}$ are symmetric functions with respect to the source indices. Einstein's summation convention is again in place. This implies that one may employ different symbolic notations, $\mathrm{Z}^{(k)}_{s_1\cdots s_k} j_{s_1}\cdots j_{s_k}$ and $\mathrm{Z}^{(k)}_{1\cdots k} j_1\cdots j_k$. The latter is employed, for instance, in Ref. [2], where the corresponding non-commutative functional calculus is explained in detail. Cvitanovic noted in Ref. [1] that the planar nature of this setting yields a different functional relation between the two planar generating functionals $\mathrm{Z}[j]$ and $\mathrm{W}[j]$, which is given in terms of the fixed point equation

$$\mathrm{Z}[j] := 1 + \mathrm{W}[j\mathrm{Z}[j]]. \tag{3}$$

Owing to the non-commutative nature of the external source, a different but equivalent form is given by $\mathrm{Z}[j] := 1 + \mathrm{W}[\mathrm{Z}[j]j]$. We will work foremost with equality (3), which up to order four yields the following relations:

$$\begin{aligned}
\mathrm{Z}^{(0)} &= 1,\\
\mathrm{Z}^{(1)}_{k_1} &= \mathrm{W}^{(1)}_{k_1},\\
\mathrm{Z}^{(2)}_{k_1k_2} &= \mathrm{W}^{(1)}_{k_1}\mathrm{Z}^{(1)}_{k_2} + \mathrm{W}^{(2)}_{k_1k_2}\mathrm{Z}^{(0)}\\
&= \mathrm{W}^{(1)}_{k_1}\mathrm{W}^{(1)}_{k_2} + \mathrm{W}^{(2)}_{k_1k_2},\\
\mathrm{Z}^{(3)}_{k_1k_2k_3} &= \mathrm{W}^{(1)}_{k_1}\mathrm{Z}^{(2)}_{k_2k_3} + \mathrm{W}^{(2)}_{k_1k_2}\mathrm{Z}^{(1)}_{k_3} + \mathrm{W}^{(2)}_{k_1k_3}\mathrm{Z}^{(1)}_{k_2}\\
&\quad + \mathrm{W}^{(3)}_{k_1k_2k_3}\mathrm{Z}^{(0)}\\
&= \mathrm{W}^{(1)}_{k_1}\mathrm{W}^{(1)}_{k_2}\mathrm{W}^{(1)}_{k_3} + \mathrm{W}^{(1)}_{k_1}\mathrm{W}^{(2)}_{k_2k_3} + \mathrm{W}^{(2)}_{k_1k_2}\mathrm{W}^{(1)}_{k_3}\\
&\quad + \mathrm{W}^{(2)}_{k_1k_3}\mathrm{W}^{(1)}_{k_2} + \mathrm{W}^{(3)}_{k_1k_2k_3},\\
\mathrm{Z}^{(4)}_{k_1k_2k_3k_4} &= \mathrm{W}^{(1)}_{k_1}\mathrm{Z}^{(3)}_{k_2k_3k_4} + \mathrm{W}^{(2)}_{k_1k_2}\mathrm{Z}^{(2)}_{k_3k_4} + \mathrm{W}^{(2)}_{k_1k_3}\mathrm{Z}^{(1)}_{k_2}\mathrm{Z}^{(1)}_{k_4}\\
&\quad + \mathrm{W}^{(2)}_{k_1k_4}\mathrm{Z}^{(2)}_{k_2k_3} + \mathrm{W}^{(3)}_{k_1k_2k_3}\mathrm{Z}^{(1)}_{k_4} + \mathrm{W}^{(3)}_{k_1k_2k_4}\mathrm{Z}^{(1)}_{k_3}\\
&\quad + \mathrm{W}^{(3)}_{k_1k_3k_4}\mathrm{Z}^{(1)}_{k_2} + \mathrm{W}^{(4)}_{k_1k_2k_3k_4}\mathrm{Z}^{(0)}.
\end{aligned} \tag{4}$$

Note that differences from the analogous polynomials in the non-planar case first appear at order four. Indeed, compare the terms $W^{(2)}_{k_1k_3}Z^{(2)}_{k_2k_4}$ and $\mathrm{W}^{(2)}_{k_1k_3}\mathrm{Z}^{(1)}_{k_2}\mathrm{Z}^{(1)}_{k_4}$ appearing in lines (2) and (4), respectively. A precise description of the combinatorial nature of the recursive structure that is on display here will be elaborated on below, in terms of a double tensor Hopf algebra equipped with a non-cocommutative unshuffle coproduct.

Our approach may be summarized as follows. In a purely algebraic manner, it captures the functional calculus employed to describe the relations between full and connected planar Green's functions. For this purpose, we depart from the functional approach and now, for example, consider $\mathrm{W}[j]$ as a *generating series* – a formal power series over a family J of variables j_{l_i}. Relations such as (3) will then be interpreted as relations between formal power series, ultimately giving rise to relations between coefficients, such as

$$\mathrm{Z}^{(2)}_{l_1l_2} = \mathrm{W}^{(1)}_{l_1}\mathrm{W}^{(1)}_{l_2} + \mathrm{W}^{(2)}_{l_1l_2}.$$

These relations hold generically. That is, they hold irrespective of the actual physical values associated to the parameters l_1, l_2. Identities such as (4) can therefore be recovered from usual formal power series analysis, analogously to the approach used in functional calculus, where they follow by applying functional derivations to the generating functionals.

More concretely, starting with an alphabet J, we demonstrate that the double tensor algebra $\bar{T}(T(J))$ and the tensor algebra $\bar{T}(J)$, equipped with suitable non-cocommutative and cocommutative unshuffle coproducts, respectively, provide the appropriate Hopf algebraic setting to "algebraize" the relations between the generating functionals for full and connected Green's functions in the planar and classical settings, respectively. To this end, the generating series for Green's functions are considered as linear forms over the two aforementioned tensor algebras, $\bar{T}(T(J))$ and $\bar{T}(J)$. In both cases, the usual relations between generating functionals are recovered in Hopf algebraic terms through linear fixed point equations of the same type, which are solved by the corresponding linear maps. In the planar case, the linear map τ_{Z} that represents $\mathrm{Z}[j]$ is a multiplicative unital map, i.e., a Hopf algebra character over $\bar{T}(T(J))$ with values in, for example, the complex numbers. Note that we completely ignore possible renormalization problems for Green's functions in QFT [20, 26, 27]. The linear map τ_{W}, representing $\mathrm{W}[j]$, is an infinitesimal Hopf algebra character. It turns out that these are related through a fundamental linear fixed point equation, defined in terms of a so-called left half-shuffle product $\prec$:

$$\tau_{\mathrm{Z}} = \varepsilon + \tau_{\mathrm{W}} \prec \tau_{\mathrm{Z}}. \tag{5}$$

Its solution is given by the exponential map

$$\tau_{\mathrm{Z}} = \exp^{\star}\big(\Omega'(\tau_{\mathrm{W}})\big).$$

The map Ω' reflects the non-commutative nature of the associative product $\star$ that is used to define the exponential. This is a natural generalization of the Magnus expansion, which is well-known in the context of linear non-autonomous initial value problems [28, 29]. In the classical case, we find that the linear maps τ_Z and τ_W solve an analogous left half-shuffle equation:

$$\tau_Z = \varepsilon + \tau_W \prec \tau_Z$$

with the exponential solution

$$\tau_Z = \exp^{\sqcup\!\sqcup}\big(\tau_W\big).$$

In this case, the exponential map is defined with respect to a commutative shuffle product $\sqcup\!\sqcup$, and the absence of the map Ω' is a consequence of the commutative nature of the underlying shuffle algebra.

Our approach has several interesting implications for planar QFT. First, it once again emphasizes the structural similarity of planar QFT with the theory of free probability – making clear, for example, how non-crossing partitions appear naturally in the expansion of planar amplitudes. Second, it permits the use of group theoretical methods to study the relations between full and connected Green's functions. The linear map corresponding to the former can be interpreted as an element in a group of Hopf algebra characters, and the relation between full and connected Green's functions is described in terms of a noncommutative generalization of the exponential map. The latter can be seen as a map between the aforementioned group of characters and its corresponding Lie algebra. Similar phenomena occur when considering 1PI Green's functions and the associated functional calculus. As already mentioned, this will be the subject of future work.

Now, we will briefly outline the organization of this article. In Section 2, we introduce various mathematical notions, ranging from classical Hopf algebras to the lesser-known unshuffle bialgebras. We explain several key properties of the latter, and illustrate these notions on the tensor and double tensor algebras. Section 3 investigates the relations between full and connected Green's functions. We also introduce the notion of non-crossing Green's functions, and show how the free probability analysis of the relations between free moments and free cumulants can be transferred to planar QFT using the unshuffle calculus that underlies our approach. We conclude the article by presenting our conclusions in Section 4.

In the following, $\mathbb{K}$ denotes a ground field of characteristic zero, e.g., $\mathbb{K}$ can be $\mathbb{C}$ or $\mathbb{R}$. All structures are described over this ground field. We also assume any algebra A to be associative and unital, unless otherwise stated. The unit in A is denoted by $\mathbf{1}_A$. Identity morphisms are denoted by id.

2 Green's functions and connected graded Hopf algebras

2.1 Hopf algebras

We begin this section by recalling a few basic facts regarding Hopf algebras, which will also serve to fix our notation. For further details, the reader is referred to [30–32]. However, one remark is first in order. As the primary object of interest in our approach to planar Green's functions, our work focuses exclusively on a particular connected graded non-commutative non-cocommutative Hopf algebra with some extra structure, defined on the double tensor algebra over an alphabet. This is related to (non-commutative) Fock spaces, and can be thought of as a kind of generalization thereof. In particular, both its algebraic structure and combinatorics are different from those underlying the modeling of the Bogoliubov recursion in the BPHZ renormalization process, by means of Hopf algebras of Feynman diagrams and Rota–Baxter algebras. We refer readers that are interested in learning about the Hopf algebraic approach to renormalization in perturbative quantum field theory of Connes and Kreimer to the original papers [33–35]. In addition, see Refs. [36–38].

A coalgebra consists of a vector space C and two maps, the coproduct $\Delta : C \to C \otimes C$, which is coassociative,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{6}$$

and the counit $\varepsilon : C \to \mathbb{K}$, subject to the relation $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$. A coalgebra is called cocommutative if $\Delta = \tau \circ \Delta$, where τ is the switch or flip map, $\tau(x \otimes y) := y \otimes x$. Iterated coproducts are described by $\Delta^0 := \mathrm{id}$ and $\Delta^n : C \to C^{\otimes n+1}$, with

$$\Delta^n := (\mathrm{id} \otimes \Delta^{n-1}) \circ \Delta.$$

A bialgebra is a vector space B that is both an algebra and a coalgebra subject to certain compatibility relations, for instance, both the algebra product $m : B \otimes B \to B$ and the unit map $e : \mathbb{K} \to B$ should be coalgebra morphisms. See Ref. [32] for further details. The unit of B is denoted by $\mathbf{1} = e(1)$. A bialgebra is called graded if there exist vector spaces B_m such that $B = \bigoplus_{n \geq 0} B_n$, where $m(B_p \otimes B_q) \subseteq B_{p+q}$ and $\Delta(B_n) \subseteq \bigoplus_{p+q=n} B_p \otimes B_q$. Elements $x \in B_n$ are said to be of degree $|x| = n$. Define $B^+ := \bigoplus_{n>0} B_n$. A graded bialgebra B is called connected if the degree zero component is one dimensional, i.e., if $B_0 = \mathbb{K}\mathbf{1}$. In this case, the coproduct for an element $x \in B^+$ of degree $|x| = n$ is of the form

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \Delta'(x) \in \bigoplus_{k+l=n} B_k \otimes B_l,$$

where $\Delta'(x) := \Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x \in B^+ \otimes B^+$ is the reduced coproduct. Furthermore, the augmentation ideal is given by $Ker(\varepsilon) = B^+$. By definition, an element $x \in B$ is called primitive if $\Delta'(x) = 0$, and it is called group-like if $\Delta(x) = x \otimes x$.

Suppose that A is an algebra, with product m_A and unit map $e_A(1) = \mathbf{1}_A$, e.g., $A = \mathbb{K}$ or $A = B$, where B is a bialgebra. Then, the vector space $Lin(B, A)$ of linear maps from B to A together with the convolution product

$$\Phi \star \Psi := m_A \circ (\Phi \otimes \Psi) \circ \Delta : B \to A, \tag{7}$$

for $\Phi, \Psi \in Lin(B, A)$, forms an associative algebra with unit $\iota := e_A \circ \varepsilon$.

A Hopf algebra is a bialgebra H that is equipped with a linear map $S : H \to H$, called the antipode, which is characterized as the inverse of the identity map $\mathrm{id} \in Lin(H, H)$ with respect to the convolution product. That is, $\mathrm{id} \star S = S \star \mathrm{id} = e \circ \varepsilon$, where

$$S \star \mathrm{id} = m_H \circ (S \otimes \mathrm{id}) \circ \Delta.$$

It is a well-known fact that any connected graded bialgebra is automatically a connected graded Hopf algebra. See, e.g., [30, 32].

Let $H = \bigoplus_{n \geq 0} H_n$ be a connected graded Hopf algebra, and suppose that A is a commutative unital algebra. The subset $g_0 \subset Lin(H, A)$ of linear maps α that send the unit to zero, $\alpha(\mathbf{1}) = 0$, forms a Lie algebra in $Lin(H, A)$. The exponential $\exp^\star(\alpha) = \sum_{j \geq 0} \frac{1}{j!} \alpha^{\star j}$ defines a bijection from g_0 onto the group $G_0 = \iota + g_0$ of linear maps γ that send the unit of H to the algebra unit, $\gamma(\mathbf{1}) = \mathbf{1}_A$. The neutral element is denoted by $\iota := e_A \circ \epsilon$, defined by $\iota(\mathbf{1}) = \mathbf{1}_A$ and $\iota(x) = 0$ for $x \in H^+ = \bigoplus_{n>0} H_n$. An infinitesimal character with values in A is a linear map $\xi \in g_0$ such that for $x, y \in H^+$, $\xi(m_H(x \otimes y)) = 0$. The linear space of infinitesimal characters is a Lie subalgebra of g_0, denoted by g_A. An element Φ in G_0 is called a character if for $x, y \in H$, $\Phi(m_H(x \otimes y)) = m_A(\Phi(x) \otimes \Phi(y))$. The set of characters is denoted by $G_A \subset G_0$. This forms a pro-unipotent group for the convolution product, with (pro-nilpotent) Lie algebra g_A. The exponential map $\exp^\star$ restricts to a bijection between g_A and G_A. The inverse of $\Phi \in G_A$ is given by composition with the Hopf algebra antipode S, i.e., $\Phi^{\star -1} = \Phi \circ S$. See Refs. [36, 37] for further details and additional results.

2.2 Tensor Hopf algebras

Next, we briefly present two examples of connected graded Hopf algebras, i.e., the tensor and double tensor algebras over an arbitrary set. These will later play a key role. Let $J := \{j_{s_1}, j_{s_2}, j_{s_3}, \ldots\}$ be a set of letters (also called an alphabet). Elements in this set can be thought of as corresponding to copies of the physical source j_s, displayed in the generating functionals of Green's functions in the introduction. Recall that the index s of the source j_s represents a list of parameters and variables characterizing external particles. In the alphabet J, we consider the s_i as formal labels. The vector space spanned by J is denoted by $\mathcal{J}$. Let us define $T(J) := \oplus_{n>0} \mathcal{J}^{\otimes n}$ to be the nonunital tensor algebra over J. The full tensor algebra is denoted by $\bar{T}(J) := \oplus_{n \geq 0} \mathcal{J}^{\otimes n}$, with $\mathcal{J}^{\otimes 0} = \mathbb{K}\mathbf{1}$. Elements in $T(J)$ are written as linear combinations of words $\boldsymbol{w} = w_1 \cdots w_l$ with letters $w_i \in J$. The degree of a word $\boldsymbol{w} = w_1 \cdots w_n$ is defined to be its length $n =: |\boldsymbol{w}|$, and we write $\boldsymbol{w} \in T_n(J) = \mathcal{J}^{\otimes n}$. The space $T(J)$ is a graded algebra, with the non-commutative product defined by concatenating words $\boldsymbol{w} = w_1 \cdots w_n \in T_n(J)$ and $\boldsymbol{w}' = w'_1 \cdots w'_m \in T_m(J)$:

$$\begin{aligned}\boldsymbol{w} \cdot \boldsymbol{w}' &:= w_1 \cdots w_n \cdot w'_1 \cdots w'_m \\ &= w_1 \cdots w_n w'_1 \cdots w'_m \in T_{n+m}(J).\end{aligned}$$

The tensor algebra $\bar{T}(J)$ becomes a unital connected non-commutative but cocommutative Hopf algebra if it is equipped with the unshuffle coproduct, which is defined by declaring the elements in $J \hookrightarrow \bar{T}(J)$ to be primitive, i.e., $\Delta^{ш}(j_{s_i}) := j_{s_i} \otimes \mathbf{1} + \mathbf{1} \otimes j_{s_i}$. This definition is extended multiplicatively to all of $\bar{T}(J)$. For instance, at length two, for the word $\boldsymbol{w} = w_1 w_2$ with letters $w_i \in J$ we find

$$\begin{aligned}\Delta^{ш}(w_1 w_2) &= \Delta^{ш}(w_1)\Delta^{ш}(w_2) \\ &= w_1 w_2 \otimes \mathbf{1} + \mathbf{1} \otimes w_1 w_2 + w_1 \otimes w_2 + w_2 \otimes w_1.\end{aligned}$$

The general form for an arbitrary word $\boldsymbol{w} \in T(J)$ is given as follows:

$$\Delta^{ш}(\boldsymbol{w}) = \sum_{\boldsymbol{u}, \boldsymbol{v}} \langle \boldsymbol{u} ш \boldsymbol{v}, \boldsymbol{w} \rangle \boldsymbol{u} \otimes \boldsymbol{v}. \tag{8}$$

The sum is taken over words $\boldsymbol{u}, \boldsymbol{v} \in T(J)$, and the coefficient on the right-hand side is defined through the linearly extended bracket $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := 1$ if $\boldsymbol{u} = \boldsymbol{v}$, and is zero otherwise. The product $ш : \bar{T}(J) \otimes \bar{T}(J) \to \bar{T}(J)$ displayed in (8) is the shuffle product of words [31], which is defined iteratively for any word $\boldsymbol{w}$, by $\mathbf{1} ш \boldsymbol{w} := \boldsymbol{w} ш \mathbf{1} := \boldsymbol{w}$. For words $\boldsymbol{u}, \boldsymbol{u}'$ and letters $w_1, w'_1 \in J$ such that $\boldsymbol{w} = w_1 \boldsymbol{u}$ and $\boldsymbol{w}' = w'_1 \boldsymbol{u}'$,

$$w_1 \boldsymbol{u} ш w'_1 \boldsymbol{u}' := w_1(\boldsymbol{u} ш \boldsymbol{w}') + w'_1(\boldsymbol{w} ш \boldsymbol{u}'). \tag{9}$$

In low degrees, for letters $w_1, w_2, w'_1, w'_2 \in J$ we find

$$\begin{aligned} w_1 ш w'_1 &= w_1 w'_1 + w'_1 w_1, \\ w_1 ш w'_1 w'_2 &= w_1 w'_1 w'_2 + w'_1(w_1 ш w'_2) \\ &= w_1 w'_1 w'_2 + w'_1 w_1 w'_2 + w'_1 w'_2 w_1, \\ w_1 w_2 ш w'_1 w'_2 &= w_1(w_2 ш w'_1 w'_2) + w'_1(w_1 w_2 ш w'_2). \end{aligned}$$

The shuffle product of words is associative and commutative.

Later, we will see that (9) constitutes the commutative version of a more general shuffle product. The name "shuffle algebra" refers to general, possibly noncommutative, shuffle algebras, and we refer explicitly to "commutative shuffle algebras" in the commutative case. In fact, we will mainly be interested in the noncommutative case (which is the relevant one for planar QFT).

Next, we augment the complexity of our word algebra $T(J)$ by defining the double tensor algebra $T(T(J)) := \oplus_{n>0} T(J)^{\otimes n}$. We employ a bar-notation to denote elements $\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n \in T(T(J))$, where $\boldsymbol{w}_i$ are words in $T(J)$, for $i = 1,\ldots,n$. The algebra $T(T(J))$ is equipped with a concatenation type product. For $\boldsymbol{a} = \boldsymbol{w}_1|\cdots|\boldsymbol{w}_n$ and $\boldsymbol{b} = \boldsymbol{w}'_1|\cdots|\boldsymbol{w}'_m$, we denote the concatenation product in $T(T(J))$ by $\boldsymbol{a}|\boldsymbol{b}$. That is, $\boldsymbol{a}|\boldsymbol{b} := \boldsymbol{w}_1|\cdots|\boldsymbol{w}_n|\boldsymbol{w}'_1|\cdots|\boldsymbol{w}'_m$. This algebra is multi-graded, i.e., $T(T(J))_{n_1,\ldots,n_k} := T_{n_1}(J) \otimes \cdots \otimes T_{n_k}(J)$, as well as graded:

$$T(T(J))_n := \bigoplus_{n_1+\cdots+n_k=n} T(T(J))_{n_1,\ldots,n_k}.$$

Similar observations hold for the unital case, $\bar{T}(T(J)) = \oplus_{n\geq 0} T(J)^{\otimes n}$, and without further comment we will identify a bar symbol such as $\boldsymbol{w}_1|\mathbf{1}|\boldsymbol{w}_2$ with $\boldsymbol{w}_1|\boldsymbol{w}_2$.

The double tensor algebra is made into a Hopf algebra by defining an additional unshuffle-type coproduct, which is a refinement of the unshuffling coproduct in (8). Given two (canonically ordered) subsets $S \subseteq U$ of the set of positive integers $\mathbb{N}^+$, we call a maximal sequence $s_1,\ldots,s_n$ in S such that there are no $1 \leq i < n$ and $u \in U$ with $s_i < u < s_{i+1}$ a connected component of S relative to U. In particular, a connected component of S in $\mathbb{N}^+$ is simply a maximal sequence of successive elements $s, s+1, \ldots, s+n$ in S.

Consider a word $\boldsymbol{w} = w_1\cdots w_n \in T(J)$ with letters $w_i \in J$. For a nonempty set $S := \{s_1,\ldots,s_p\} \subseteq [n]$, we define $\boldsymbol{w}_S := w_{s_1}\cdots w_{s_p}$ (in particular, $w_\emptyset := \mathbf{1}$). Denoting the connected components of $[n]-S$ by $J_1,\ldots,J_k$, we also set $\boldsymbol{w}_{J^S_{[n]}} := \boldsymbol{w}_{J_1}|\cdots|\boldsymbol{w}_{J_k}$. More generally, for $S \subseteq U \subseteq [n]$ set $\boldsymbol{w}_{J^S_U} := \boldsymbol{w}_{J_1}|\cdots|\boldsymbol{w}_{J_k}$, where J_i are now the connected components of $U - S$ in U.

Definition 1. *For a word $\boldsymbol{w} = w_1\cdots w_n \in T(J)$, the map $\delta : T(J) \to \bar{T}(J) \otimes \bar{T}(T(J))$ is defined by*

$$\delta(w_1\cdots w_n) := \sum_{S\subseteq[n]} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J_1}|\cdots|\boldsymbol{w}_{J_k} = \sum_{S\subseteq[n]} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J^S_{[n]}}. \tag{10}$$

The coproduct is then extended multiplicatively to all of $\bar{T}(T(J))$, i.e., for $\boldsymbol{w}_1|\cdots|\boldsymbol{w}_m \in T(T(J))$ with $\boldsymbol{w}_i \in T(J)$ for $i = 1,\ldots,m$,

$$\delta(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_m) := \delta(\boldsymbol{w}_1)\cdots\delta(\boldsymbol{w}_m),$$

with $\delta(\mathbf{1}) := \mathbf{1}\otimes\mathbf{1}$.

Theorem 2. [17, 18] *The graded algebra $\bar{T}(T(J))$ equipped with the coproduct (10) is a connected graded noncommutative and non-cocommutative Hopf algebra.*

2.3 Splitting of unshuffle coproducts

As we have already indicated, both of the coproducts (8) and (10) are considered to be of unshuffle-type. In the latter case, we keep track of the subsets of letters that have been extracted from a word $\boldsymbol{w}$ in $T(J)$ by "filling in the holes" by bars. We will show that this simple operation, i.e., going from $T(J)$ to $T(T(J))$, is sufficient to understand the different natures of the relations between full and connected Green's functions in the context of planar field theories. In fact, we will see that these relations, whether in the planar or classical case, are encoded by a particular linear fixed point equation defined on either $T(J)$ or $T(T(J))$, which is derived from a rather natural splitting of the coproducts (8) and (10).

Indeed, the unshuffle-type coproducts (8) and (10) share the coalgebraic property that they can both be split into a sum of left and right unshuffle half-coproducts. Let us introduce first this splitting for the coproduct (8), where it is easily defined by noting that when applied to a word $\boldsymbol{w} = w_1\cdots w_n \in T(J)$ with $w_i \in J$, (8) can be written in more set-theoretic terms as

$$\Delta^{ш}(w_1\cdots w_n) = \sum_{I\subseteq[n]} \boldsymbol{w}_I \otimes \boldsymbol{w}_{[n]-I}. \tag{11}$$

As previously, for subsets $S = \{s_1,\ldots,s_k\} \subset [n]$, $\boldsymbol{w}_S$ denotes the word $w_{s_1}\cdots w_{s_k}$. Now, we define the left unshuffle half-coproduct $\Delta^{ш}_{\prec} : T(J) \to T(J) \otimes \bar{T}(J)$:

$$\Delta^{ш}_{\prec}(\boldsymbol{w}) = \sum_{\substack{I\subseteq[n]\\ 1\in I}} \boldsymbol{w}_I \otimes \boldsymbol{w}_{[n]-I}. \tag{12}$$

The right unshuffle half-coproduct is defined by $\Delta^{ш} - \Delta^{ш}_{\prec} =: \Delta^{ш}_{\succ} : T(J) \to \bar{T}(J) \otimes T(J)$. In an explicit form, this is given by

$$\Delta^{ш}_{\succ}(\boldsymbol{w}) = \sum_{\substack{I\subset[n]\\ 1\notin I}} \boldsymbol{w}_I \otimes \boldsymbol{w}_{[n]-I}, \tag{13}$$

so that the splitting of the unshuffle coproduct (8) in terms of these two operations becomes

$$\Delta^{ш} = \Delta^{ш}_{\succ} + \Delta^{ш}_{\prec}. \tag{14}$$

This gives rise to an unshuffle bialgebra structure on $\bar{T}(J)$, the formal definition of which will be presented further below. The fine structure of this mathematical notion is studied in Ref. [39]. Note that in the following, we shall use both appellations, the (left) right unshuffle half-coproduct and (left) right half-unshuffle.

Before we move on to the double tensor algebra $T(T(J))$, let us provide a few examples for words $\boldsymbol{w} = w_1 \cdots w_l$ in $T(J)$ of up to order four, with letters w_i in J:

$$\begin{aligned}
\Delta^{\sqcup\!\sqcup}_{\prec}(w_1) &= w_1 \otimes \mathbf{1},\\
\Delta^{\sqcup\!\sqcup}_{\prec}(w_1w_2) &= w_1 \otimes w_2 + w_1w_2 \otimes \mathbf{1},\\
\Delta^{\sqcup\!\sqcup}_{\prec}(w_1w_2w_3) &= w_1 \otimes w_2w_3 + w_1w_2 \otimes w_3\\
&\quad + w_1w_3 \otimes w_2 + w_1w_2w_3 \otimes \mathbf{1},\\
\Delta^{\sqcup\!\sqcup}_{\prec}(w_1w_2w_3w_4) &= w_1 \otimes w_2w_3w_4 + w_1w_2 \otimes w_3w_4\\
&\quad + w_1w_3 \otimes w_2w_4 + w_1w_4 \otimes w_2w_3\\
&\quad + w_1w_2w_3 \otimes w_4 + w_1w_2w_4 \otimes w_3\\
&\quad + w_1w_3w_4 \otimes w_2 + w_1w_2w_3w_4 \otimes \mathbf{1},
\end{aligned}$$

which should be compared with (2). For the right half-unshuffle $\Delta^{\sqcup\!\sqcup}_{\succ}$, for words $\boldsymbol{w} = w_1 \cdots w_n$ with $w_j \in J$ up to order three, we find

$$\begin{aligned}
\Delta^{\sqcup\!\sqcup}_{\succ}(w_1) &= \mathbf{1} \otimes w_1,\\
\Delta^{\sqcup\!\sqcup}_{\succ}(w_1w_2) &= w_2 \otimes w_1 + \mathbf{1} \otimes w_1w_2,\\
\Delta^{\sqcup\!\sqcup}_{\succ}(w_1w_2w_3) &= w_2 \otimes w_1w_3 + w_3 \otimes w_1w_2\\
&\quad + w_2w_3 \otimes w_1 + \mathbf{1} \otimes w_1w_2w_3.
\end{aligned}$$

Remark 1. Observe that $\tau \circ \Delta^{\sqcup\!\sqcup}_{\succ} = \Delta^{\sqcup\!\sqcup}_{\prec}$, which amounts to the cocommutativity of the coproduct (8), $\tau \circ \Delta^{\sqcup\!\sqcup} = \Delta^{\sqcup\!\sqcup}$.

Let us now consider the coproduct (10), which splits into left and right unshuffle half-coproducts as

$$\delta = \delta_{\succ} + \delta_{\prec}, \tag{15}$$

analogously to (14). The left unshuffle half-coproduct $\delta_{\prec} : T(J) \to T(J) \otimes \bar{T}(T(J))$ is defined for words $\boldsymbol{w} = w_1 \cdots w_n$ in $T(J)$ by

$$\begin{aligned}
\delta_{\prec}(w_1 \cdots w_n) :&= \sum_{\substack{S \subseteq [n]\\ 1 \in S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J_1} | \cdots | \boldsymbol{w}_{J_k}\\
&= \sum_{\substack{S \subseteq [n]\\ 1 \in S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J^S_{[n]}},
\end{aligned} \tag{16}$$

and extended to $\bar{T}(T(J))$ by $\delta_{\prec}(\boldsymbol{w}_1 | \cdots | \boldsymbol{w}_n) := \delta_{\prec}(\boldsymbol{w}_1)\delta(\boldsymbol{w}_2) \cdots \delta(\boldsymbol{w}_n)$. The right unshuffle half-coproduct $\delta_{\succ} : T(J) \to \bar{T}(J) \otimes T(T(J))$ is defined by

$$\begin{aligned}
\delta_{\succ}(w_1 \cdots w_n) :&= \sum_{\substack{S \subset [n]\\ 1 \notin S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J_1} | \cdots | \boldsymbol{w}_{J_k}\\
&= \sum_{\substack{S \subset [n]\\ 1 \notin S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J^S_{[n]}},
\end{aligned} \tag{17}$$

and similarly extended to $\bar{T}(T(J))$.

Remark 2. Note that by symmetry considerations, one can define a companion left half-unshuffle:

$$\begin{aligned}
\delta_{\overline{\prec}}(w_1 \cdots w_n) :&= \sum_{\substack{S \subseteq [n]\\ n \in S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J_1} | \cdots | \boldsymbol{w}_{J_k}\\
&= \sum_{\substack{S \subseteq [n]\\ n \in S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J^S_{[n]}},
\end{aligned} \tag{18}$$

and correspondingly an additional right half-unshuffle $\delta_{\overline{\succ}}(w_1 \cdots w_n) := \sum_{\substack{S \subset [n]\\ n \notin S}} \boldsymbol{w}_S \otimes \boldsymbol{w}_{J^S_{[n]}}$.

Let us provide a few examples for words $\boldsymbol{w} = w_1 \cdots w_n \in T(J)$ of length up to $n = 4$:

$$\begin{aligned}
\delta_{\prec}(w_1) &= w_1 \otimes \mathbf{1},\\
\delta_{\prec}(w_1w_2) &= w_1 \otimes w_2 + w_1w_2 \otimes \mathbf{1},\\
\delta_{\prec}(w_1w_2w_3) &= w_1 \otimes w_2w_3 + w_1w_2 \otimes w_3\\
&\quad + w_1w_3 \otimes w_2 + w_1w_2w_3 \otimes \mathbf{1},\\
\delta_{\prec}(w_1w_2w_3w_4) &= w_1 \otimes w_2w_3w_4 + w_1w_2 \otimes w_3w_4\\
&\quad + w_1w_3 \otimes w_2 | w_4 + w_1w_4 \otimes w_2w_3\\
&\quad + w_1w_2w_3 \otimes w_4 + w_1w_2w_4 \otimes w_3\\
&\quad + w_1w_3w_4 \otimes w_2 + w_1w_2w_3w_4 \otimes \mathbf{1}.
\end{aligned}$$

Note the difference between the terms $w_1w_3 \otimes w_2 | w_4 \in T(J) \otimes T(T(J))$ and $w_1w_3 \otimes w_2w_4 \in T(J) \otimes T(J)$, which distinguishes the two left unshuffle half-coproducts $\delta_{\prec}$ and $\Delta^{\sqcup\!\sqcup}_{\prec}$ at order four. This distinction parallels the one that we already observed (in (2) and (4)) between the expansions at order four of the full Green's functions in terms of the connected ones, in the planar and non-planar cases.

For the right unshuffle half-coproduct for words $\boldsymbol{w} = w_1 \cdots w_n$ of length up to $n = 4$, we find

$$\begin{aligned}
\delta_{\succ}(w_1) &= \mathbf{1} \otimes w_1,\\
\delta_{\succ}(w_1w_2) &= w_2 \otimes w_1 + \mathbf{1} \otimes w_1w_2,\\
\delta_{\succ}(w_1w_2w_3) &= w_2 \otimes w_1 | w_3 + w_3 \otimes w_1w_2\\
&\quad + w_2w_3 \otimes w_1 + \mathbf{1} \otimes w_1w_2w_3,\\
\delta_{\succ}(w_1w_2w_3w_4) &= w_2 \otimes w_1 | w_3w_4 + w_3 \otimes w_1w_2 | w_4\\
&\quad + w_4 \otimes w_1w_2w_3 + w_2w_3 \otimes w_1 | w_4\\
&\quad + w_2w_4 \otimes w_1 | w_3 + w_3w_4 \otimes w_1w_2\\
&\quad + w_2w_3w_4 \otimes w_1 + \mathbf{1} \otimes w_1w_2w_3w_4.
\end{aligned}$$

Remark 3. Note that, contrary to Remark 1, we already observe at order three that $\tau \circ \delta_{\succ} \neq \delta_{\prec}$. This difference, i.e., the fact that (8) is cocommutative but (10) is not, algebraically distinguishes the non-planar from the planar setting of QFTs.

In the light of coassociativity (6) of the coproducts (8) and (10), the respective splittings in (14) and (15) imply general properties that should be satisfied by the left and

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right unshuffle half-coproducts, which will be stated in the following two definitions [40].

Definition 3. *A counital unshuffle coalgebra is a coaugmented coassociative coalgebra* $\bar{C} = C \oplus \mathbb{K}\mathbf{1}$ *with coproduct*

$$\Delta(c) := \bar{\Delta}(c) + c \otimes \mathbf{1} + \mathbf{1} \otimes c, \tag{19}$$

such that on C *the reduced coproduct splits as* $\bar{\Delta} = \bar{\Delta}_{\prec} + \bar{\Delta}_{\succ}$*, with*

$$(\bar{\Delta}_{\prec} \otimes \mathrm{id}) \circ \bar{\Delta}_{\prec} = (\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta}_{\prec}, \tag{20}$$

$$(\bar{\Delta}_{\succ} \otimes \mathrm{id}) \circ \bar{\Delta}_{\prec} = (\mathrm{id} \otimes \bar{\Delta}_{\prec}) \circ \bar{\Delta}_{\succ}, \tag{21}$$

$$(\bar{\Delta} \otimes \mathrm{id}) \circ \bar{\Delta}_{\succ} = (\mathrm{id} \otimes \bar{\Delta}_{\succ}) \circ \bar{\Delta}_{\succ}. \tag{22}$$

The maps $\bar{\Delta}_{\prec}$ *and* $\bar{\Delta}_{\succ}$ *are called augmented left and right unshuffle half-coproducts, respectively. A cocommutative unshuffle coalgebra satisfies* $\bar{\Delta}_{\prec} = \tau \circ \bar{\Delta}_{\succ}$.

Definition 4. *An unshuffle bialgebra is a unital and counital bialgebra* $\bar{B} = B \oplus \mathbb{K}\mathbf{1}$ *with product* $m_B(x \otimes y) =: x \cdot_B y$ *and coproduct* Δ*, as well as a counital unshuffle coalgebra such that* $\bar{\Delta} = \bar{\Delta}_{\prec} + \bar{\Delta}_{\succ}$*. Moreover, the following compatibility relations hold:*

$$\Delta_{\prec}(a \cdot_B b) = \Delta_{\prec}(a) \cdot_B \Delta(b), \tag{23}$$

$$\Delta_{\succ}(a \cdot_B b) = \Delta_{\succ}(a) \cdot_B \Delta(b), \tag{24}$$

where

$$\Delta_{\prec}(a) := \bar{\Delta}_{\prec}(a) + a \otimes \mathbf{1}, \tag{25}$$

$$\Delta_{\succ}(a) := \bar{\Delta}_{\succ}(a) + \mathbf{1} \otimes a, \tag{26}$$

and $\Delta = \Delta_{\prec} + \Delta_{\succ}$.

For example, together with the concatenation product, (11) defines the structure of a cocommutative unshuffle bialgebra on $\bar{T}(J)$.

Theorem 5. [17] *With its coproduct* (10) *split into left and right unshuffle half-coproducts* $\delta = \delta_{\prec} + \delta_{\succ}$*, as defined in* (16) *and* (17)*, the bialgebra* $\bar{T}(T(J))$ *is an unshuffle bialgebra.*

Remark 4. The left unshuffle half-coproduct (16) can be further split as

$$\bar{\delta}_{\prec}(w_1 \cdots w_n) = \hat{\delta}_{\prec}(w_1 \cdots w_n) + \tilde{\delta}_{\prec}(w_1 \cdots w_n), \tag{27}$$

where $\hat{\delta}_{\prec}(w_1 \cdots w_n) \in T(J) \otimes \bar{T}(J)$ is the right-linear part, and the remaining part $\tilde{\delta}_{\prec}(w_1 \cdots w_n) \in \bar{T}(J) \otimes \bigoplus_{n>1} T(T(J))_n$. The right-linear part is best described in terms of intervals:

$$\hat{\delta}_{\prec}(w_1 \cdots w_n) = \sum_{\substack{I_1 \coprod I_2 \coprod I_3 = [n] \\ 1 \in I_1, I_2 \neq \emptyset}} \boldsymbol{w}_{I_1 \coprod I_3} \otimes \boldsymbol{w}_{I_2}. \tag{28}$$

Here, I_1, I_2, I_3 are three disjoint intervals such that $I_1 \coprod I_2 \coprod I_3 = [n]$ and I_3 is possibly empty. Moreover, the minimal elements of each interval satisfy $\min(I_1) = 1 < \min(I_2) < \min(I_3)$.

2.4 Green's functions revisited

Recall that the main mathematical purpose of QFT is to calculate Green's functions, from which physical quantities can ultimately be computed. A Green's function of order n, also called an n-point correlation function, is defined as the vacuum expectation value of the time-ordered product of n field operators [22]. In the following, wherever physical Green's functions occur the reader should consider the base field $\mathbb{K}$ to be $\mathbb{C}$.

As mentioned in the introduction, in this article we interpret the generating functional for Green's functions of planar (respectively classical) field theories as elements in the dual space $Lin(\bar{T}(T(J)), \mathbb{K}) =: \bar{T}^*(T(J))$ of linear maps from the Hopf algebra $\bar{T}(T(J))$ to $\mathbb{K}$ (respectively $Lin(\bar{T}(J), \mathbb{K}) =: \bar{T}^*(J)$). This approach allows algebraic identities to be considered between Green's functions using the machinery of Hopf algebras and unshuffle bialgebras.

Note that the problem of (ultraviolet) divergences and its solution in terms of renormalization [26, 27, 41] requires regularization procedures, which involve replacing the field of complex numbers as the target space of linear maps by some commutative unital $\mathbb{C}$-algebra A. For example, the algebra of Laurent series $\mathbb{C}[\epsilon^{-1}, \epsilon]]$ in a dimensional regularization parameter ϵ. We point out that changing the target algebra from $\mathbb{C}$ to such an algebra A would not change the underlying algebraic and combinatorial framework. Thus, the forthcoming developments also apply in this more general setting.

To make this more precise, we first consider $Lin(T(J), \mathbb{K}) = T^*(J)$. A linear form F over $T(J)$ can then be encoded by the formal power series $\bar{F} := \sum_{\boldsymbol{w}} F(\boldsymbol{w})\ \boldsymbol{w}$, where $\boldsymbol{w} = w_1 \cdots w_n$ runs over words of letters from the alphabet J. From this perspective, generating functionals of Green's functions are viewed as elements of $T^*(J)$, with the full and connected Green's functions as coefficients. For a word $\boldsymbol{w} = w_1 \cdots w_l$ with $w_k = j_{s_{i_k}} \in J$,

$$\tau_Z(\boldsymbol{w}) := Z^{(l)}_{s_{i_1} \cdots s_{i_l}},$$

$$\tau_W(\boldsymbol{w}) := W^{(l)}_{s_{i_1} \cdots s_{i_l}}.$$

We call these linear maps the (formal) classical full and connected Green's functions, respectively. "Formal" refers to the fact that these coefficients, i.e., $W^{(l)}_{s_{i_1} \cdots s_{i_l}}$ or $Z^{(l)}_{s_{i_1} \cdots s_{i_l}}$, are in fact functions whose variables s_{i_k} run implicitly over the set of all possible physical parameters characterizing the particles (or fields) of the theory.

Let us now focus on the planar case. Here, one may again apply the above interpretation of generating series of Green's functions as linear forms. When restricted to words in $T(J) \hookrightarrow T(T(J))$, one sets

$$\tau_{\mathrm{Z}}(\boldsymbol{w}) := \mathrm{Z}^{(m)}_{s_{i_1} \cdots s_{i_m}},$$

$$\tau_{\mathrm{W}}(\boldsymbol{w}) := \mathrm{W}^{(m)}_{s_{i_1}\cdots s_{i_m}},$$

The critical step consists in extending these maps to all of $\bar{T}(T(J))$. Indeed, the first step in clarifying the relations between these linear maps from a Hopf algebra point of view is to consider full Green's functions as a multiplicative map on $\bar{T}(T(J))$. That is, to extend the linear map of full Green's functions τ_{Z} multiplicatively to all of $\bar{T}(T(J))$:

$$\tau_{\mathrm{Z}}(\mathbf{1}) = 1, \quad \tau_{\mathrm{Z}}(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) := \tau_{\mathrm{Z}}(\boldsymbol{w}_1)\cdots\tau_{\mathrm{Z}}(\boldsymbol{w}_n).$$

For $\tau_{\mathrm{Z}} \in G_{\mathbb{K}}$, which denotes the group of characters on $\bar{T}(T(J))$, we shall see later that it is natural to require that τ_{W} be an infinitesimal character.

Recall the convolution product (7), introduced above in the context of the space of linear maps on a Hopf algebra with values in a commutative unital algebra, for instance, the complex numbers. This makes $\bar{T}^*(T(J))$ into a non-commutative unital algebra, where the product is defined in terms of the coproduct (10), i.e., for $\alpha, \beta \in Lin(\bar{T}(T(A)), \mathbb{K})$,

$$\alpha \star \beta := m_{\mathbb{K}} \circ (\alpha \otimes \beta) \circ \delta. \tag{29}$$

Here, $m_{\mathbb{K}}$ denotes the product map in $\mathbb{K}$.

The splitting of the coproduct (10) into left and right unshuffle half-coproducts (15) can be lifted to the algebra $\bar{T}^*(T(J))$. Considering this, we define the left and right half-shuffle convolution products

$$\alpha \prec \beta := m_{\mathbb{K}} \circ (\alpha \otimes \beta) \circ \delta_{\prec}, \tag{30}$$
$$\alpha \succ \beta := m_{\mathbb{K}} \circ (\alpha \otimes \beta) \circ \delta_{\succ}, \tag{31}$$

such that (15) implies that

$$\alpha \star \beta = \alpha \succ \beta + \alpha \prec \beta. \tag{32}$$

Remark 5. The same construction applies on the dual space of an arbitrary unshuffle bialgebra.

Definition 3 implies the following relations for the binary operations $\succ$ and $\prec$. Note that, with the aim of emphasizing the shuffle-type behavior, we replace the notation $\star$ for the product by the classical notation $\sqcup\!\sqcup$ for the shuffle product. That is, $\star = \sqcup\!\sqcup$.

$$(a \prec b) \prec c = a \prec (b \sqcup\!\sqcup c), \tag{33}$$
$$(a \succ b) \prec c = a \succ (b \prec c), \tag{34}$$
$$a \succ (b \succ c) = (a \sqcup\!\sqcup b) \succ c, \tag{35}$$

where

$$a \sqcup\!\sqcup b := a \prec b + a \succ b. \tag{36}$$

These are the axioms defining the abstract notion of a *shuffle*, or *dendrimorphic* algebra [17, 18], which is a vector space D together with two bilinear compositions $\prec$ and $\succ$, the so-called left and right half-shuffle products, that satisfy (33), (34), and (35). In fact, these axioms imply that any shuffle algebra is an associative algebra for the shuffle product defined in (36), and we call $\sqcup\!\sqcup$ the shuffle product on D.

A *commutative shuffle algebra* is a shuffle algebra for which the left and right half-shuffles are identified as follows:

$$x \succ y = y \prec x,$$

so that, in particular, the shuffle product $\sqcup\!\sqcup$ is commutative, $x \sqcup\!\sqcup y = x \prec y + x \succ y = y \sqcup\!\sqcup x$. The standard example of a commutative shuffle algebra structure was introduced on $\bar{T}(J)$ in (9). We remark that the same structure can be obtained as the dual to the one on $\bar{T}(J)$, when equipped with the concatenation product and the unshuffle coproduct.

Shuffle algebras are not naturally unital. This is because it is impossible to "split" the unit equation $\mathbf{1} \sqcup\!\sqcup a = a \sqcup\!\sqcup \mathbf{1} = a$ into two equations involving the left and right half-shuffle products $\succ$ and $\prec$. This issue can be circumvented by using the "Schützenberger trick". That is, for a shuffle algebra D, $\bar{D} := D \oplus \mathbb{K}\mathbf{1}$ denotes the shuffle algebra augmented by a unit $\mathbf{1}$ such that for $a \in D$,

$$a \prec \mathbf{1} := a =: \mathbf{1} \succ a \quad \mathbf{1} \prec a := 0 =: a \succ \mathbf{1}, \tag{37}$$

which implies that $a \sqcup\!\sqcup \mathbf{1} = \mathbf{1} \sqcup\!\sqcup a = a$. By convention, $\mathbf{1} \sqcup\!\sqcup \mathbf{1} = \mathbf{1}$. However, $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ cannot be defined consistently in the context of the axioms of shuffle algebras. In light of (30) and (31), we arrive at the next result.

Proposition 6. *The space* $(Lin(\bar{T}(T(J)), \mathbb{K}), \prec, \succ)$ *is a shuffle algebra.*

Remark 6. We state the proposition for $\bar{T}(T(J))$, because we always attempt to place the emphasis on planar QFT. However, the proof depends only on the fact that $\bar{T}(T(J))$ is an unshuffle bialgebra, and therefore applies for an arbitrary unshuffle bialgebra B with $\bar{B} = B \oplus \mathbb{K}\mathbf{1}$. In particular, the property holds for $Lin(\bar{T}(J), \mathbb{K})$.

Proof. For arbitrary $\alpha, \beta, \gamma \in T^*(T(J))$,

$$\begin{aligned}(\alpha \prec \beta) \prec \gamma &= m_{\mathbb{K}^{[3]}} \circ ((\alpha \prec \beta) \otimes \gamma) \circ \delta_{\prec} \\ &= m_{\mathbb{K}^{[3]}} \circ (\alpha \otimes \beta \otimes \gamma) \circ (\delta_{\prec} \otimes \mathrm{id}) \circ \delta_{\prec},\end{aligned}$$

where $m_{\mathbb{K}^{[3]}}$ denotes the product map from $\mathbb{K}^{\otimes 3}$ to $\mathbb{K}$. Similarly,

$$\begin{aligned}\alpha \prec (\beta \sqcup\!\sqcup \gamma) &= m_{\mathbb{K}} \circ (\alpha \otimes (\beta \sqcup\!\sqcup \gamma)) \circ \delta_{\prec} \\ &= m_{\mathbb{K}^{[3]}} \circ (\alpha \otimes \beta \otimes \gamma) \circ (\mathrm{id} \otimes \overline{\delta}) \circ \delta_{\prec},\end{aligned}$$

where $\overline{\delta}(u) = \delta(u) - u \otimes \mathbf{1} - \mathbf{1} \otimes u$ is the reduced coproduct. Then, the identity $(\alpha \prec \beta) \prec \gamma = \alpha \prec (\beta \sqcup\!\sqcup \gamma)$ follows from $(\delta_{\prec}\otimes\mathrm{id})\otimes\delta_{\prec} = (\mathrm{id}\otimes\bar{\delta})\circ\delta_{\prec}$, and the other identities characterizing shuffle algebras follow in a similar manner.

We equip the shuffle algebra $(T^*(T(J)), \prec, \succ)$ with the unit ε. Recall that ε is the canonical projection on the scalars $T(J)^{\otimes 0} \cong \mathbb{K}\mathbf{1}$. That is, for an arbitrary α in $\bar{T}^*(T(J))$,

$$\alpha \prec \varepsilon = \alpha = \varepsilon \succ \alpha, \quad \varepsilon \prec \alpha = 0 = \alpha \succ \varepsilon.$$

□

Let us introduce some useful notation. Let $L_{a\succ}(b) := a \succ b =: R_{\succ b}(a)$. The shuffle axioms yield that

$$L_{a\succ}L_{b\succ} = L_{a \,\sqcup\!\sqcup\, b\succ}, \quad R_{\prec a}R_{\prec b} = R_{\prec b \,\sqcup\!\sqcup\, a}.$$

Recall that a left pre-Lie algebra [42, 43] is a vector space V equipped with a bilinear product $\rhd \colon V\otimes V \to V$, such that for arbitrary $a, b, c \in V$,

$$a \rhd (b \rhd c) - (a \rhd b) \rhd c = b \rhd (a \rhd c) - (b \rhd a) \rhd c. \tag{38}$$

This implies that the bracket $[a, b] := a \rhd b - b \rhd a$ satisfies the Jacobi identity. Right pre-Lie algebras are defined similarly. Note that any associative algebra is pre-Lie. For several reasons, pre-Lie algebras play a key role, e.g., in the understanding of recursive equations such as Bogoliubov's counterterm formula in perturbative QFT [44, 45]. The next lemma follows directly from the axioms (33)–(35) of shuffle products.

Lemma 7. *Let D be a shuffle algebra. Then, the product $\rhd \colon D \otimes D \to D$*

$$a \rhd b := a \succ b - b \prec a$$

is left pre-Lie. We write its left action as $L_{a\rhd}(b) = a \rhd b = L_{a\succ} - R_{\prec a}$.

Note that $[a, b] = a \rhd b - b \rhd a = a \,\sqcup\!\sqcup\, b - b \,\sqcup\!\sqcup\, a$ for all $a, b \in D$. The pre-Lie product is trivial (null) on commutative shuffle algebras, because then we have that $a \succ b = b \prec a$.

The following set of left and right half-shuffle words in $\bar{D}$ are defined recursively for fixed elements $x_1, \ldots, x_n \in D$, $n \in \mathbb{N}^+$:

$$\begin{aligned} w_{\prec}^{(0)}(x_1, \ldots, x_n) &:= \mathbf{1} =: w_{\succ}^{(0)}(x_1, \ldots, x_n), \\ w_{\prec}^{(n)}(x_1, \ldots, x_n) &:= x_1 \prec \big(w_{\prec}^{(n-1)}(x_2, \ldots, x_n)\big), \\ w_{\succ}^{(n)}(x_1, \ldots, x_n) &:= \big(w_{\succ}^{(n-1)}(x_1, \ldots, x_{n-1})\big) \succ x_n. \end{aligned}$$

In the case that $x_1 = \cdots = x_n = x$, we simply write $x^{\prec n} := w_{\prec}^{(n)}(x, \ldots, x)$ and $x^{\succ n} := w_{\succ}^{(n)}(x, \ldots, x)$.

In the unital algebra $\bar{D}$, both the exponential and logarithm maps are defined in terms of the associative product (36):

$$\exp^{\sqcup\!\sqcup}(x) := \mathbf{1} + \sum_{n>0} \frac{x^{\sqcup\!\sqcup\, n}}{n!} \quad \text{resp.}$$
$$\log^{\sqcup\!\sqcup}(\mathbf{1} + x) := -\sum_{n>0} (-1)^n \frac{x^{\sqcup\!\sqcup\, n}}{n}. \tag{39}$$

Notice that we do not consider convergence issues. In practice, we will apply such formal power series computations either in a purely algebraic setting (formal convergence arguments would then apply), or when dealing with graded algebras (then, the series will reduce to a finite number of nonzero terms when restricted to a given graded component).

It is also convenient to introduce the "(time-)ordered" exponential:

$$\exp^{\prec}(x) := \mathbf{1} + \sum_{n>0} x^{\prec n}.$$

Similarly, we also define $\exp^{\succ}(x) := \mathbf{1} + \sum_{n>0} x^{\succ n}$. This corresponds to the usual time-ordered exponential in physics, when the shuffle product is defined with respect to products of, say, matrix- or operator-valued iterated integrals [46]. Notice that $X = \exp^{\prec}(a)$ and $Z = \exp^{\succ}(a)$ are respectively the formal solutions of the two linear fixed point equations:

$$\begin{aligned} X &= \mathbf{1} + a \prec X, \\ Z &= \mathbf{1} + Z \succ a. \end{aligned} \tag{40}$$

The solution of equation (40) can also be written in terms of the exponential map (39). In what follows, we will see that (40) is the key ingredient in our approach to a Hopf algebraic description of the functional relations between (non-)planar Green's functions. This point of view paves the way to new formal results on the combinatorics of Green's functions.

Lemma 8. *Let D be a shuffle algebra, and let $\bar{D}$ be its augmentation by a unit $\mathbf{1}$. For $x \in D$, we have that*

$$\exp^{\succ}(-x) \,\sqcup\!\sqcup\, \exp^{\prec}(x) = \mathbf{1}.$$

Proof. Indeed, we see that

$$\begin{aligned} &\exp^{\succ}(-x) \,\sqcup\!\sqcup\, \exp^{\prec}(x) - \mathbf{1} \\ &\quad = \sum_{n+m\geq 1} (-1)^n \big\{(x^{\succ n}) \prec (x^{\prec m}) + (x^{\succ n}) \succ (x^{\prec m})\big\} \\ &\quad = \sum_{n>0, m\geq 0} (-1)^n (x^{\succ n}) \prec (x^{\prec m}) \\ &\qquad + \sum_{n\geq 0, m>0} (-1)^n (x^{\succ n}) \succ (x^{\prec m}). \end{aligned}$$

Now, because $(-1)^n (x^{\succ n}) \prec (x^{\prec m}) = (-1)^n ((x^{\succ n-1}) \succ x) \prec (x^{\prec m}) = (-1)^n (x^{\succ n-1}) \succ (x^{\prec m+1})$, the result follows. □

Another useful result follows from the computation of the composition inverse of the time-ordered exponential.

Lemma 9. *Let D be a shuffle algebra, and let $\bar{D}$ be its augmentation by a unit $\mathbf{1}$. Then, for $x \in D$ and $Y := \exp^{\prec}(x) - \mathbf{1}$,*

$$x = Y \prec \Big(\sum_{n\geq 0}(-1)^n Y^{\sqcup\!\sqcup n}\Big).$$

Proof. We follow Ref. [39]. From $X := \mathbf{1} + \sum_{n>0} x^{\prec n}$, by solving $X = \mathbf{1} + x \prec X$ we obtain that $X - \mathbf{1} = Y = x \prec X$. On the other hand, the (formal) inverse of X *for the shuffle product* is given by $X^{-1} = \frac{1}{1+Y} = \sum_{k\geq 0}(-1)^k Y^{\sqcup\!\sqcup k}$. Finally, we obtain that

$$\begin{aligned} x &= x \prec \mathbf{1} = x \prec (X \sqcup\!\sqcup X^{-1}) = (x \prec X) \prec X^{-1} \\ &= Y \prec \Big(\sum_{n\geq 0}(-1)^n Y^{\sqcup\!\sqcup n}\Big). \end{aligned}$$

□

2.5 Towards a group theoretical view on planar field theory

Recall that $\bar{T}^*(T(J)) := Lin(\bar{T}(T(J)), \mathbb{K})$. A linear form $\phi \in \bar{T}^*(T(J))$ is called a character if it is unital, i.e., $\phi(\mathbf{1}) = 1$, and multiplicative, i.e., for all $\boldsymbol{a}, \boldsymbol{b} \in \bar{T}(T(J))$, $\phi(\boldsymbol{a}|\boldsymbol{b}) = \phi(\boldsymbol{a})\phi(\boldsymbol{b})$. A linear form $\kappa \in \bar{T}^*(T(J))$ is called an infinitesimal character if $\kappa(\mathbf{1}) = 0$ and $\kappa(\boldsymbol{a}|\boldsymbol{b}) = 0$ for all $\boldsymbol{a}, \boldsymbol{b} \in T(T(J))$. Characters and infinitesimal characters are bijectively related through the exponential map defined with respect to the convolution product. We write $Ch(\kappa)$ for the obvious extension of a linear form on $T(J)$ (e.g., the restriction to $T(J)$ of an infinitesimal character) to a character, defined by $Ch(\kappa)(\mathbf{1}) := 1$ and $Ch(\kappa)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_k) := \kappa(\boldsymbol{w}_1)\cdots\kappa(\boldsymbol{w}_k)$. Conversely, for an arbitrary $F \in \bar{T}^*(T(J))$ let us write $Res(F)$ for the infinitesimal character, which is defined by the restriction of F to $T(J)$ and the null map on other tensor powers of $T(J)$ in $\bar{T}(T(J))$.

The linear fixed point equation (40) is characterized in the following theorem.

Theorem 10. *There exists an additional natural bijection between the group of characters and the Lie algebra of infinitesimal characters on $\bar{T}(T(J))$. Indeed, for a character ϕ there exists an unique infinitesimal character κ such that*

$$\phi = \varepsilon + \kappa \prec \phi, \tag{41}$$

which implies that $\phi = \exp^{\prec}(\kappa)$. Conversely, for an infinitesimal character κ, the linear map

$$\phi := \exp^{\prec}(\kappa)$$

is a character.

In the following, we will employ the "Hopf- or Sweedler-type" notation $\delta_{\prec}(\boldsymbol{w}) =: \boldsymbol{w}^{1,\prec} \otimes \boldsymbol{w}^{2,\prec}$ as a shorthand.

Proof. We know from Lemma 9 that the implicit equation $\phi = \varepsilon + \kappa \prec \phi$ yields that $\phi = \exp^{\prec}(\kappa)$, and has a unique solution κ in $\bar{T}^*(T(J))$. Let us consider the infinitesimal character $\mu := Res(\kappa)$, and let us show that μ also solves $\phi = \varepsilon + \mu \prec \phi$. Then, the first part of the statement in the theorem will follow.

Indeed, for an arbitrary $\boldsymbol{w} = \boldsymbol{w}_1|\cdots|\boldsymbol{w}_n \in T(T(J))$, $\boldsymbol{w}_i \in T(J)$, notice first that from definition of the product $\prec$ and the fact that μ vanishes on any tensor power $T(J)^{\otimes k}$ for $k \neq 1$, we have that

$$\begin{aligned} (\mu \prec \phi)(\boldsymbol{w}) &= \mu(\boldsymbol{w}_1^{1,\prec})\phi(\boldsymbol{w}_1^{2,\prec}|\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n) \\ &= \kappa(\boldsymbol{w}_1^{1,\prec})\phi(\boldsymbol{w}_1^{2,\prec}|\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n). \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \phi(\boldsymbol{w}_1) &= (\varepsilon + \kappa \prec \phi)(\boldsymbol{w}_1) = \kappa(\boldsymbol{w}_1^{1,\prec})\phi(\boldsymbol{w}_1^{2,\prec}) \\ &= \mu(\boldsymbol{w}_1^{1,\prec})\phi(\boldsymbol{w}_1^{2,\prec}), \end{aligned}$$

so we immediately obtain that

$$\begin{aligned} \phi(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) &= \phi(\boldsymbol{w}_1)\phi(\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n) \\ &= \mu(\boldsymbol{w}_1^{1,\prec})\phi(\boldsymbol{w}_1^{2,\prec}|\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n)) \\ &= (\varepsilon + \mu \prec \phi)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n), \end{aligned}$$

for any $i > 1$. Then, the desired property follows. Conversely, we have that

$$\begin{aligned} \exp^{\prec}(\kappa)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) &= (\varepsilon + \kappa \prec \exp^{\prec}(\kappa))(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) \\ &= \kappa(\boldsymbol{w}_1^{1,\prec})\exp^{\prec}(\kappa)(\boldsymbol{w}_1^{2,\prec}|\cdots|\boldsymbol{w}_n). \end{aligned}$$

Assuming by induction that the property $\exp^{\prec}(\kappa)(\boldsymbol{w}'_1|\cdots|\boldsymbol{w}'_k) = \exp^{\prec}(\kappa)(\boldsymbol{w}'_1)\cdots\exp^{\prec}(\kappa)(\boldsymbol{w}'_k)$ holds for elements $\boldsymbol{w}'_1|\cdots|\boldsymbol{w}'_k \in T(T(J))$ of total degree less than the degree of $\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n$ yields that

$$\begin{aligned} &\exp^{\prec}(\kappa)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) \\ &\quad= \kappa(\boldsymbol{w}_1^{1,\prec})\exp^{\prec}(\kappa)(\boldsymbol{w}_1^{2,\prec})\exp^{\prec}(\kappa)(\boldsymbol{w}_2)\cdots\exp^{\prec}(\kappa)(\boldsymbol{w}_n) \\ &\quad= \exp^{\prec}(\kappa)(\boldsymbol{w}_1)\exp^{\prec}(\kappa)(\boldsymbol{w}_2)\cdots\exp^{\prec}(\kappa)(\boldsymbol{w}_n). \end{aligned}$$

□

The next result shows that Eq. (40) has a solution in terms of the exponential map defined in (39) with respect to the convolution (29), which splits as a shuffle product (32) in the sense of (36). We recall that $L_{a\rhd}(b) := a \rhd b = a \succ b - b \prec a$, where the product $a \rhd b$ satisfies the pre-Lie relation (38). See Refs. [42, 43] for further details and results concerning pre-Lie algebras.

Theorem 11. [47, 48] *Equation* (41) *admits the exponential solution*

$$\phi = \exp^{\star}\big(\Omega'(\kappa)\big),$$

where $\Omega'(\kappa)$ is the pre-Lie Magnus expansion, which obeys the following recursive equation:

$$\Omega'(\kappa) = \frac{L_{\Omega'\triangleright}}{\exp(L_{\Omega'\triangleright}) - 1}(\kappa) = \sum_{m\geq 0} \frac{B_m}{m!} L^m_{\Omega'\triangleright}(\kappa).$$

Here, the B_l terms are the Bernoulli numbers, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$, where $B_{2k+1} = 0$ for $k \geq 1$.

For a proof of this theorem, we refer the reader to [47, 48]. Note that for a commutative shuffle algebra the pre-Lie product is the null product, and $\Omega'(\kappa)$ reduces to the identity map. In this case, the solution to (41) is given by $\phi = \exp^\star(\kappa)$. Let us mention that the pre-Lie Magnus expansion $\Omega'(\kappa)$ can also be understood from the point of view of enveloping algebras of pre-Lie algebras [49]. As a historical remark, we mention that in Ref. [29] Wilhelm Magnus proposed a particular differential equation for the matrix valued function $\Omega(s; A)$, such that the solution of the linear non-autonomous initial value problem $\dot{Y} = AY$, $Y(0) = Y_0$ is given by $X(t) = \exp(\int_0^t \dot{\Omega}(x; A)dx)Y_0$, where $\Omega(0; A) = 0$ and

$$\begin{aligned}\dot{\Omega}(s; A) &= A(s) + \sum_{n>0} \frac{B_n}{n!} ad^{(n)}_{\int_0^s \dot{\Omega}(x;A)dx}(A(s)) \\ &= \frac{ad_{\Omega(s;A)}}{\exp(ad_{\Omega(s;A)}) - 1}(A(s)).\end{aligned}$$

Iterated Lie brackets are denoted by $ad^{(n)}_U(W) := [U, [U, \cdots [U, W]] \cdots]$. See Ref. [28] for details.

3 From full to connected and noncrossing planar Green's functions

3.1 From full to connected Green's functions

Recall the fixed point equation relating the generating functionals $\mathrm{Z}[j]$ and $\mathrm{W}[j]$ in the planar context:

$$\mathrm{Z}[j] = 1 + \mathrm{W}[j\mathrm{Z}[j]]. \tag{42}$$

When expanded to compute the planar n-point function $\mathrm{Z}^{(n)}_{i_1\cdots i_n}$, the equation reads:

$$\begin{aligned}\mathrm{Z}^{(n)}_{i_1\cdots i_n} = \sum_{A=\{1=a_1,\ldots,a_k\}\subset[n]} & \mathrm{W}^{(k)}_{i_{a_1}i_{a_2}\cdots i_{a_k}} \mathrm{Z}^{(a_2-a_1-1)}_{i_{a_1+1}\cdots i_{a_2-1}} \\ & \cdots \mathrm{Z}^{(n-a_k)}_{i_{a_k+1}\cdots i_n}.\end{aligned} \tag{43}$$

We employ the half-(un)shuffle machinery to rewrite this convoluted relation in a rather natural way. Recall that $\bar{T}^*(T(J))$ is a unital shuffle algebra for the left and right half-shuffle products $\prec, \succ$, defined in terms of the unshuffle half-coproducts (16) and (17) respectively.

Let the linear map $\tau_{\mathrm{W}} \in \bar{T}^*(T(J))$ associated to connected planar n-point functions be defined by $\tau_{\mathrm{W}}(\boldsymbol{w}) := \mathrm{W}^{(n)}_{s_{i_1}\cdots s_{i_n}}$ for any word $\boldsymbol{w} = w_1 \cdots w_n \in T_n(J)$ with $\tau_{\mathrm{W}}(\mathbf{1}) = 0$, where τ_{W} is zero on the components $T_n(T(J))$ for $n \geq 2$. The latter requirement is natural, in view of the desired connectedness. Hence, the map τ_{W} defines an infinitesimal character with respect to the Hopf algebra $\bar{T}(T(J))$. On the other hand, recall that the full planar Green's function $\tau_{\mathrm{Z}}(w_1 \cdots w_n) := \mathrm{Z}^{(n)}_{s_{i_1}\cdots s_{i_n}}$ is supposed to be multiplicative. Then, we find that by the definition of the left half-shuffle product $\prec$, the relation between full and connected planar Green's functions (42) is equivalently encoded, from the point of view of generating series, by the linear fixed point equation

$$\tau_{\mathrm{Z}} = \varepsilon + \tau_{\mathrm{W}} \prec \tau_{\mathrm{Z}}. \tag{44}$$

This claim follows immediately from the unfolding of (44) when evaluated on the word $\boldsymbol{w} = w_1 \cdots w_n \in T_n(J)$. For simplicity, we choose $w_i = j_{s_i} \in J$. Indeed, the definition of the left half-unshuffle (16) on $\bar{T}(T(J))$, together with the multiplicativity of τ_{Z}, implies that

$$\begin{aligned}\mathrm{Z}^{(n)}_{s_1\cdots s_n} &= \tau_{\mathrm{Z}}(w_1 \cdots w_n) = (\tau_{\mathrm{W}} \prec \tau_{\mathrm{Z}})(w_1 \cdots w_n) \\ &= \sum_{K=\{1=k_1,\ldots,k_l\}\subset[n]} \tau_{\mathrm{W}}(\boldsymbol{w}_K) \\ &\qquad \tau_{\mathrm{Z}}(w_{k_1+1} \cdots w_{k_2-1}) \cdots \tau_{\mathrm{Z}}(w_{k_l+1} \cdots w_n) \qquad (45) \\ &= \sum_{K=\{1=k_1,\ldots,k_l\}\subset[n]} \mathrm{W}^{(l)}_{s_{k_1}s_{k_2}\cdots s_{k_l}} \\ &\qquad \mathrm{Z}^{(k_2-k_1-1)}_{s_{k_1+1}\cdots s_{k_2-1}} \cdots \mathrm{Z}^{(n-k_l)}_{s_{k_l+1}\cdots s_n}. \qquad (46)\end{aligned}$$

This exemplifies the naturalness of the (half-)unshuffle structure in the context of the relation between full and connected planar Green's functions.

Remark 7. (1) The companion equation $\mathrm{Z}[j] = 1 + \mathrm{W}[\mathrm{Z}[j]j]$ is encoded in terms of the second left half-unshuffle defined in Remark 2, Eq. (18).

(2) Theorem 11 implies that the full planar Green's function can be written as an exponential in terms of the connected planar Green's function:

$$\tau_{\mathrm{Z}} = \exp^\star\left(\Omega'(\tau_{\mathrm{W}})\right).$$

Recall that the associative shuffle product $a \star b = a \prec b + a \succ b$, which is the convolution product in $\bar{T}^*(T(J))$, is non-commutative.

(3) The last item should be regarded from the viewpoint of the non-planar setting, in which the full and connected Green's functions are also related through the exponential map. We refer the reader to subsection 3.5.

Let us illustrate equation (46) by expanding its solution in terms of the time ordered exponential of the infinitesimal character of connected planar Green's functions, $\tau_{\mathrm{Z}} = \exp^{\prec}(\tau_{\mathrm{W}})$, up to order four (compare with the identities obtained using the relation (42) at the beginning of this article). At order one, for the one letter word $\boldsymbol{w} = w_1$, we have that

$$\begin{aligned}\tau_{\mathrm{Z}}(w_1) &= (\varepsilon + \tau_{\mathrm{W}} + \tau_{\mathrm{W}} \prec \tau_{\mathrm{W}} + \tau_{\mathrm{W}} \prec (\tau_{\mathrm{W}} \prec \tau_{\mathrm{W}}) + \cdots)(w_1)\\ &= \tau_{\mathrm{W}}(w_1),\end{aligned}$$

because $\varepsilon(w_1) = 0$, as are powers of left half-shuffle products beyond order one applied to the letter w_1, because $\tau_{\mathrm{W}}(\mathbf{1}) = 0$. For the two letter word $w = w_1 w_2$, we get that $\delta_{\prec}(w_1 w_2) = w_1 w_2 \otimes \mathbf{1} + w_1 \otimes w_2$, so that

$$\tau_{\mathrm{Z}}(w_1 w_2) = \tau_{\mathrm{W}}(w_1 w_2) + \tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2).$$

For the order three word $w = w_1 w_2 w_3$ with three letters $w_i \in J$, we get that

$$\begin{aligned}\tau_{\mathrm{Z}}(w_1 w_2 w_3) = {}& \tau_{\mathrm{W}}(w_1 w_2 w_3) + \tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2 w_3)\\ &+\tau_{\mathrm{W}}(w_1 w_3)\tau_{\mathrm{W}}(w_2) + \tau_{\mathrm{W}}(w_1 w_2)\tau_{\mathrm{W}}(w_3)\\ &+\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2)\tau_{\mathrm{W}}(w_3).\end{aligned}$$

In addition, for the order four word $w = w_1 w_2 w_3 w_4$ with letters $w_i \in J$, the left half-unshuffle product results in the lengthy expansion

$$\begin{aligned}&\tau_{\mathrm{Z}}(w_1 w_2 w_3 w_4)\\ &= \tau_{\mathrm{W}}(w_1 w_2 w_3 w_4) + \tau_{\mathrm{W}}(w_1 w_2)\tau_{\mathrm{Z}}(w_3 w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1 w_3)\tau_{\mathrm{Z}}(w_2)\tau_{\mathrm{Z}}(w_4) + \tau_{\mathrm{W}}(w_1 w_4)\tau_{\mathrm{Z}}(w_2 w_3)\\ &\quad+\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{Z}}(w_2 w_3 w_4) + \tau_{\mathrm{W}}(w_1 w_2 w_3)\tau_{\mathrm{Z}}(w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1 w_2 w_4)\tau_{\mathrm{Z}}(w_3) + \tau_{\mathrm{W}}(w_1 w_3 w_4)\tau_{\mathrm{Z}}(w_2)\\ &= \tau_{\mathrm{W}}(w_1 w_2 w_3 w_4) + \tau_{\mathrm{W}}(w_1 w_2)\tau_{\mathrm{W}}(w_3 w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1 w_4)\tau_{\mathrm{W}}(w_2 w_3) + \tau_{\mathrm{W}}(w_1 w_3)\tau_{\mathrm{W}}(w_2)\tau_{\mathrm{W}}(w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1 w_2)\tau_{\mathrm{W}}(w_3)\tau_{\mathrm{W}}(w_4) + \tau_{\mathrm{W}}(w_1 w_4)\tau_{\mathrm{W}}(w_2)\tau_{\mathrm{W}}(w_3)\\ &\quad+\tau_{\mathrm{W}}(w_3 w_4)\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2) + \tau_{\mathrm{W}}(w_2 w_3)\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_4)\\ &\quad+\tau_{\mathrm{W}}(w_2 w_4)\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_3)\\ &\quad+\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2)\tau_{\mathrm{W}}(w_3)\tau_{\mathrm{W}}(w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1)\tau_{\mathrm{W}}(w_2 w_3 w_4) + \tau_{\mathrm{W}}(w_1 w_2 w_3)\tau_{\mathrm{W}}(w_4)\\ &\quad+\tau_{\mathrm{W}}(w_1 w_2 w_4)\tau_{\mathrm{W}}(w_3) + \tau_{\mathrm{W}}(w_1 w_3 w_4)\tau_{\mathrm{W}}(w_2). \qquad (47)\end{aligned}$$

Recall the definitions of τ_{W} and τ_{Z}, and compare lines (47) and (4). Note that the multiplicativity of τ_{Z} enters at order four in the term $\tau_{\mathrm{W}}(w_1 w_3)\tau_{\mathrm{Z}}(w_2)\tau_{\mathrm{Z}}(w_4)$ corresponding to the tensor product $w_1 w_3 \otimes w_2 | w_4 \in T(J) \otimes T_2(T(J))$.

3.2 A bialgebra of non-crossing partitions

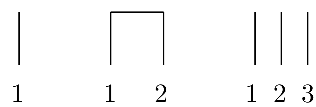
The following sections aim to further develop the combinatorics underlying planar theories, through the notion of non-crossing partitions. Recall that a partition L of a (finite) set $[n] := \{1, \ldots, n\}$ consists of a collection of (non-empty) subsets $L = \{L_1, \ldots, L_b\}$ of $[n]$, called blocks, which are mutually disjoint, i.e., $L_i \cap L_j = \emptyset$ for all $i \neq j$, and whose union $\cup_{i=1}^{b} L_i = [n]$. See Refs. [4, 24] for details. The number of blocks of the partition L is denoted by $|L| := b$, and $|L_i|$ is the number of elements in the ith block L_i. Given $p, q \in [n]$, we will write that $p \sim_L q$ if and only if they belong to the same block. The lattice of set partitions of $[n]$ is denoted by P_n. This has a partial order of refinement: $L \leq K$ if L is a finer partition than K. The partition $\hat{1}_n = \{L_1\}$ consists of a single block, i.e., $|L_1| = n$, and is the maximum element in P_n. The partition $\hat{0}_n = \{L_1, \ldots, L_n\}$ has n singleton blocks, and is the minimum partition in P_n. A set partition $L = \{L_1, \ldots, L_k\}$ of $[n]$ ($L_1 \coprod \cdots \coprod L_k = [n]$) is called non-crossing if for $p_1, p_2, q_1, q_2 \in [n]$, the following property does not occur:

$$1 \leq p_1 < q_1 < p_2 < q_2 \leq n$$

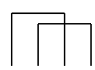
and

$$p_1 \sim_L p_2 \nsim_L q_1 \sim_L q_2.$$

The set of non-crossing partitions of $[n]$ will be denoted by NC_n, and we set $NC := \cup_{n \in \mathbb{N}} NC_n$. The reader is referred to the standard reference [12] for further details. In addition, see Refs. [13, 14, 24]. The common pictorial representation of (non-crossing) partitions is invoked. For example,



The first represents the singleton $\hat{0}_1 = \hat{1}_1 = \{1\}$ in P_1. The second is the single block partition, i.e., the maximal element $\hat{1}_2 = \{1, 2\} \in P_2$. The third represents the minimal element in P_3, i.e., the partition of the set [3] into singletons, $\hat{0}_3 = \{\{1\}, \{2\}, \{3\}\}$. The partition $\{\{1, 3\}, \{2, 4\}\}$ is represented by

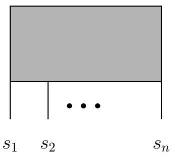


and is not a non-crossing partition, whereas $\{\{1, 9\}, \{2, 6, 8\}, \{3, 5\}, \{4\}, \{7\}\}$ and $\{\{1, 3, 7\}, \{2\}, \{4, 5, 6\}\}$ are proper non-crossing partitions, i.e., partitions without crossings.

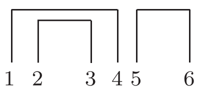
Non-crossing partitions of arbitrary subsets of the integers are defined similarly. For example, $\{\{1, 6, 10\}, \{2\}, \{7, 9\}\}$ is a non-crossing partition of $\{1, 2, 6, 7, 9, 10\}$. We will implicitly use various elementary properties of non-crossing partitions [12]. In particular, we will use the fact that if L is a non-crossing partition of $[n]$, then its restriction to an arbitrary subset S of $[n]$ (by intersecting the blocks of L with S) defines a non-crossing partition of S.

The full n-point Green's function with its external legs decorated by $(s_1, \ldots, s_n)$ is usually represented by

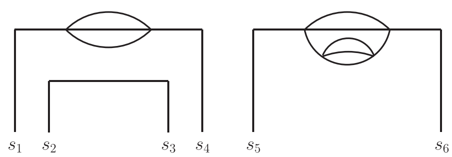
$$\mathrm{Z}^{(n)}_{s_1\cdots s_n} = \qquad (48)$$



Recall that the indices $s_1, \ldots, s_n$ may represent discrete and continuous variables, as well as parameters specifying the amplitude. In the perturbative approach, this diagram represents the sum of all Feynman diagrams with n legs and external parameters $s_1, \ldots, s_n$. Then, the planarity constraint in QFT translates into the property that the various propagators joining external sources in the diagrammatic expansion of planar Green's functions never cross. In particular, given a Feynman graph $\Gamma_{s_1\cdots s_n}$ in the expansion of the full n-point Green's function $\mathrm{Z}^{(n)}_{s_1\cdots s_n}$ with k connected components, the partition of the external legs according to them belonging to one common connected component of the graph induces a non-crossing partition $N(\Gamma_{s_1\cdots s_n}) = (L_1, \ldots, L_k)$ of $[n]$. For example, in planar Φ^4 theory, for the non-crossing partition of the set [6] given by



we have the following Feynman diagram:



We denote a Feynman diagram with n external legs by $\Gamma^{(n)}_{s_1\cdots s_n}$. Its corresponding amplitude $\gamma^{(n)}_{s_1\cdots s_n}$ is given by Feynman rules, and is normalized by the appropriate symmetry factor. The sum of all amplitudes when $\Gamma^{(n)}_{s_1\cdots s_n}$ runs over all diagrams with n external legs gives the full Green's function $\mathrm{Z}^{(n)}_{s_1\cdots s_n}$, which is depicted as a coefficient in (48) (resp. the connected Green's function $\mathrm{W}^{(n)}_{s_1\cdots s_n}$ when the sum runs over connected diagrams).

Summing up all these amplitudes $\gamma^{(n)}_{s_1\cdots s_n}$ of the Feynman diagrams associated to a given non-crossing partition $L = \{L_1, \ldots, L_k\}$ of $[n]$ defines a new Green's function,

$$\mathrm{N}^{(n)}_{s_1\cdots s_n}(L) = \mathrm{N}^{(n)}_{s_1\cdots s_n}(L_1, \ldots, L_k) := \sum_{N(\gamma)=L} \gamma^{(n)}_{s_1\cdots s_n},$$

so that by setting $\mathrm{W}^K := \mathrm{W}^{(p)}_{s_{k_1}\cdots s_{k_p}}$ for $K = \{k_1, \ldots, k_p\} \subset [n]$, one obtains that

$$\mathrm{N}^{(n)}_{s_1\cdots s_n}(L_1, \ldots, L_k) = \prod_{i=1}^{k} \mathrm{W}^{L_i}.$$

Furthermore, full Green's functions split according to non-crossing partitions

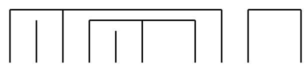
$$\mathrm{Z}^{(n)}_{s_1\cdots s_n} = \sum_{L \in NC_n} \mathrm{N}^{(n)}_{s_1\cdots s_n}(L).$$

This is the phenomenon that will be investigated in this section from a combinatorial point of view.

Let $L = \{L_1, \ldots, L_k\}$ be an arbitrary non-crossing partition of $[n] := \{1, \ldots, n\}$ with $\inf(L_i) < \inf(L_{i+1})$ for $i = 1, \ldots, n-1$. Let us write $L_i < L_j$ if $\forall a \in L_i$ and $\forall b \in L_j$ we have that $a < b$. This defines a first partial order, the linear ordering of blocks of the partition. We then define a second partial order $<_L$ on the blocks L_i, the nesting order, by $L_i <_L L_j$ if and only if for all $m \in L_i$, $\inf(L_j) < m < \sup(L_j)$. The definition of non-crossing partitions ensures that this partial order is well-defined. Moreover, given two distinct blocks $L_i, L_j \in L$, exactly one of the following inequalities holds:

$$L_i < L_j, \quad L_j < L_i, \quad L_i <_L L_j, \quad L_j <_L L_i.$$

As an example, we consider the particular non-crossing partition $L \in P_{10}$ with five blocks $L = \{L_1, L_2, L_3, L_4, L_5\} = \{\{1,3,8\}, \{2\}, \{4,6,7\}, \{5\}, \{9,10\}\}$:



The block $L_5 > L_i$ for $i = 1, 2, 3, 4$. In addition, $L_2 <_L L_1$ and $L_4 <_L L_3 <_L L_1$.

A splitting (which from now on will be called a *cut*) of the blocks of L into two (possibly empty) subsets

$$L = Q \coprod T = \{Q_1, \ldots, Q_i\} \coprod \{T_1, \ldots, T_{k-i}\}$$

will be called *admissible* if and only if for all $p \leq i$ and $q \leq k - i$, $Q_p \not<_L T_q$. That is, either $T_q <_L Q_p$ or the two subsets of $[n]$ are incomparable for the partial order. Then, we write $L = Q \coprod_{\mathrm{adm}} T$. Admissible cuts of non-crossing partitions of arbitrary finite subsets S of the integers are defined accordingly. Returning to the above example, we have that (note that the list is not exhaustive)

$$\begin{aligned} L &= \{L_1, L_2, L_3, L_4\} \coprod_{\mathrm{adm}} \{L_5\} \\ &= \{L_1, L_2, L_5\} \coprod_{\mathrm{adm}} \{L_3, L_4\} \\ &= \{L_1, L_5\} \coprod_{\mathrm{adm}} \{L_2, L_3, L_4\}. \end{aligned}$$

Given two (canonically ordered) subsets $S \subseteq U$ of the set of positive integers $\mathbb{N}^+$, recall that a connected component of S relative to U is a maximal sequence $s_1, \ldots, s_n$ in S such that there exist no $1 \leq i < n$ and $t \in U$ with $s_i < t < s_{i+1}$. In particular, a connected component of S in $\mathbb{N}^+$ is simply a maximal sequence of successive elements $s, s+1, \ldots, s+n$ in S.

For an admissible cut $L = Q \coprod_{\mathrm{adm}} U$ as above, we consider the connected components $J_1, \ldots, J_{k(L,Q)}$ of $[n] - (Q_1 \cup \cdots \cup Q_i)$, for which we will slightly abuse terminology by calling the *connected components* of $[n] - Q$ from now on. The definition of $<_L$ implies that $J_i \cap U_j$ is empty or equal to U_j. We write $J_i^{L,Q}$ for the set of all non-empty intersections $J_i \cap U_j$, $j = 1, \ldots, k-i$, and notice that because L is a non-crossing partition of $[n]$, $J_i^{L,Q}$ is, by restriction, a non-crossing partition of the component J_i. For the same reason, Q is a non-crossing partition of $Q_1 \cup \cdots \cup Q_i$.

Let us recall now that given a finite subset S of cardinality n of the integers, the standardization map st is the (necessarily unique) increasing bijection between S and $[n]$. By extension, we also write st for the induced map on the various objects associated to $[n]$ (such as partitions). For example, the standardization of the non-crossing partition $L := \{\{3,6,10\},\{4,5\},\{8\}\}$ of the set $\{3,4,5,6,8,10\}$ is the non-crossing partition $st(L) := \{\{1,4,6\},\{2,3\},\{5\}\}$ of $[6] = st(\{3,4,5,6,8,10\})$.

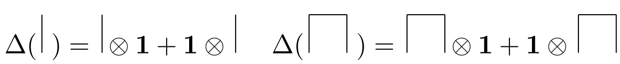
The linear span $\mathcal{NC}$ of all non-crossing partitions can then be equipped with a coproduct map Δ from $\mathcal{NC}$ to $\mathcal{NC} \otimes T(\mathcal{NC})$, defined by (using our previous notations as well as the bar-$|$ notation for elements in $T(\mathcal{NC})$)

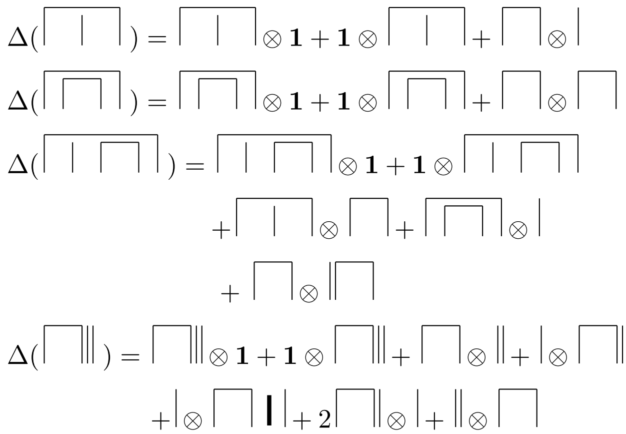
$$\Delta(L) = \sum_{Q \coprod_{\mathrm{adm}} U = L} st(Q) \otimes \big(st(J_1^{L,Q})|\cdots|st(J_{k(L,Q)}^{L,Q})\big).$$

A few examples are in order at this stage.

$$\begin{aligned}
&\Delta(\{\{1,4\},\{2,3\}\})\\
&= \{\{1,4\},\{2,3\}\} \otimes \mathbf{1} + \mathbf{1} \otimes \{\{1,4\},\{2,3\}\}\\
&\quad + \{1,2\} \otimes \{1,2\} \Delta(\{\{1,5\},\{2\},\{3,4\}\})\\
&= \{\{1,5\},\{2\},\{3,4\}\} \otimes \mathbf{1} + \mathbf{1} \otimes \{\{1,5\},\{2\},\{3,4\}\}\\
&\quad + \{\{1,3\},\{2\}\} \otimes \{1,2\} + \{\{1,4\},\{2,3\}\} \otimes \{1\}\\
&\quad + \{1,2\} \otimes \{\{1\},\{2,3\}\},\\
&\Delta(\{\{1,2\},\{3\},\{4\}\})\\
&= \{\{1,2\},\{3\},\{4\}\} \otimes \mathbf{1} + \mathbf{1} \otimes \{\{1,2\},\{3\},\{4\}\}\\
&\quad + \{1,2\} \otimes \{\{1\}\{2\}\} + 2\{\{1,2\},\{3\}\} \otimes \{1\}\\
&\quad + \{1\} \otimes \{\{1,2\},\{3\}\} + \{1\} \otimes \{1,2\}|\{1\}\\
&\quad \{\{1\},\{2\}\} \otimes \{1,2\}
\end{aligned}$$

The graphical notation of the coproduct simplifies the calculus, because it is automatically standardized (note that the bar notation is formatted in bold to distinguish it from the single element partition)





The map Δ is then extended multiplicatively to a coproduct on $\bar{T}(\mathcal{NC})$, as

$$\Delta(L_1|\cdots|L_n) := \Delta(L_1)\cdots\Delta(L_n), \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}.$$

Here, $\bar{T}(\mathcal{NC})$ is equipped with the structure of a free associative algebra over $\mathcal{NC}$ by the concatenation map, $(L_1|\cdots|L_k)\cdot(L_{k+1}|\cdots|L_n) := (L_1|\cdots|L_k|L_{k+1}|\cdots|L_n)$.

Theorem 12. [18] *The graded algebra $\bar{T}(\mathcal{NC})$ equipped with the coproduct Δ is a connected graded non-commutative and non-cocommutative Hopf algebra.*

This coproduct can be split into two parts as follows. On $\mathcal{NC}$, define the *left half-coproduct* by

$$\Delta_{\prec}(L) = \sum_{\substack{Q \coprod_{\mathrm{adm}} U = L\\ 1 \in Q_1}} st(Q) \otimes \big(st(J_1^{L,Q})|\cdots|st(J_{k(L,Q)}^{L,Q})\big), \tag{49}$$

and

$$\bar{\Delta}_{\prec}(L) := \Delta_{\prec}(L) - L \otimes \mathbf{1}. \tag{50}$$

The *right half-coproduct* is defined by

$$\Delta_{\succ}(L) = \sum_{\substack{Q \coprod_{\mathrm{adm}} U = L\\ 1 \notin Q_1}} st(Q) \otimes \big(st(J_1^{L,Q})|\cdots|st(J_{k(L,Q)}^{L,Q})\big), \tag{51}$$

and

$$\bar{\Delta}_{\succ}(L) := \Delta_{\succ}(L) - \mathbf{1} \otimes L. \tag{52}$$

This yields that $\Delta = \Delta_{\prec} + \Delta_{\succ}$, and for $L \in \mathcal{NC}$,

$$\Delta(L) = \bar{\Delta}_{\prec}(L) + \bar{\Delta}_{\succ}(L) + L \otimes \mathbf{1} + \mathbf{1} \otimes L.$$

This is extended to $\bar{T}(\mathcal{NC})$ by defining

$$\Delta_{\prec}(L_1|\cdots|L_m) := \Delta_{\prec}(L_1)\Delta(L_2)\cdots\Delta(L_m) \tag{53}$$

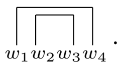
$$\Delta_{\succ}(L_1|\cdots|L_m) := \Delta_{\succ}(L_1)\Delta(L_2)\cdots\Delta(L_m). \tag{54}$$

Theorem 13. [18] *The bialgebra $\bar{T}(\mathcal{NC})$ equipped with $\Delta_{\succ}$ and $\Delta_{\prec}$ is an unshuffle bialgebra.*

Frontiers of Physics

3.3 Non-crossing Green's functions

At the beginning of this section, we have introduced the notion of Green's functions parametrized by non-crossing partitions $L \in NC_n$ and $\mathrm{N}^{(n)}_{s_1\cdots s_n}(L)$. We will simply call these non-crossing Green's functions. These Green's functions are naturally parametrized by the pair of a non-crossing partition $L \in NC_n$ of $[n]$ and a word of n letters, $\boldsymbol{w} = w_1 \cdots w_n$, $w_i \in J$, which will be denoted from now on by $L(w_1 \cdots w_n)$, and called decorated non-crossing partitions. The latter are represented graphically by adding decorations to the graphical representations of non-crossing partitions. For example, for $L = \{\{1,3\},\{2,4\}\}$, $L(w_1 \cdots w_4)$ is represented by



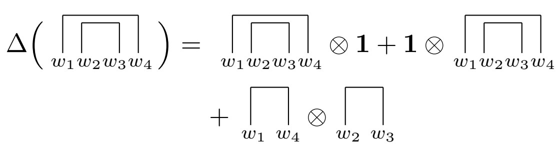
In the present section, we will demonstrate how, for such data, to mimic the analogue of the calculus that has been developed previously in order to understand the link between full and connected Green's functions from a Hopf algebraic and group-theoretical point of view.

For an alphabet $J = \{j_{s_i}\}_{i>0}$, let us write $\mathcal{NC}(J)$ for the graded vector space spanned by J-decorated (or simply decorated) non-crossing partitions. For $S = \{k_1,\ldots,k_l\} \subset [n]$ and $\boldsymbol{w} := w_1 \cdots w_n$, we set $\boldsymbol{w}_K := w_{k_1} \cdots w_{k_l}$. Similarly, for a non-crossing partition $Q = \{Q_1,\ldots,Q_i\}$ of the set K, we write $\boldsymbol{w}_Q := \boldsymbol{w}_S$.

The definitions and results in the previous paragraph carry over to decorated non-crossing partitions in a straightforward way. For example, the coproduct map Δ is defined on $\mathcal{NC}(J)$ by

$$\Delta(L(w_1\cdots w_n)) = \sum_{Q \coprod_{\mathrm{adm}} U = L} \big(st(Q) \otimes \boldsymbol{w}_Q\big) \otimes \big(st(J_1^{L,Q}) \otimes (\boldsymbol{w}_{J_1^{L,Q}}) | \cdots | st(J_{k(L,Q)}^{L,Q}) \otimes (\boldsymbol{w}_{J_{k(L,Q)}^{L,Q}})\big),$$

where L is a non-crossing partition of $[n]$. As an example we calculate:



This is then extended to $T(\mathcal{NC}(J))$ multiplicatively, as in the previous sections. The other structural maps on $T(\mathcal{NC})$ are extended similarly to $T(\mathcal{NC}(J))$, and are written using the same symbols as before. Finally, we obtain the following theorem.

Theorem 14. *The bialgebra $\bar{T}(\mathcal{NC}(J))$ equipped with $\Delta_\succ$ and $\Delta_\prec$ is an unshuffle bialgebra.*

Because $\bar{T}(\mathcal{NC}(J))$ is in particular a Hopf algebra, the set of linear maps $Lin(\bar{T}(\mathcal{NC}(J)),\mathbb{K})$ is an algebra with respect to the convolution product, which is defined in terms of the coproduct Δ, i.e., for $f, g \in Lin(\bar{T}(\mathcal{NC}(J)),\mathbb{K})$,

$$f \star g := m_{\mathbb{K}} \circ (f \otimes g) \circ \Delta,$$

where $m_{\mathbb{K}}$ stands for the product map in $\mathbb{K}$. Note that, motivated by the next proposition, we will also later employ a shuffle notation for this product, $f \star g =: f \sqcup\!\sqcup g$. We define accordingly the left and right convolution half-products on $Lin(T(\mathcal{NC}(J)),\mathbb{K})$:

$$f \prec g := m_{\mathbb{K}} \circ (f \otimes g) \circ \Delta_\prec,$$
$$f \succ g := m_{\mathbb{K}} \circ (f \otimes g) \circ \Delta_\succ.$$

Proposition 15. *The space $T^*(\mathcal{NC}(J)) := Lin(T(\mathcal{NC}(J)),\mathbb{K})$ equipped with $(\prec,\succ)$ is a shuffle algebra.*

For completeness, and in view of the importance of this proposition for forthcoming developments, we briefly recall its proof, which is analogous to the proof of Proposition 8. For arbitrary $f, g, h \in T^*(\mathcal{NC}(J))$,

$$(f \prec g) \prec h = m_{\mathbb{K}} \circ ((f \prec g) \otimes h) \circ \Delta_\prec$$
$$= m_{\mathbb{K}^{[3]}} \circ (f \otimes g \otimes h) \circ (\Delta_\prec \otimes I) \circ \Delta_\prec,$$

where $m_{\mathbb{K}^{[3]}}$ stands for the product map from $\mathbb{K}^{\otimes 3}$ to $\mathbb{K}$. Similarly,

$$f \prec (g \sqcup\!\sqcup h) = m_{\mathbb{K}} \circ (f \otimes (g \sqcup\!\sqcup h)) \circ \Delta_\prec$$
$$= m_{\mathbb{K}^{[3]}} \circ (f \otimes g \otimes h) \circ (I \otimes \overline{\Delta}) \circ \Delta_\prec,$$

so that the identity $(f \prec g) \prec h = f \prec (g \sqcup\!\sqcup h)$ follows from $(\Delta_\prec \otimes I) \otimes \Delta_\prec = (I \otimes \overline{\Delta}) \circ \Delta_\prec$. The other identities characterizing shuffle algebras follow similarly.

As usual, we equip the shuffle algebra $T^*(\mathcal{NC}(J))$ with a unit. That is, we set $\bar{T}^*(\mathcal{NC}(J)) := T^*(\mathcal{NC}(J)) \oplus \mathbb{K}\mathbf{1} \cong Lin(\bar{T}(\mathcal{NC}(J)),\mathbb{K})$, where in the last isomorphism the unit $\mathbf{1} \in \bar{T}^*(\mathcal{NC}(J))$ is identified with the augmentation map $\varepsilon \in Lin(\bar{T}(\mathcal{NC}(J)),\mathbb{K})$, which is the projection on the scalars $\mathcal{NC}(J)^{\otimes 0} \cong \mathbb{K}$ orthogonal to $T(\mathcal{NC}(J))$. Moreover, for an arbitrary f in $T^*(\mathcal{NC}(J))$,

$$f \prec \varepsilon = f = \varepsilon \succ f, \quad \varepsilon \prec f = 0 = f \succ \varepsilon.$$

Now, let ϕ be a linear form on $\mathcal{NC}(J)$. For example, we have τ_{nc} associated with non-crossing Green's functions, and defined for the word $\boldsymbol{w} = w_1 \cdots w_n$, $w_l = j_{s_{i_l}}$ by

$$\tau_{\mathrm{nc}}(L(\boldsymbol{w})) := \mathrm{N}^{(n)}_{s_{i_1}\cdots s_{i_n}}(L).$$

This extends uniquely to a multiplicative linear form Φ on $T(\mathcal{NC}(J))$ (still written as τ_{nc} when $\phi = \tau_{\mathrm{nc}}$), by setting

$$\Phi(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) := \phi(\boldsymbol{w}_1)\cdots\phi(\boldsymbol{w}_n).$$

(It also extends to a unital and multiplicative linear form on $\bar{T}(\mathcal{NC}(J))$, by setting $\Phi(\mathbf{1}) := 1$.) Conversely, any such multiplicative map Φ gives rise to a linear form on $\mathcal{NC}(J)$ by restriction of its domain.

Definition 16. *A linear form $\Phi \in \bar{T}(\mathcal{NC}(J))$ is called a character if it is unital, i.e., $\Phi(\mathbf{1}) = 1$, and multiplicative, i.e., for all $\boldsymbol{a}, \boldsymbol{b} \in T(\mathcal{NC}(J))$, $\Phi(\boldsymbol{a}|\boldsymbol{b}) = \Phi(\boldsymbol{a})\Phi(\boldsymbol{b})$. A linear form $\kappa \in \bar{T}(\mathcal{NC}(J))$ is called an infinitesimal character if $\kappa(\mathbf{1}) = 0$ and for all $\boldsymbol{a}, \boldsymbol{b} \in T(\mathcal{NC}(J))$, $\kappa(\boldsymbol{a}|\boldsymbol{b}) = 0$.*

We write G_{NC} for the set of characters in $\bar{T}(\mathcal{NC}(J))$, and g_{NC} for the corresponding set of infinitesimal characters. Notice that, by its definition, an infinitesimal character is entirely determined by its restriction to $\mathcal{NC}(J) \subset T(\mathcal{NC}(J))$.

Theorem 17. [18] *There exists a natural bijection between G_{NC}, the set of characters, and g_{NC}, the set of infinitesimal characters, on $\bar{T}(\mathcal{NC}(J))$. More precisely, for $\Phi \in G_{NC}, \exists!\kappa \in g_{NC}$ such that $\Phi = \varepsilon + \kappa \prec \Phi$, and conversely, for $\kappa \in g_{NC}(J)$,*

$$\begin{aligned}\Phi :=& \varepsilon + \kappa + \kappa \prec \kappa + \kappa \prec (\kappa \prec \kappa) \\ &+ \kappa \prec (\kappa \prec (\kappa \prec \kappa)) + \cdots \\ =:& \exp^{\prec}(\kappa)\end{aligned}$$

is a character.

Recall that the analogous result in Theorem 13 holds for the group $G_{\mathbb{C}}$ of characters and Lie algebra $g_{\mathbb{C}}$ of infinitesimal characters with target space the complex numbers.

In view of the importance of Theorem 17 for our forthcoming developments, we include a sketch of the proof. Because we are dealing with graded structures, the implicit equation $\Phi = \varepsilon + \kappa \prec \Phi$ can be shown recursively to have a unique solution κ in $\bar{T}^*(\mathcal{NC}(J))$. Let us consider the infinitesimal character $\mu := Res(\kappa)$, and let us show that μ also solves $\Phi = \varepsilon + \mu \prec \Phi$. The first part of the theorem will follow.

Indeed, for an arbitrary $\boldsymbol{w} = \boldsymbol{w}_1|\cdots|\boldsymbol{w}_n \in T(\mathcal{NC}(J))$, we have that

$$\begin{aligned}\Phi(\boldsymbol{w}_1) &= (\varepsilon + \kappa \prec \Phi)(\boldsymbol{w}_1) = \kappa(\boldsymbol{w}_1^{1,\prec})\Phi(\boldsymbol{w}_1^{2,\prec}) \\ &= \mu(\boldsymbol{w}_1^{1,\prec})\Phi(\boldsymbol{w}_1^{2,\prec}),\end{aligned}$$

so that for any $n > 1$,

$$\begin{aligned}\Phi(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) &= \Phi(\boldsymbol{w}_1)\Phi(\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n) \\ &= \mu(w_1^{1,\prec})\Phi(\boldsymbol{w}_1^{2,\prec}|\boldsymbol{w}_2|\cdots|\boldsymbol{w}_n)) \\ &= (\varepsilon + \mu \prec \Phi)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n),\end{aligned}$$

from which the property follows. Conversely,

$$\begin{aligned}\exp^{\prec}(\kappa)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) &= (\varepsilon + \kappa \prec \exp^{\prec}(\kappa))(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) \\ &= \kappa(\boldsymbol{w}_1^{1,\prec})\exp^{\prec}(\kappa)(\boldsymbol{w}_1^{2,\prec}|\cdots|\boldsymbol{w}_n).\end{aligned}$$

Assuming (by induction on the total tensor degree of the expressions) that $\exp^{\prec}$ acts as a character on $(\boldsymbol{w}_1^{2,\prec}|\cdots|\boldsymbol{w}_n)$, we obtain:

$$\begin{aligned}&\exp^{\prec}(\kappa)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) \\ &= (\kappa(\boldsymbol{w}_1^{1,\prec})\exp^{\prec}(\kappa)(\boldsymbol{w}_1^{2,\prec}))\exp^{\prec}(\kappa)(\boldsymbol{w}_2)\cdots\exp^{\prec}(\kappa)(\boldsymbol{w}_n) \\ &= \exp^{\prec}(\kappa)(\boldsymbol{w}_1)\exp^{\prec}(\kappa)(\boldsymbol{w}_2)\cdots\exp^{\prec}(\kappa)(\boldsymbol{w}_n).\end{aligned}$$

3.4 From full to non-crossing Green's functions

We now turn to the relationship between full and connected planar Green's functions viewed through the prism of non-crossing partitions.

Definition 18. *The splitting map Sp is the map from $T(J)$ to $\mathcal{NC}(J)$ defined on $\boldsymbol{w} = w_1 \cdots w_n \in T(J)$ as follows:*

$$Sp(w_1 \cdots w_n) := \sum_{L \in NC_n} L(w_1 \cdots w_n).$$

This is extended multiplicatively to a unital map Sp from $\bar{T}(T(J))$ to $\bar{T}(\mathcal{NC}(J))$, i.e., for $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$,

$$Sp(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_k) := Sp(\boldsymbol{w}_1)|\cdots|Sp(\boldsymbol{w}_k).$$

The name "splitting map" is chosen because on dual spaces it permits the "splitting" of the value of a linear form ϕ on $T(J)$ (typically the full Green's function) into a sum of terms indexed by non-crossing partitions (typically, the non-crossing Green's functions).

Theorem 19. [18] *The map Sp from $\bar{T}(T(J))$ to $\bar{T}(\mathcal{NC}(J))$ is an unshuffle bialgebra morphism.*

The previous constructions are dual to each other. That is, the linear dual of an unshuffle coalgebra is a shuffle algebra, and a morphism f between two unshuffle coalgebras induces a morphism of shuffle algebras, written f^*, between the linear duals. In particular, we have the following.

Lemma 20. *The map Sp from $T(T(J))$ to $T(\mathcal{NC}(J))$ induces a morphism, Sp^*, of shuffle algebras with units, from $\bar{T}^*(\mathcal{NC}(J))$ to $\bar{T}^*(T(J))$.*

Lemma 21. *The map Sp^* restricts to maps from G_{NC} to $G_{\mathbb{K}}$ and from g_{NC} to $g_{\mathbb{K}}$.*

Indeed, the map Sp from $T(T(J))$ to $T(\mathcal{NC}(J))$ is induced multiplicatively by a map from $T(J)$ to $\mathcal{NC}(J)$. With the same notation as at the beginning of this section, we find that $Sp(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) = Sp(\boldsymbol{w}_1)|\cdots|Sp(\boldsymbol{w}_n)$. Therefore, for $\Phi \in G_{NC}$ we have that

$$\begin{aligned}Sp^*(\Phi)(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) &= \Phi \circ Sp(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_n) \\ &= \Phi(Sp(\boldsymbol{w}_1)|\cdots|Sp(\boldsymbol{w}_n)) \\ &= \Phi \circ Sp(\boldsymbol{w}_1)\cdots\Phi \circ Sp(\boldsymbol{w}_n) \\ &= Sp^*(\Phi)(\boldsymbol{w}_1)\cdots Sp^*(\Phi)(\boldsymbol{w}_n),\end{aligned}$$

and $Sp^*(\Phi) \in G_{\mathbb{K}}$. A similar argument holds for infinitesimal characters.

Notice that elements in G_{NC} and g_{NC} are entirely characterized by their restrictions to $\mathcal{NC}(J)$. Similarly, elements in $G_{\mathbb{K}}$ and $g_{\mathbb{K}}$ are characterized by their restrictions to $T(J)$. It follows that any section σ of the map Sp^* from $Lin(\mathcal{NC}(J),\mathbb{K})$ to $Lin(T(J),\mathbb{K})$ induces a section to the map Sp^* from G_{NC} (resp. g_{NC}) to $G_{\mathbb{K}}$ (resp. $g_{\mathbb{K}}$). The existence of such sections, and the surjectivity of Sp^*, follow by direct inspection.

Theorem 22. *The bijections of Theorems 10 and 17 commute with Sp^*, in the sense that given Φ and κ,*

$$Sp^*(\Phi) = \varepsilon + Sp^*(\kappa) \prec Sp^*(\Phi) = \exp^{\prec}(Sp^*(\kappa)).$$

The theorem follows from Lemma 20 and Lemma 21.

In view of Theorems 22, 17, and 10, and of the previous results in this section, one can use any section σ of the map Sp^* from $Lin(\mathcal{NC}(J),\mathbb{K})$ to $Lin(T(J),\mathbb{K})$ to lift the equation $\tau_{\mathrm{Z}} = \varepsilon + \tau_{\mathrm{W}} \prec \tau_{\mathrm{Z}}$ relating full and connected Green's functions to non-crossing partitions. That is, to $Lin(\mathcal{NC}(J),\mathbb{K})$.

For that reason, we introduce a so-called "standard section". In free probability, this leads to a new presentation of the classical Möbius-inversion type relations between free moments and free cumulants [18]. Other choices of sections are theoretically possible, which would lead to different lifts for non-crossing partitions of the relations between full and connected planar Green's functions. Whether or not such sections may yield interesting result, for example from an algebraic or combinatorial point of view, is yet to be investigated. However, for the subject of the present article, planar QFT, the choice of a section is governed by physics, and seems to be essentially unique from this point of view. Introducing sections than the "standard section" that follows would probably be pointless.

Definition 23. *Let κ be a unital map from $T(J)$ to $\mathbb{K}$. We call the linear form $sd(\kappa)$ on $\mathcal{NC}(J)$ the standard section of κ, which for $\boldsymbol{w} = w_1 \cdots w_n$ is defined by*

$$sd(\kappa)(L(\boldsymbol{w})) := \kappa(w_1 \cdots w_n)$$

if L is the trivial non-crossing partition ($L = [n]$), and zero otherwise.

For an arbitrary non-crossing partition $L = \{L_1, \ldots, L_k\}$ of $[n]$, we write $\kappa^L(w_1 \cdots w_n) := \prod_{i=1}^k \kappa(\boldsymbol{w}_{L_i})$. We write ϕ for the solution to $\phi = \varepsilon + \kappa \prec \phi$ with the notation of Theorem 10.

Proposition 24. *The solution Ψ of the equation $\Psi = \varepsilon + sd(\kappa) \prec \Psi$ in $Lin(\bar{T}(\mathcal{NC}(J)),\mathbb{K})$ satisfies, for an arbitrary non-crossing partition $L \in NC_n$,*

$$\Psi(L(w_1 \cdots w_n)) = \kappa^L(w_1 \cdots w_n). \tag{55}$$

Therefore, we obtain the following identity:

$$\begin{aligned}\Psi(Sp(w_1 \cdots w_n)) &= \sum_{L \in NC_n} \kappa^L(w_1 \cdots w_n) \\ &= \phi(w_1 \cdots w_n).\end{aligned} \tag{56}$$

The identity $\Psi(Sp(w_1 \cdots w_n)) = \phi(w_1 \cdots w_n)$ follows from Theorem 22, because

$$\begin{aligned}\Psi \circ Sp = Sp^*(\Psi) &= \varepsilon + Sp^* \circ sd(\kappa) \prec Sp^*(\Psi) \\ &= \varepsilon + \kappa \prec Sp^*(\Psi),\end{aligned}$$

from which we obtain (by unicity of the solution) that $Sp^*(\Psi) = \phi$.

The first identity, from which $\Psi(Sp(w_1 \cdots w_n)) = \sum_{L \in NC_n} \kappa^L(w_1 \cdots w_n)$ is deduced, follows by induction on $[n]$. Let us assume that the identity (55) holds for non-crossing partitions of $[p]$ with $p < n$. Then, we obtain:

$$\begin{aligned}&\Psi(L(w_1 \cdots w_n)) \\ &= \sum_{Q \coprod_{\mathrm{adm}} P = L} sd(\kappa)\big(st(Q)(\boldsymbol{w}_Q)\big) \\ &\quad \Psi\big(st(J_1^{L,Q})(\boldsymbol{w}_{J_1^{L,Q}}) | \cdots | st(J_{k(L,Q)}^{L,Q})(\boldsymbol{w}_{J_{k(L,Q)}^{L,Q}})\big).\end{aligned}$$

However, because $sd(\kappa)$ vanishes on all non-crossing partitions except for trivial ones, terms on the right hand-side vanish except when Q is the component of L containing 1, and the expression finally reduces to

$$\begin{aligned}&\Psi(L(w_1 \cdots w_n)) \\ &= sd(\kappa)(st(L_1)(\boldsymbol{w}_{L_1})\Psi(st(L_2)(\boldsymbol{w}_{L_2})| \cdots |st(L_k)(\boldsymbol{w}_{L_k})) \\ &= \kappa(\boldsymbol{w}_{L_1})\Psi(st(L_2)(\boldsymbol{w}_{L_2})| \cdots |st(L_k)(\boldsymbol{w}_{L_k})),\end{aligned}$$

where $L = \{L_1, \ldots, L_k\}$. From the induction hypothesis and the multiplicativity of Ψ, we obtain the expected identity:

$$\Psi(L(w_1 \cdots w_n)) = \prod_{i=1}^k \kappa(\boldsymbol{w}_{L_i}).$$

Corollary 25. *Let κ be the infinitesimal character on $T(J)$, defined by the connected Green's function:*

$$\kappa(w_1 \cdots w_n) := \mathrm{W}^{(n)}_{s_{i_1} \cdots s_{i_n}}.$$

We have $sd(\kappa)(L(w_1 \cdots w_n)) = \mathrm{W}^{(n)}_{s_{i_1} \cdots s_{i_n}}$ if $L = [n]$, and it is equal to zero otherwise. Moreover, the solution to $\Psi = \varepsilon + sd(\kappa) \prec \Psi$ is the non-crossing Green's function; that is,

$$\Psi(L(w_1 \cdots w_n)) = \mathrm{N}^{(n)}_{s_{i_1} \cdots s_{i_n}}(L).$$

In addition, we recover from the properties of the splitting map Sp the relation linking full and non-crossing Green's functions:

$$\mathrm{Z}^{(n)}_{s_{i_1} \cdots s_{i_n}} = \Psi(Sp(w_1 \cdots w_n)) = \sum_{L \in NC_n} \mathrm{N}^{(n)}_{i_1 \cdots i_n}(L).$$

3.5 From full to connected (classical case)

Recall that in the classical setting, the relation between full and connected Green's functions in the functional framework is given by

$$Z(j) = \exp\big(W(j)\big). \tag{57}$$

By functional derivation, we obtain that

$$\frac{\partial}{\partial j_{k_1}} Z(j) = (\frac{\partial}{\partial j_{k_1}} W(j)) Z(j).$$

Because the underlying algebra is commutative, the usual Leibniz rule of differential calculus applies, and we obtain that

$$\begin{aligned} Z^{(n)}_{i_1\cdots i_n} &= \frac{\partial^n}{\partial_{j_{i_n}}\cdots\partial_{j_{i_1}}}\bigg|_{j=0} \exp\big(W(j)\big) \\ &= \frac{\partial^{n-1}}{\partial_{j_{i_n}}\cdots\partial_{j_{i_2}}}\Big((\partial_{j_{i_1}} W(j))Z(j)\Big)\bigg|_{j=0} \\ &= \sum_{S=\{1,s_2,\ldots,s_k\}\subset[n]} W^{(k)}_{i_1 i_{s_2}\cdots i_{s_k}} \\ &\quad Z^{(n-k)}_{i_2\cdots i_{s_2-1} i_{s_2+1}\cdots i_{s_k-1} i_{s_k+1}\cdots i_n} \\ &= \sum_{S=\{1=s_1,\ldots,s_k\}\subset[n]} W^{(k)}_{i_{s_1}\cdots i_{s_k}} Z^{(n-k)}_{i_{l_1}\cdots i_{l_{n-k}}}, \end{aligned}$$

where $L = \{l_1, \ldots, l_{n-k}\}$ and $L \coprod S = [n]$.

We will now show that the relation between classical full and connected Green's functions is naturally described in the context of the cocommutative unshuffle bialgebra $\bar{T}(J)$ with coproduct (14). Indeed, the cocommutativity of the coproduct $\Delta^{ш}_{\succ} = \tau \circ \Delta^{ш}_{\prec}$ implies that for arbitrary $\alpha, \beta \in T^*(J)$, we have that

$$\alpha \prec \beta = \beta \succ \alpha.$$

Thus, $\bar{T}^*(J) = Lin(\bar{T}(J), \mathbb{K})$ is a unital commutative shuffle algebra for the left and right half-shuffle products $\prec$ and $\succ$ defined in (12) and (13), respectively. Now, let $\phi : \bar{T}(J) \to \mathbb{K}$ be a unital map in $\bar{T}^*(J)$, and consider the linear fixed point equation

$$\phi = \iota + \tau \prec \phi. \tag{58}$$

Here, the map $\iota : \bar{T}(J) \to \mathbb{K}$ is the projection onto $\mathbb{K}\mathbf{1}$ orthogonally to $T(J)$. Let $w = j_i \in J \hookrightarrow T(J)$ be a letter. Then, $\Delta^{ш}_{\prec}(j_i) = j_i \otimes \mathbf{1}$. With $\phi(\mathbf{1}) = 1$, we find that

$$\phi(j_{k_1}) = \tau(j_{k_1}).$$

Next, we examine the word $w = w_1 w_2 \in T_2(J)$. Recall that we have the left half-unshuffle $\Delta^{ш}_{\prec}(w_1 w_2) = w_1 w_2 \otimes \mathbf{1} + w_1 \otimes w_2$, such that

$$\phi(w_1 w_2) = \tau(w_1 w_2) + \tau(w_1)\tau(w_2).$$

For the order three word $w = w_1 w_2 w_3 \in T_3(J)$, the left half-unshuffle yields that

$$\begin{aligned} \phi(w_1 w_2 w_3) = \tau(w_1 w_2 w_3) &+ \tau(w_1)\tau(w_2 w_3) \\ &+\tau(w_1 w_3)\tau(w_2) + \tau(w_1 w_2)\tau(w_3) \\ &+\tau(w_1)\tau(w_2)\tau(w_3). \end{aligned}$$

In addition, for the order four word $w = w_1 w_2 w_3 w_4 \in T_4(J)$, the left half-unshuffle gives that

$$\begin{aligned} &\phi(w_1 w_2 w_3 w_4) \\ &= \tau(w_1 w_2 w_3 w_4) + \tau(w_1 w_2)\phi(w_3 w_4) \\ &\quad + \tau(w_1 w_3)\phi(w_2 w_4) + \tau(w_1 w_4)\phi(w_2 w_3) \\ &\quad + \tau(w_1)\phi(w_2 w_3 w_4) + \tau(w_1 w_2 w_3)\phi(w_4) \\ &\quad + \tau(w_1 w_2 w_4)\phi(w_3) + \tau(w_1 w_3 w_4)\phi(w_2) \\ &= \tau(w_1 w_2 w_3 w_4) + \tau(w_1 w_2)\tau(w_3 w_4) \\ &\quad + \tau(w_1 w_4)\tau(w_2 w_3) + \tau(w_1 w_3)\tau(w_2 w_4) \\ &\quad + \tau(w_1 w_3)\tau(w_2)\tau(w_4) + \tau(w_1 w_2)\tau(w_3)\tau(w_4) \\ &\quad + \tau(w_1 w_4)\tau(w_2)\tau(w_3) + \tau(w_3 w_4)\tau(w_1)\tau(w_2) \\ &\quad + \tau(w_2 w_3)\tau(w_1)\tau(w_4) + \tau(w_2 w_4)\tau(w_1)\tau(w_3) \\ &\quad + \tau(w_1)\tau(w_2)\tau(w_3)\tau(w_4) + \tau(w_1)\tau(w_2 w_3 w_4) \\ &\quad + \tau(w_1 w_2 w_3)\tau(w_4) + \tau(w_1 w_2 w_4)\tau(w_3) \\ &\quad + \tau(w_1 w_3 w_4)\tau(w_2). \end{aligned}$$

Next, observe that by defining the linear map $\tau(w) := \tau_W(w) := W^{(|w|)}_{i_1\cdots i_n}$ for $\boldsymbol{w} = w_1 \cdots w_n \in T_n(J)$ and $w_k = j_{i_k} \in J$, with $\tau(\mathbf{1}) := 0$, we obtain that the above relations are the same as the relations (2) between the classical full and connected Green's functions up to order four, i.e., $\phi(w) = \tau_Z(w) := Z^{(|w|)}_{i_1\cdots i_n}$.

Hence, analogously to the planar case, we obtain that the relation between classical full and connected Green's functions τ_Z and τ_W, respectively, is encoded by the linear fixed point equation

$$\tau_Z = \iota + \tau_W \prec \tau_Z, \tag{59}$$

in $\bar{T}^*(J)$. Indeed, by the definition of the left half-shuffle product $\prec$, this equation gives that for any word $\boldsymbol{w} = w_1 \cdots w_n \in T_n(J)$,

$$\begin{aligned} Z^{(n)}_{i_1\ldots i_n} &= \tau_Z(\boldsymbol{w}) = (\tau_W \prec \tau_Z)(\boldsymbol{w}) \\ &= \sum_{S=\{1=s_1,\ldots,s_k\}\subset[n]} \tau_W(\boldsymbol{w}_S)\tau_Z(\boldsymbol{w}_{[n]-S}) \\ &= \sum_{S=\{1=s_1,\ldots,s_k\}\subset[n]} W^{(k)}_{i_{s_1} i_{s_2}\cdots i_{s_k}} \\ &\quad Z^{(n-k)}_{i_2\cdots i_{s_2-1} i_{s_2+1}\cdots i_{s_k-1} i_{s_k+1}\cdots i_n}. \end{aligned} \tag{60}$$

The solution to this fixed point equation is given by exponentials.

Theorem 26. *Expanding the linear fixed point equation* (59) *yields that*

$$\tau_Z = \exp^{\prec}\left(\tau_W\right). \tag{61}$$

A closed solution is given in terms of the exponential map, defined with respect to the commutative shuffle product $ш : \bar{T}^*(J) \otimes \bar{T}^*(J) \to \bar{T}^*(J)$*:*

$$\tau_Z = \exp^{ш}\left(\tau_W\right), \tag{62}$$

which is the convolution product in $\bar{T}^*(J)$*, defined in terms of the left and right half-shuffles defined in* (12) *and* (13) *respectively, i.e.,* $ш = \prec + \succ$.

Proof. The first statement is obvious, and follows from comparing the expansions of (61) with the iteration of (59). The second statement follows from the fact that the unshuffle coproduct (8) is cocommutative. In the resulting commutative shuffle algebra, Theorem 11 gives that $\Omega'(\tau_W) = \tau_W$. It is easy to verify that the shuffle algebra identity $a^{ш n} = n!\, a^{\prec n}$ holds.

Note the following interesting identity for exponentials:

$$Z^{(n)}_{i_1\cdots i_n} = \frac{\partial^n}{\partial_{j_{i_n}}\cdots\partial_{j_{i_1}}}\bigg|_{j=0} \exp\left(W(j)\right) = \exp^{ш}(\tau_W)(\boldsymbol{w}),$$

for $\boldsymbol{w} = w_1\cdots w_n$. This follows immediately from the Leibniz rule.

3.6 Partition Green's function (classical case)

We have seen that the relation between full and connected Green's functions can be refined in the planar case by introducing non-crossing Green's functions. The corresponding relations can be refined in a similar manner in the classical case. The approach we develop here appears to be new. We will sketch the arguments, which follow along the same lines as for the planar case.

A Feynman graph Γ in a given QFT splits into $n \geq k \geq 1$ connected components. Grouping external legs according to when they belong to a common connected component defines a partition $L = \{L_1, \ldots, L_k\}$ of $[n]$. From now on, we assume that L_i are ordered according to their minimal elements, so that $1 \in L_1$. Summing over all partitions of $[n]$ (the set of partitions of $[n]$ is written P_n), we get that

$$Z^{(n)}_{i_1\cdots i_n} = \sum_{L\in P_n}\prod_{i=1}^{k} W^{|L_i|}_{l^i_1\cdots l^i_{|L_i|}},$$

where $L_i = \{l^i_1, \ldots, l^i_{|L_i|}\}$. We abbreviate $W^{|L_i|}_{l^i_1\cdots l^i_{|L_i|}}$ to W^{L_i} from now on, and define the partitioned Green's functions as

$$P^{(n)}_{s_1\cdots s_n}(L) := \prod_{i=1}^{k} W^{L_i},$$

so that

$$Z^{(n)}_{s_1\cdots s_n} = \sum_{L\in P_n} P^{(n)}_{s_1\cdots s_n}(L).$$

Notice that we deliberately employ a similar notation as for the planar case, because it will always be clear from the context whether we are working in the setting of a planar or non-planar theory.

The corresponding bialgebra of partitions is defined as follows. We write $\mathcal{P}$ (resp. $\bar{\mathcal{P}}$) for the linear span of the P_n, for $n \in \mathbb{N}^+$ (resp. in $\mathbb{N}$). The product is the shifted concatenation of partitions. For a partition $[n]$ of $L = \{L_1, \ldots, L_k\}$ and a partition $[m]$ of $K = \{K_1, \ldots, K_l\}$, $L\cdot K$ is the partition $\{L_1, \ldots, L_k, K_1 + n, \ldots, K_l + n\}$ of $[n+m]$, where for a subset $S = \{s_1, \ldots, s_p\}$ of the integers, $S + n$ stands for $\{s_1 + n, \ldots, s_p + n\}$. The coproduct acting on partitions is simply the (standardized) unshuffling coproduct:

$$\Delta(L) = \Delta(\{L_1, \ldots, L_k\}) = \sum_{I\coprod J=[n]} st(L_I)\otimes st(L_J),$$

where $L_I = \{L_{i_1}, \ldots, L_{i_k}\}$ for $I = \{i_1, \ldots, i_k\}$. This coproduct is a map of algebras from $\mathcal{P}$ to $\mathcal{P}\otimes\mathcal{P}$, and is coassociative:

$$\begin{aligned}(\Delta\otimes\mathrm{id})\otimes\Delta(L) &= \sum_{I\coprod J\coprod K=[n]} st(L_I)\otimes st(L_J)\otimes st(L_K)\\ &= (\mathrm{id}\otimes\Delta)\circ\Delta(L).\end{aligned}$$

It splits naturally into two half-coproducts, as

$$\Delta_{\prec}(L) := \sum_{\substack{I\coprod J=[n]\\ 1\in I}} st(L_I)\otimes st(L_J),$$

and $\Delta_{\succ} := \Delta - \Delta_{\prec}$.

Decorated partitions are defined exactly as in the case of non-crossing partitions, and the previously given constructions also hold in the presence of decorations. Finally, we obtain the following theorem.

Theorem 27. *The algebra* $\mathcal{P}$ *and its decorated version* $\mathcal{P}(J)$ *equipped with the concatenation product and the coproduct* $\Delta = \Delta_{\prec} + \Delta_{\succ}$ *are unshuffle bialgebras.*

Using *mutatis mutandis* the same notations as for the planar case, the splitting map Sp is now defined as a map from $T(J)$ to $\mathcal{P}(J)$ as

$$Sp(w_1\cdots w_n) := \sum_{L\in P_n} L(w_1\cdots w_n),$$

and we obtain the following proposition (the proof of which is left to the reader).

Proposition 28. *The splitting map Sp is a map of unshuffle bialgebras from* $\bar{T}(J)$ *to* $\bar{\mathcal{P}}(J)$.

The standard section of a unital map κ from $T(J)$ to the scalars is defined for a word $\boldsymbol{w} = w_1 \cdots w_n \in T(J)$, by

$$st(\kappa)(L(w_1 \cdots w_n)) := \kappa(w_1 \cdots w_n)$$

if L is the trivial partition ($L = [n]$), and zero otherwise. For an arbitrary partition $L = \{L_1, \dots, L_k\}$ of $[n]$, we write

$$\kappa^L(w_1 \cdots w_n) := \prod_{i=1}^{k} \kappa(\boldsymbol{w}_{L_i}).$$

Finally, we obtain the following result.

Proposition 29. *Let* κ *be the linear form on* $T(J)$, *defined by the connected Green's function*

$$\kappa(w_1 \cdots w_n) := W^{(n)}_{i_1 \cdots i_n}.$$

Then, we have that $sd(\kappa)(L(w_1 \cdots w_n)) = W^{(n)}_{i_1 \cdots i_n}$ *if* $L = [n]$, *and it is zero otherwise. Moreover, the solution to the linear fixed point equation*

$$\Psi = \iota + sd(\kappa) \prec \Psi$$

gives the partitioned Green's function. That is,

$$\Psi(L(w_1 \cdots w_n)) = P^{(n)}_{i_1 \cdots i_n}.$$

From the properties of the splitting map Sp, we recover the relation linking full and partitioned Green's functions:

$$Z^{(n)}_{i_1 \cdots i_n} = \Psi(Sp(w_1 \cdots w_n)) = \sum_{L \in P_n} P^{(n)}_{i_1 \cdots i_n}.$$

To simplify notation, we have assumed that for the word $\boldsymbol{w} = w_1 \cdots w_n \in T(J)$, the letters $w_k = j_{i_k} \in J$. Recall that $Z^{(l)}_{i_1 \cdots i_l}$ are actually functions, whose variables i_k run implicitly over the set of all possible physical parameters characterizing the particles (or fields) of the theory.

4 Conclusion

It has recently been shown [17, 18] that Hopf algebra and half-shuffle techniques can be used to shed new light on the relationship between moments and cumulants in free probability. In the present article, the same approach has been applied to field theories in the planar setting. Generating series for the full and connected Green's functions are related through a linear fixed point equation, with respect to a suitably defined half-shuffle product. This approach allows the notion of non-crossing Green's functions to be introduced naturally. This also facilitates an understanding of the combinatorics of theories in the planar sector, from a group theoretical point of view. That is, full Green's functions are now viewed as the components of a character on an appropriate Hopf algebra. The same approach can be developed for classical field theories, leading to similar results. Finally, we remark that one-particle irreducible (1PI) Green's functions can be analyzed using similar techniques. These will be the subject of forthcoming work.

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