Front. Phys. 11(5), 114204 (2016)
DOI 10.1007/s11467-016-0569-2

Research article

Multi-peak solitons in PT-symmetric Bessel optical lattices with defects

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Received August 18, 2015; accepted January 26, 2016

This paper presents a theoretical analysis of the existence and stability of multi-peak solitons in parity–time-symmetric Bessel optical lattices with defects in nonlinear media. The results demonstrate that there always exists a critical propagation constant μ_c for the existence of multi-peak solitons regardless of whether the nonlinearity is self-focusing or self-defocusing. In self-focusing media, multi-peak solitons exist when the propagation constant $\mu > \mu_c$. In the self-defocusing case, solitons exist only when $\mu < \mu_c$. Only low-power solitons can propagate stably when random noise perturbations are present. Positive defects help stabilize the propagation of multi-peak solitons when the nonlinearity is self-focusing. When the nonlinearity is self-defocusing, however, multi-peak solitons in negative defects have wider stable regions than those in positive defects.

Keywords lattice solitons, multi-peak solitons, PT-symmetry, defect solitons

PACS numbers 42.65.Tg, 42.65.Jx, 42.65.Wi

1 Introduction

Since the successful formulation of optical systems with parity–time (PT)-symmetric potentials [1], numerous studies have focused on light propagation in PT-symmetric systems [2, 3]. In optical systems with PT-symmetry, the gain and loss are formulated carefully so that the potential satisfies $V(\boldsymbol{x}) = V^*(-x)$, where $\boldsymbol{x}$ is the spatial coordinate and $*$ denotes complex conjugation. That is, the refractive-index profile of the medium is even, and the gain–loss profile is odd. Despite the presence of gain and loss, this kind of wave system can admit completely real linear spectrum.

In optics, there is growing interest in PT symmetry because it is believed that optics may be a testing ground for PT-symmetric systems, and such PT optical complex potentials have been realized both theoretically and experimentally [4–8]. The possibility of using PT-symmetric potentials to observe soliton phenomena obeying the nonlinear Schrödinger equation (NLSE) is of great interest due to its potential applications in all-optical switches and optical signal routing [9, 10]. PT-symmetric potentials were first introduced into the NLSE by Musslimani *et al.* in 2008 [4], who found that solitons can exhibit stability over a wide range of potential parameters in PT-symmetric potentials and that the transverse power will flow from gain to loss regions. A number of subsequent theoretical studies have focused on the existence and stability of nonlinear localized modes and soliton dynamics supported by different nonlinearities with PT-symmetric potentials, such as Kerr nonlinearities [11–13], competing cubic–quintic media [14], saturable nonlinearities [15, 16], nonlocal nonlinearities [17, 18], and so on. So far, fundamental [19–22], out-of-phase dipole [19] and in-phase multi-hump [23], gray [24], discrete [25, 26], multi-peak [27, 28], and vector [29] solitons have been reported in different kinds of PT-symmetric potentials, such as PT-symmetric periodic potentials [4, 11–13], Gaussian potentials [19], parabolic potentials, and Scarff potentials [4], among others. PT-symmetric Bessel potentials have also proven able to support fundamental and multi-peak solitons in Kerr nonlinearities [30]. Very recently, we also studied the existence and stability of fundamental solitons and dipole solitons when defects were introduced into Bessel potentials [31]. It was found that all of the fundamental solitons could propagate stably, while only low-power dipole solitons were stable. In this paper, we discuss the existence and stability of multi-peak solitons in self-focusing and self-defocusing nonlinearities in PT-symmetric Bessel potentials with defects and describe the impacts of the lattice depth and defects on the formation and stability of multi-peak solitons.

2 Theoretical model

In Kerr nonlinear media with PT-symmetric optical lattices, the evolution of an optical field can be described by the following dimensionless NLSE:

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial x^2}+[v(x)+\mathrm{i}w(x)]u+\sigma|u|^2u=0, \tag{1}$$

where u is the complex optical field, $\sigma=\pm1$ is the sign of the nonlinearity, and x and z are the dimensionless transverse and longitudinal coordinates, respectively. $V(x)$ and $W(x)$ are the real and imaginary parts of the PT-symmetric potential, respectively. In a PT-symmetric Bessel optical lattice with defects, the real and imaginary parts of the complex potential can be respectively expressed as

$$v(x)=V_0J_0(x)[1+\varepsilon\exp(-x^8/128)] \tag{2}$$

and

$$w(x)=W_0J_1(x), \tag{3}$$

where V_0 and W_0 represent the depths of the real and imaginary parts of the PT potential, respectively, and ε is the defect intensity modulation parameter. To facilitate quantitative investigation of defect solitons and their stability, we employed $V_0=20$ and $W_0=4$. In addition, we set ε equal to +0.5 or –0.5 to generate positive and negative defects, respectively. The profiles of PT-symmetric Bessel lattices with these specific positive and negative defects are displayed in Fig. 1.

We searched for stationary solutions to Eq. (1) of the form $u(x)=f(x)\exp(\mathrm{i}bz)$, where $f(x)$ is a complex function and μ is the corresponding real propagation constant. By substituting this form into Eq. (1), it becomes evident that $f(x)$ should satisfy the following equation:

$$-bf+\frac{\partial^2 f}{\partial x^2}+[v(x)+\mathrm{i}w(x)]f+\sigma|f^2|f=0. \tag{4}$$

The solutions for the defect solitons can be obtained numerically from Eq. (4) by using a modified squared-

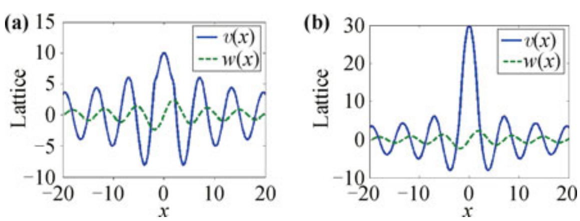

Fig. 1 Profiles of PT-symmetric Bessel lattices with (a) $\varepsilon=-0.5$ and (b) $\varepsilon=+0.5$.

operator method. To elucidate the stability of the defect solitons, we searched for a perturbed solution to Eq. (1) with the form $u(x,z)=\exp(\mathrm{i}\mu z)\{f(x)+g(x)\exp(\delta z)+t^*(x)\exp(\delta^* z)\}$, where $|g(x)|$ and $|t(x)|$ are both much less than $|f(x)|$, and δ is the instability growth rate. By substituting this form into Eq. (1) and linearizing, the following eigenvalue problem can be obtained:

$$\mathrm{i}\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix} g \\ t\end{pmatrix}=\delta\begin{pmatrix} g \\ t\end{pmatrix}, \tag{5}$$

where $L_{11}=\mu+\partial_{xx}+v(x)+\mathrm{i}w(x)+2\sigma|f|^2$, $L_{12}=\sigma f^2$, $L_{21}=-\sigma(f^2)^*$, and $L_{22}=-\lfloor\mu+\partial_{xx}+v(x)-\mathrm{i}w(x)+2\sigma|f|^2\rfloor$. This eigenvalue problem can be solved by using the Newton–conjugate gradient method (for individual discrete eigenvalues). If $\mathrm{Re}(\delta)>0$, small perturbations will undergo exponential growth, and the solitons will be linearly unstable. Otherwise, they will be linearly stable.

3 Multi-peak solitons in self-focusing media

In this section, we discuss the existence and stability of multi-peak solitons in a PT-symmetric Bessel optical lattice with defects when the nonlinearity is of the self-focusing type. Figure 2 shows the soliton power $P=\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x$ versus the propagation constant μ with different numbers and types of defects. Similar to our previous results for fundamental and dipole solitons, this figure reveals that there are critical constants μ_c for the existence of three-, four-, and five-peak solitons. In other words, the multi-peak solitons can only exist in self-focusing nonlinear media when $\mu>\mu_c$. For defect solitons with the same number of peaks, μ_c is higher when the defects are positive than when they are negative, and the solitons in positive defects have higher μ for the same P. The lattice depth also significantly influences the properties that enable multiple solitons to exist when defects are present. Compared to the results reported in Ref. [30], a higher lattice depth was found to increase μ_c in this study. Because $\mu_c<0$ when $P=0$, the five-peak solitons in negative defects always have $P\neq0$ when $\mu>0$. Except for the five-peak solitons in negative defects, all of the other multi-peak solitons can propagate stably in self-focusing media at low powers. Furthermore, the multi-peak solitons in positive defects have wider stability regions than those in negative defects; this observation has a straightforward explanation. In self-focusing nonlinearity, a positive defect can be viewed as a high-index waveguide that can even guide linear modes without nonlinearity, while a negative defect acts as a low-index waveguide and requires light with more power to compensate for the index difference

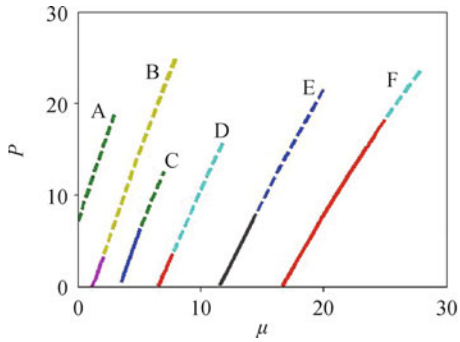

Fig. 2 P versus μ fordefect solitons in self-focusing media. (A) Five-, (B) four-, (C) three-peak solitons in negative defects, and (D) five-, (E) four-, and (F) three-peak solitons in positive defects.

and to enable the light to be guided. Therefore, the central peaks of optical solitons in negative defects have intensities greater than those of the side humps, while in positive defects, the central peaks have intensities less than those of the side humps. Furthermore, the solitons with relatively high powers in positive defects may have sufficient strength to overcome the modulation instability in self-focusing nonlinearity. To confirm the results of the linearly stability analysis, we added 1% white noise to the exact soliton solutions and simulated the propagation behaviors of the multi-peak solitons by using Eq. (4). In other words, the input condition was $u(x,0) = f(x) + \kappa\eta(x)$, where $\eta(x)$ is a random function with a value between 0 and 1 and κ is a perturbation constant, which was set to 0.01 in the simulation.

As shown in Fig. 2, the region in which three-peak solitons are stable in self-focusing media with negative defects is $3.45 < \mu < 5.11$. To facilitate demonstration, we took $\mu = 5.0$ and $\mu = 7.0$ as stable and unstable examples, respectively, of three-peak solitons in self-focusing media with negative defects. The real parts, imaginary parts, and intensity profiles are shown in Figs. 3(b) and (e) for $\mu = 5.0$ and $\mu = 7.0$, respectively. From these profiles, it is evident that the intensity of the central hump is higher than those of the side humps and that the adjacent two humps are out of phase. As shown in Fig. 3(a), the instability growth rate is zero when $\mu = 5.0$; therefore, the soliton can propagate stably in the self-focusing medium with negative defects [Fig. 3(c)]. For the unstable case with $\mu = 7.0$, however, there exist two pairs of quadruple complex eigenvalues in the instability growth spectrum, as shown in Fig. 3(d). Due to the large instability growth rate, the white noise undergoes strong modulation instability, and the intensity increases exponentially.

If the sign of the defect parameter is changed to positive, i.e., $\varepsilon = 0.5$, the phase between the adjacent humps remains π, but the intensity of the central hump becomes less than those of the side humps [Figs. 4(b) and (e)]. Three-peak solitons can propagate stably in self-focusing media with positive defects when $16.68 < \mu < 25.05$, while they are unstable in the high-power region ($\mu > 25.05$). We took $\mu = 21.0$ as an example of a stable soliton. The eigenvalues are purely imaginary, so the soliton can propagate stably. When $\mu = 26.0$, the soliton solution has a quadruple complex eigenvalue [Fig. 4(d)] and exhibits unstable propagation [Fig. 4(f)].

Next, we discuss the stability of four-peak solitons in Bessel lattices with defects under self-focusing nonlinearity. When the defect parameter is negative ($\varepsilon = -0.5$), the region in which the four-peak solitons are stable is $1.318 < \mu < 2.25$. Figures 5(a), (b), and (c) show the instability growth spectrum, soliton profiles, and evolution, respectively, for a stable four-peak soliton when $\mu = 2.0$.

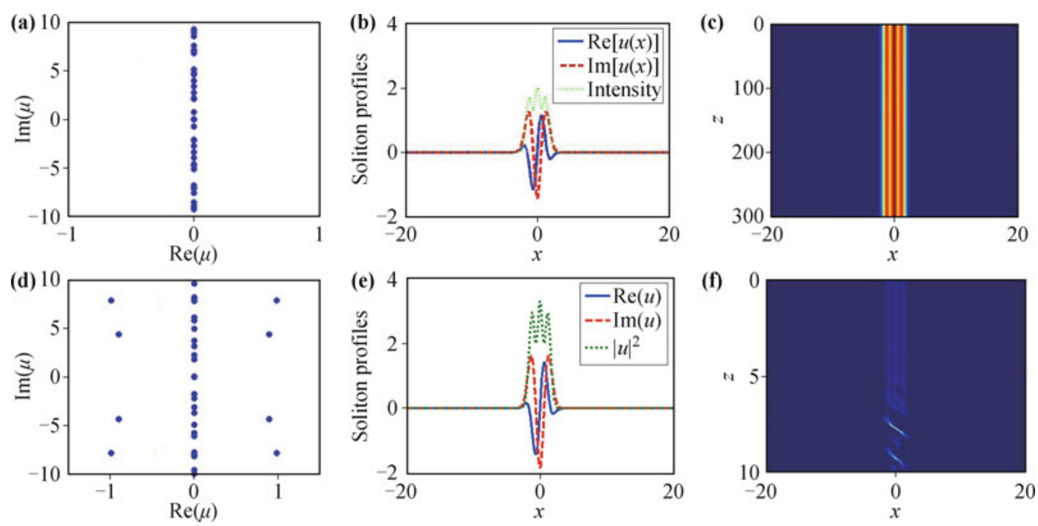

Fig. 3 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of three-peak soliton in self-focusing medium with negative defects and $\mu = 5.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 7.0$.

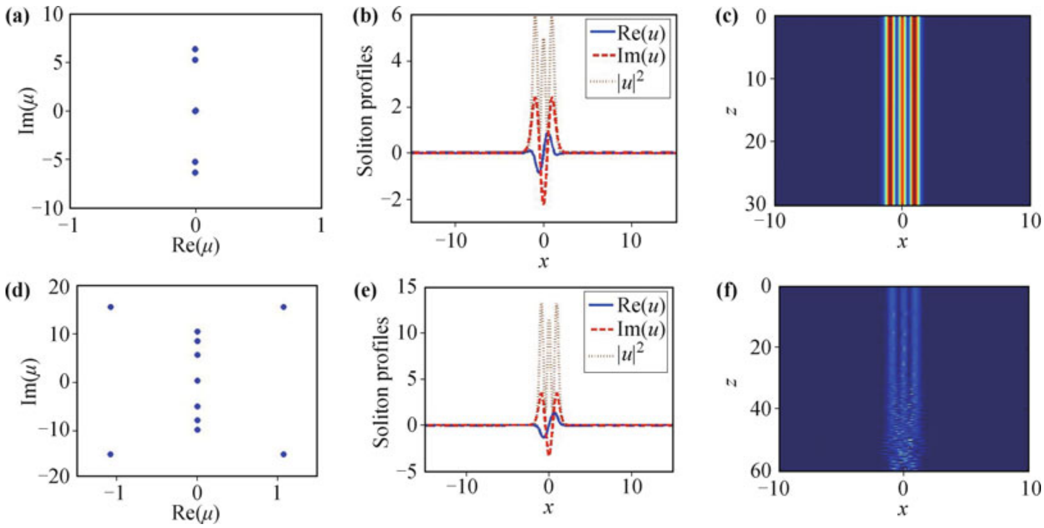

Fig. 4 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of three-peak soliton in self-focusing medium with positive defects and $\mu = 21.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 26.0$.

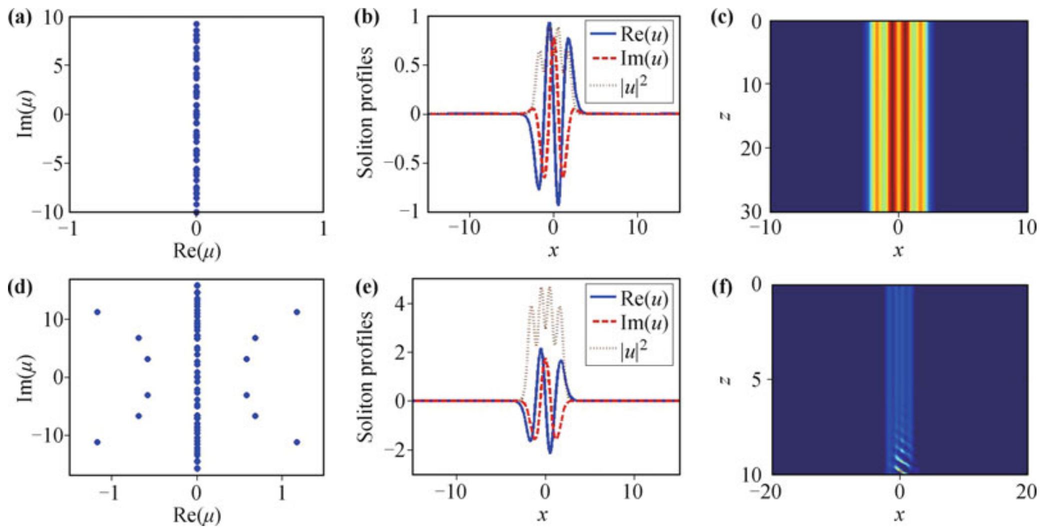

Fig. 5 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of four-peak soliton in self-focusing medium with negative defects and $\mu = 2.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 5.0$.

The two humps in the center have the same peak intensity, which is greater than that of the side humps. For the unstable case with $\mu = 5.0$, the two central humps have a peak intensity less than that of the side humps, as shown in Fig. 5(e). Due to the three quadruple complex eigenvalues shown in Fig. 5(d), the soliton breaks out, and the intensity increases rapidly.

Figure 6 shows the soliton profiles and stabilities of four-peak solitons in PT-symmetric Bessel potentials with positive defects when $\mu = 13.0$ and $\mu = 19.0$. Contrary to their counterparts in negative defects, the peak intensities of the two central humps are less than those of the side humps [Figs. 6(b) and (e)]. As shown in Fig. 6(a), the stable four-peak soliton solution with $\mu = 13.0$ has purely imaginary eigenvalues, and the instability growth rate is zero. This soliton can propagate very stably under perturbation by 2% random noise. The stable region of propagation is $11.55 < \mu < 16.58$, which is wider than the stable regions observed when negative defects were present. When $\mu > 16.58$, two quadruple eigenvalues bifurcate out, as shown in the instability growth spectrum of the four-peak soliton when $\mu = 19.0$ in a potential with self-focusing nonlinearity and positive defects [Fig. 6(d)]. This unstable four-peak soliton breaks out rapidly and cannot maintain its original shape during its propagation.

Five-peak solitons were also found in a PT-symmetric Bessel lattice with both negative and positive defects and self-focusing nonlinearity. In negative defects, μ_c for the existence of five-peak solitons is below zero. In order to ensure that μ was above zero, a minimum P of 6.7865 was found. All of the five-peak solitons in lattices with

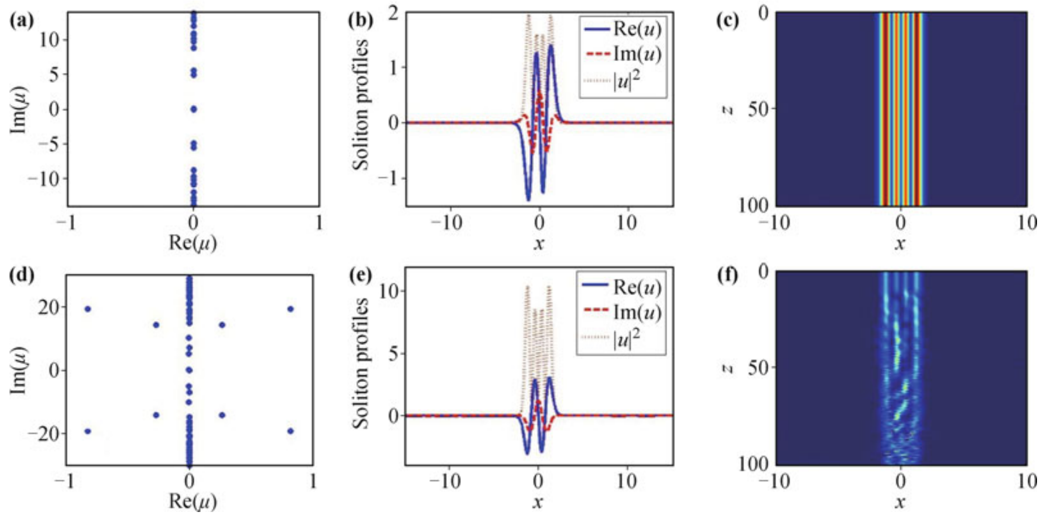

Fig. 6 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of four-peak soliton in self-focusing medium with positive defects and $\mu = 13.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 19.0$.

self-focusing nonlinearity and negative defects were found to be unstable. Figure 7(a) shows the instability growth spectrum of the five-peak soliton when $\mu = 1.0$, which has several quadruple complex eigenvalues. Due to the strong modulation instability, the soliton intensity profile rapidly loses its original shape, and P increases simultaneously.

When $\varepsilon = 0.5$, μ_c for the existence of five-peak solitons in a PT-symmetric Bessel lattice with self-focusing nonlinearity changes to 6.59. That is to say, the normalized power of five-peak solitons can exist when $\mu > 6.59$. Those solitons with low powers ($6.59 < \mu < 9.8$) are

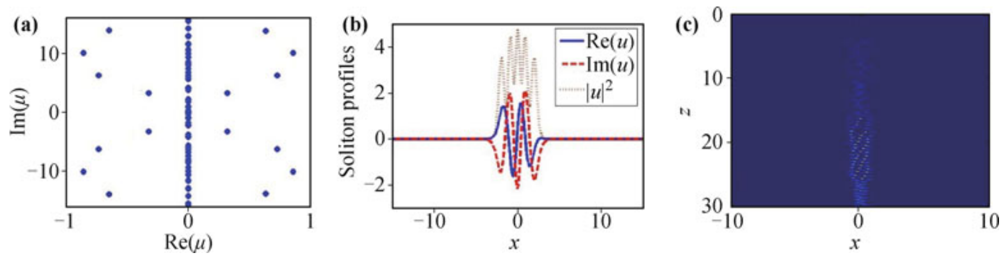

Fig. 7 **(a)** Instability growth spectrum, **(b)** soliton profiles, and **(c)** propagation of five-peak solitons in self-focusing medium with negative defects at $\mu = 1.0$.

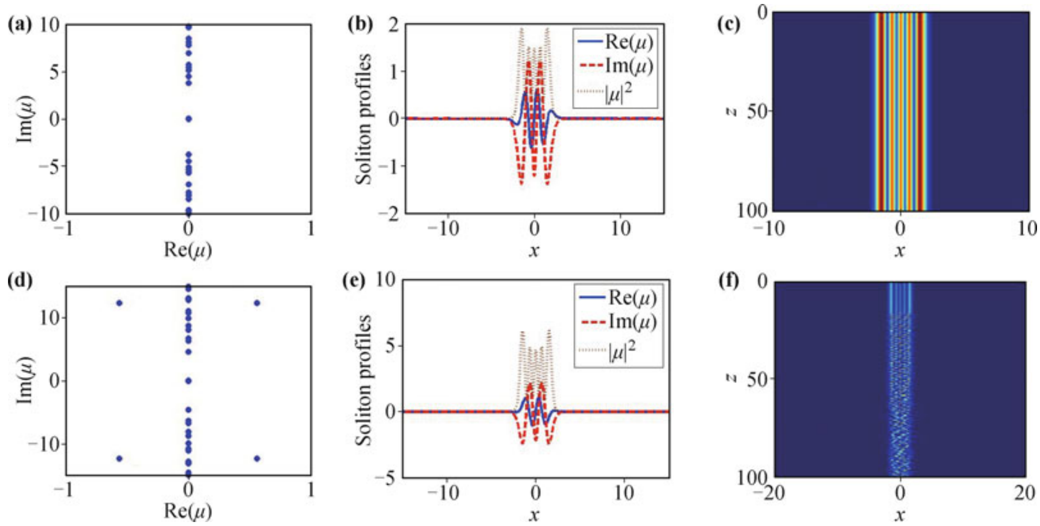

Fig. 8 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of five-peak soliton in self-focusing medium with positive defects and $\mu = 8.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 11.0$.

stable under modulation by small perturbations. As an example, the intensity profile and optical field of a five-peak soliton in a lattice with positive defects when $\mu = 8.0$ are presented in Fig. 8(b). As shown in Fig. 8(a), the instability growth rate eigenvalues are purely imaginary, and this soliton is sufficiently strong to overcome the perturbation by white noise; therefore, it propagates stably in the optical lattice with positive site defects.

4 Multi-peak solitons in self-defocusing media

In the section, we focus on the case in which $\sigma = -1$ and discuss the existence and stability of multi-peak solitons in PT-symmetric Bessel potentials with defects under self-defocusing nonlinearity. Since the propagation constant threshold is below zero, five-peak solitons cannot exist in a PT-symmetric Bessel lattice when $\varepsilon = -0.5$ and $\sigma = -1$. Figure 9 shows the relation between P and μ for multi-peak solitons in PT-symmetric Bessel potentials under self-defocusing nonlinearity. All of the multi-peak solitons can exist below the existence threshold μ_0. It is interesting to note that μ_0 is exactly equal to the critical constant of self-focusing nonlinearity. This result can be directly deduced from the fact that μ_0 is the eigenvalue of a bound state existing in the absence of nonlinearity. When the sign of the nonlinearity changes, the direction of the eigenvalue of the nonlinear model is also reversed. As shown in Fig. 9, the defect solitons in positive defects have higher P than the defect solitons with the same peaks in negative defects for the same μ. The negative defects also stabilize the propagation of the multi-peak solitons in the PT-symmetric Bessel lattice. In other words, the stable region for multi-peak solitons in negative defects is wider than that for solitons in positive defects when the nonlinearity is self-defocusing. In self-defocusing nonlinear media, a lattice with negative defects behaves as a normal lattice superimposed with a bright beam soliton. Therefore, negative defects can support stable solitons with relatively higher P than positive defects can. Optical solitons in positive defects will suffer more modulation instability than those in negative defects, and solitons in negative defects can have wider stability regions than those in positive defects when the nonlinearity is self-defocusing. It can also be concluded that, for all multi-peak solitons, P decreases as μ increases in a self-defocusing medium.

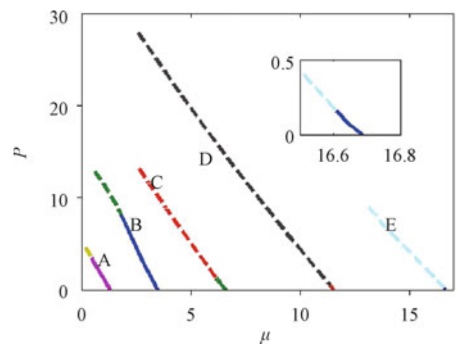

Fig. 9 P versus μ for defect solitons in PT-symmetric Bessel lattice with defects under self-defocusing nonlinearity. (A) Four-and (B) three-peak solitons in negative defects and (C) five-, (D) four-, and (E) three-peak solitons in positive defects. Inset depicts P for three-peak solitons in positive defects when $16.50 < \mu < 16.68$.

To validate the stability analysis, we added 1% white

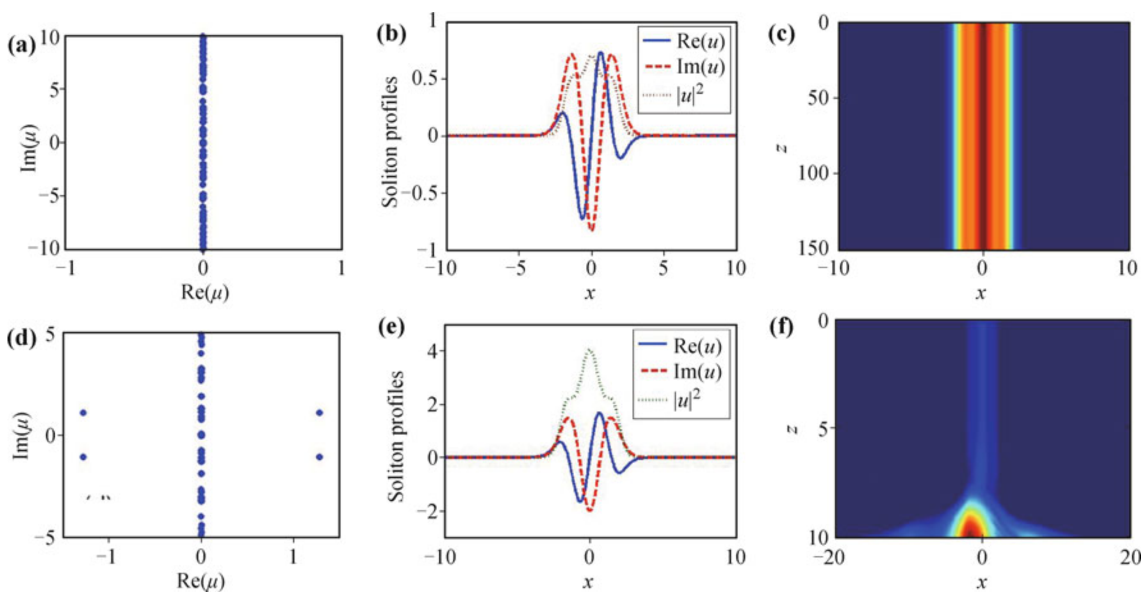

Fig. 10 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of three-peak soliton in self-defocusing medium with negative defects and $\mu = 2.0$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 1.0$.

noise to the soliton solutions and simulated the propagation directly. From Fig. 9, it can be seen that the stable region for three-peak solitons in a Bessel lattice with negative defects under self-defocusing nonlinearity is $1.795 < \mu < 3.505$. Figure 10 presents the instability growth rate, soliton profiles, and propagation of a stable three-peak soliton when $\mu = 2.0$ and an unstable three-peak soliton when $\mu = 1.0$; for both solitons, $\varepsilon = -0.5$ and $\sigma = -1$ were used. From Figs. 10(b) and (e), it is evident that the intensity of the central peak is greater than those of the two side peaks. Each complete intensity profile resembles that of "embedded solitons." When $\mu = 2.0$, which lies in the stable region, all of the eigenvalues of the perturbation remain purely imaginary [Fig. 10(a)], and the intensity profile retains its shape during the entire propagation of 150 units. When $\mu = 1.0$, however, a quadruple eigenvalue bifurcates out [Fig. 10(d)]. The perturbed soliton breaks out, and the power increases rapidly after propagating a short distance due to the strong effect of the large modulation instability.

If the coefficient of the defect depth is changed to 0.5, the peak intensity of the central peak becomes less than that of the side humps, as can be seen from paring Figs. 11(b) and (e). The stable region becomes $16.50 < \mu < 16.68$, which is narrower than that in negative defects. When $\mu = 16.62$, the modulation growth rate is purely imaginary, and the noise is not amplified [Fig. 11(a)]. Therefore, the soliton can propagate stably without changing shape [Fig. 11(c)]. When $\mu = 15$, two quadruple complex eigenvalues appear [Fig. 11(d)], and the propagation is unstable.

Out-of-phase defect solitons with four peaks can also be found in PT-symmetric Bessel potentials with defects under self-defocusing nonlinearity. When the defect parameter is negative, four-peak solitons can exist when $\mu < 1.321$, and the stable range is $0.465 < \mu < 1.321$. Figure 12 presents two examples of four-peak solitons in negative defects when the nonlinearity is negative. The soliton profiles of the four-peak defect soliton when $\mu = 1.0$ are plotted in Fig. 12(b). All of the adjacent peaks are out of phase. All of the eigenvalues of the modulation growth rate are purely imaginary [Fig. 12(a)], and the soliton is sufficiently strong to overcome the perturbations due to white noise. When $\mu = 0.2$, which is out of the stable region, two quadruple eigenvalues bifurcate out [Fig. 12(d)]. As shown in Fig. 12(f), the intensity profiles begin to oscillate strongly after propagating a short distance, and the total power increases exponentially during propagation.

In a PT-symmetric Bessel potential with positive defects under self-defocusing nonlinearity, out-of-phase four-peak solitons can exist when $\mu < 11.55$. As shown in Fig. 9, the stable range for four-peak solitons in positive defects is $11.39 < \mu < 11.55$. Obviously, four-peak solitons in negative defects are more stable than those in positive defects under self-defocusing nonlinearity. A typical example of a stable four-peak soliton in a lattice with positive defects and $\mu = 11.42$ is shown in Fig. 13(b). This soliton can propagate very stably under perturbation by random noise. When $\mu = 9.0$, however, a quadruple complex eigenvalue bifurcates out, and the propagation is unstable.

As demonstrated above, five-peak solitons do not exist in a PT-symmetric Bessel lattice when $\varepsilon = -0.5$ under self-defocusing nonlinearity. Therefore, we discuss the existence and stability of five-peak solitons in positive defects only. Under the effect of positive defects, five-peak

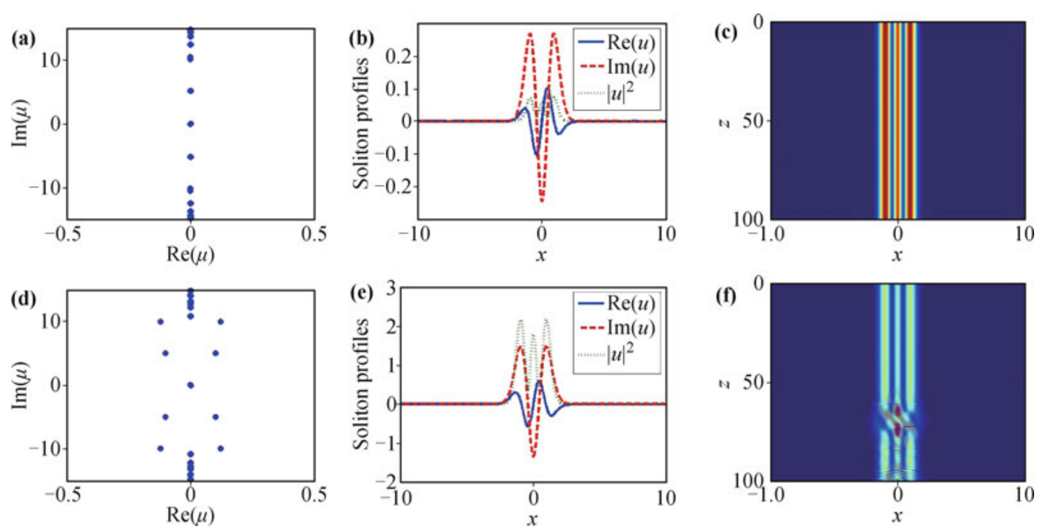

Fig. 11 **(a)** Instability growth spectrum, **(b)** profiles, and **(c)** propagation of three-peak soliton in self-defocusing medium with positive defects and $\mu = 16.62$. **(d, e, f)** Same as (a), (b), and (c), respectively, but with $\mu = 15.0$.

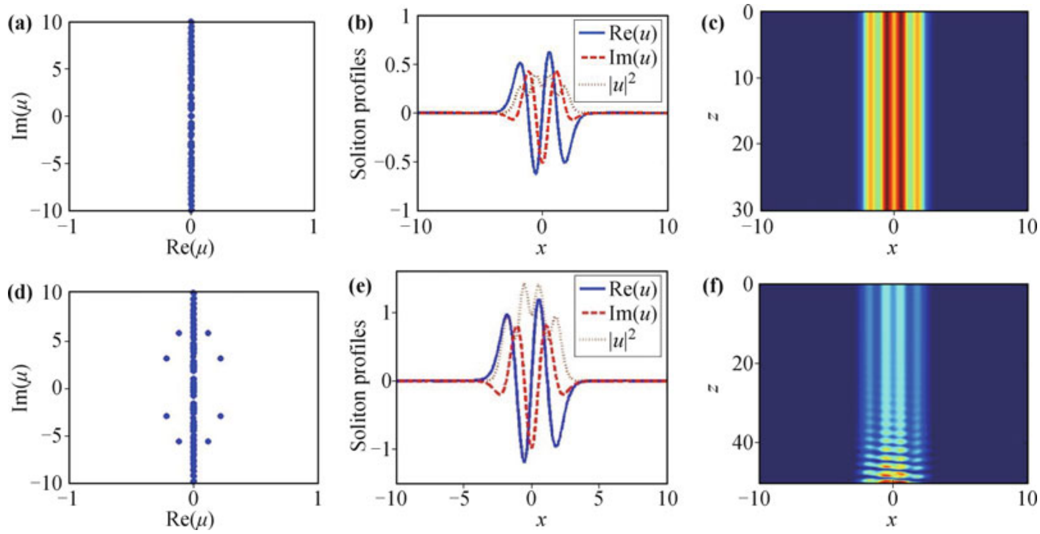

Fig. 12 (**a**) Instability growth spectrum, (**b**) profiles, and (**c**) propagation of four-peak soliton in self-defocusing medium with negative defects and $\mu = 1.0$. (**d, e, f**) Same as (a), (b), and (c), respectively, but with $\mu = 2.0$.

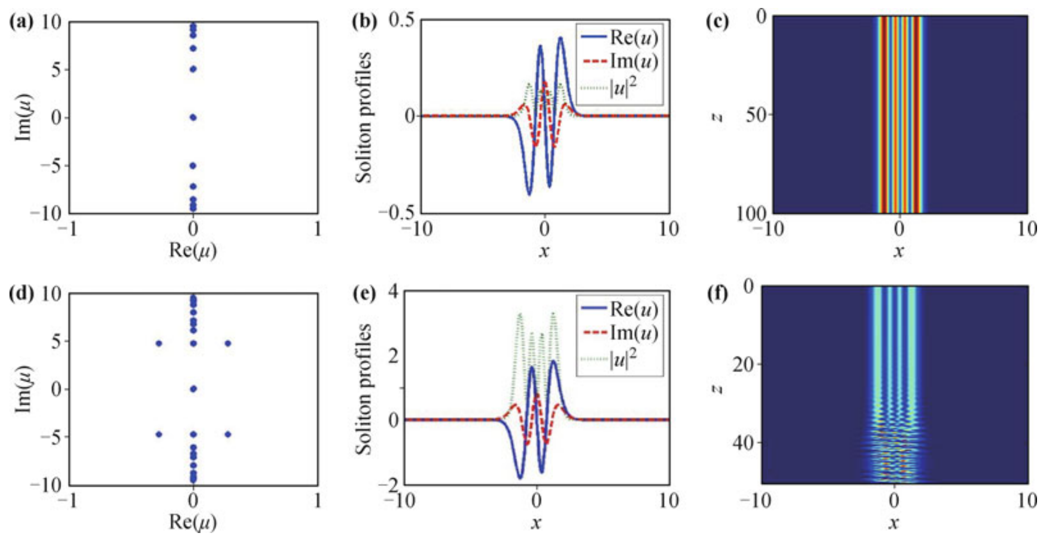

Fig. 13 (**a**) Instability growth spectrum, (**b**) profiles, and (**c**) propagation of four-peak soliton in self-defocusing medium with positive defects and $\mu = 11.42$. (**d, e, f**) Same as (a), (b), and (c), respectively, but with $\mu = 9.0$.

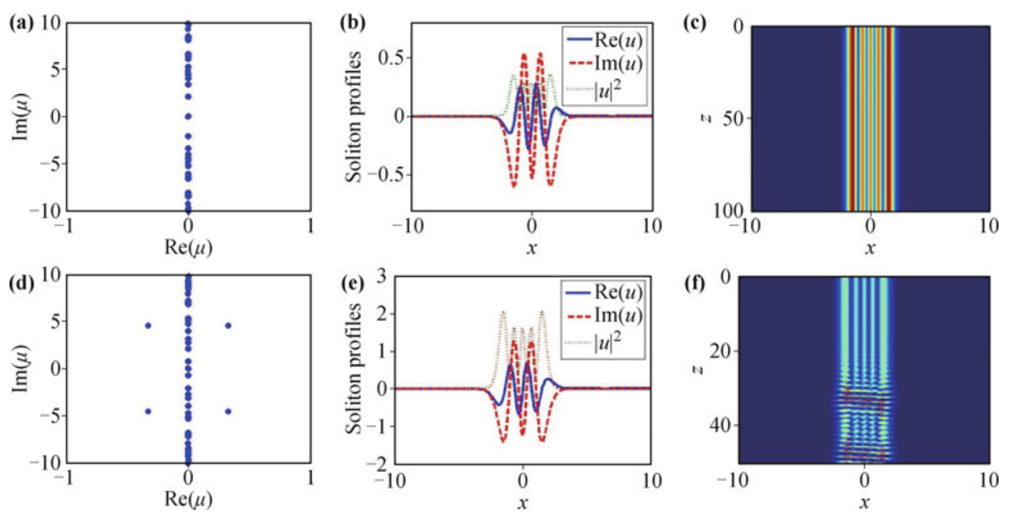

Fig. 14 (**a**) Instability growth spectrum, (**b**) profiles, and (**c**) propagation of five-peak soliton in self-defocusing medium with positive defects and $\mu = 6.35$. (**d, e, f**) Same as (a), (b), and (c), respectively, but with $\mu = 5.0$.

solitons can exist when $\mu < 6.59$, and the stable region is $6.12 < \mu < 6.59$. As a stable example, the soliton profiles of a five-peak soliton at $\mu = 6.35$ are presented in Fig. 14(b). The corresponding instability growth rate and propagation are shown in Figs. 14(a) and (c), respectively. All of the eigenvalues are purely imaginary, and the soliton is sufficiently strong to overcome the perturbation by random noise. For $\mu = 5.0$, the corresponding instability growth rate, soliton profiles, and propagation are depicted in Figs. 14(d), (e), and (f), respectively. As shown, a quadruple eigenvalue appears in the instability growth rate, and this soliton cannot propagate stably in the nonlinear optical lattice.

5 Conclusion

In conclusion, we investigated the existence and stability of multi-peak solitons in PT-symmetric Bessel optical lattices with defects embedded in nonlinear media. In self-focusing media, multi-peak solitons can exist when $\mu > \mu_0$, and P increases monotonically as μ increases. Only low-power multi-peak solitons are stable. Furthermore, they can be more stable in positive defects than in negative defects under self-focusing nonlinearity. If the nonlinearity is self-defocusing, however, the solitons exist when $\mu < \mu_0$, and P decreases as μ increases. Similar to self-focusing media, only low-power multi-peak solitons are stable in self-defocusing media. Unlike in the case of self-focusing media, negative defects can help stabilize the multi-peak solitons in self-defocusing media.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 61308019) and the Foundation for Distinguished Young Talents in Higher Education of Guangdong (Grant No. Yq2013157). H. Wang also acknowledges the financial support from China Scholarship Council.

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