

Physical basis for the symmetries in the Friedmann–Robertson–Walker metric

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Modern cosmological theory is based on the Friedmann–Robertson–Walker (FRW) metric. Often written in terms of co-moving coordinates, this well-known solution to Einstein’s equations owes its elegant and highly practical formulation to the cosmological principle and Weyl’s postulate, upon which it is founded. However, there is physics behind such symmetries, and not all of it has yet been recognized. In this paper, we derive the FRW metric coefficients from the general form of the spherically symmetric line element and demonstrate that, because the co-moving frame also happens to be in free fall, the symmetries in FRW are valid only for a medium with zero active mass. In other words, the spacetime of a perfect fluid in cosmology may be correctly written as FRW only when its equation of state is $\rho + 3p = 0$, in terms of the *total* pressure p and *total* energy density ρ . There is now compelling observational support for this conclusion, including the Alcock–Paczyński test, which shows that only an FRW cosmology with zero active mass is consistent with the latest model-independent baryon acoustic oscillation data.

Keywords cosmological parameters, cosmological observations, cosmological theory, gravitation

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1 Introduction

The gravitational collapse (or expansion) of a spherically symmetric distribution of matter and energy was first considered in Ref. [1]. Since then, several important generalizations have been made in Refs. [2–4], among others, each of which introduced essential physical ingredients, such as the influence of non-zero pressure.

Perhaps because the work of Birkhoff [5] had not yet been fully appreciated, the development of what we now call the Friedmann–Robertson–Walker (FRW) metric took a different approach from that of the general problem of gravitational expansion and contraction [6]. The corollary to Birkhoff’s theorem, however, states that even for an infinite, isotropic medium — be it dynamic or static — the spacetime within a spherical shell is independent of what lies beyond the enclosed spherical volume.¹⁾ It is therefore not difficult to convince oneself that the general relativistic description of the universal

expansion of a shell at radius R relative to an observer at the origin of the coordinates should closely mirror the formalism employed in problems of stellar collapse or explosion involving a body of the same size.

The difference between the two approaches is highlighted by the fact that, whereas the dynamical equations of gravitational collapse are obtained by solving Einstein’s equations with a general form of the metric, the Friedmann equations are derived after all the symmetries have been used to greatly simplify the FRW metric *before* it is introduced into the field equations. For example, the lapse function in this metric is conventionally set equal to one without considering possible dilation effects due to an accelerated expansion of the spatial coordinates. However, as we shall show in this paper, such an approach bypasses at least one important condition that the stress-energy tensor must satisfy in order to permit this simple choice of metric. In so doing, we will demonstrate that the FRW spacetime is actually valid only for a perfect fluid with zero active mass.

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¹⁾Weinberg [7] provides an excellent, though brief, description of this phenomenon. See also Ref. [8] for a more recent discussion.

This critical re-evaluation of the applicability of the FRW spacetime to arbitrary constituents of the cosmic fluid is motivated in large part by the ever-improving precision of cosmological measurements [9], which are refining our view of the cosmic equation of state. The current standard model is an FRW cosmology with a relatively unconstrained blend of constituents, including matter (ρ_m), radiation (ρ_r), and an unknown dark energy (ρ_{de}), and their associated pressures, p_m , p_r , and p_{de} , respectively. The existence of ρ_{de} has been demonstrated convincingly by meticulous analysis of Type Ia SN data, which prove beyond any doubt that the expansion of the Universe is not slowing down, an otherwise unavoidable outcome if matter and radiation were acting alone [10–13]. When dark energy is assumed to be a cosmological constant, Λ , with $p_{de} = -\rho_{de}$, the model is referred to as the Λ cold dark matter (Λ CDM) model; otherwise, the conventional designation is w CDM, where $w \equiv p_{de}/\rho_{de}$ characterizes the dark energy equation of state.

Over the past several decades, Λ CDM/ w CDM has been quite successful in accounting for the observations, thanks chiefly to the flexibility it enjoys owing to a rather large number of free parameters. These include w ; the Hubble constant H_0 , which represents the expansion rate today; the scaled matter (Ω_m) and dark energy (Ω_{de}) densities, where $\Omega_i \equiv \rho_i/\rho_c$ and $\rho_c \equiv 3c^2 H_0^2/(8\pi G)$ is the so-called critical density; and the partitioning of ρ_m into baryonic and dark matter. The scaled radiation energy density, $\Omega_r \equiv \rho_r/\rho_c$, is not considered to be a free parameter because we can measure the temperature of the cosmic microwave background (CMB) (blackbody) radiation very accurately. All told, the standard model has at least five unspecified parameters, all of which can be adjusted to fit the data.

Given the wide latitude permitted by this parametrization, optimization of the model parameters by fitting the observations (most impressively through measurements of the CMB [14–16]) is revealing a very surprising result: over a Hubble time (i.e., H_0^{-1}), the Universe expanded by an amount equal to the amount under constant expansion, despite the fact that the combination of ρ_m , ρ_r , and ρ_{de} should have produced periods of deceleration and acceleration. In other words, the average acceleration of the Universe up to this point in time is zero within the measurement errors. A more meaningful way to say this is that averaged over a Hubble time, the quantity p/ρ , where $p = p_r + p_m + p_{de}$ and $\rho = \rho_r + \rho_m + \rho_{de}$, yields $\langle p/\rho \rangle = -1/3$.

What makes this result even more striking is that, in the context of Λ CDM, the Universe is open and infinite. However, the condition $\langle p/\rho \rangle = -1/3$ can be achieved only once in its entire (presumably infinite) history, and

it is happening right now, just when we are looking. Such an astonishing coincidence begs for a physical explanation. By demonstrating that the symmetries in FRW require zero active mass in the cosmic fluid, we will show in this paper that $\rho + 3p$ is in fact always zero, and that the result $\langle p/\rho \rangle = -1/3$ is therefore independent of time. Instead, it must be an imperfect (or incomplete) parametrization of the standard model that leads to an inferred variable expansion rate. Further, to maintain consistency with the condition $\langle p/\rho \rangle = -1/3$, it is therefore the optimized parameter values that must change depending on when the fits to the data are made.

2 General relativistic expansion and contraction

The FRW metric for a spatially homogeneous and isotropic three-dimensional space is usually written as

$$ds^2 = c^2 dt^2 - a^2(t)[dr^2(1 - kr^2)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1)$$

in terms of the cosmic time t , co-moving radius r , universal expansion factor $a(t)$, and angular coordinates θ and ϕ in the co-moving frame. The spatial curvature constant k takes the values $(-1, 0, +1)$ for an open, flat, or closed universe, respectively.

Clearly, not only is the fluid at rest in this coordinate system (ct, r, θ, ϕ) , but the corresponding frame must also be in free fall, because $g_{tt} = 1$. To see this, one need only remember Einstein's demonstration that his theory of gravity, based on the equivalence principle, correctly reduces to Newton's law in the weak-field limit if

$$g_{tt} = 1 + \frac{2\Phi}{c^2}, \quad (2)$$

where Φ is the gravitational potential. Obviously, if $g_{tt} = 1$, an observer in this co-moving frame sees no gravity. Thus, we should immediately ask ourselves under what conditions the ansatz in Eq. (1) is justified when we apply it to cosmology, where one typically assumes a perfect fluid with the stress-energy tensor

$$T_{\alpha\beta} = \left(\rho_m + \frac{p}{c^2}\right) u_\alpha u_\beta - pg_{\alpha\beta} \quad (3)$$

in terms of the co-moving energy density $\rho = \rho_m c^2$ (where ρ_m is the equivalent mass density), pressure p , and four-velocity u_α . For example, we might wonder whether the symmetries built into Eq. (1) place any constraints on the pressure, which in fact they do, as p must be homogeneous and isotropic.

Let us now take a step backward and, instead of using the FRW metric as given in Eq. (1), treat it as a special

case of the more general, spherically symmetric, diagonal form of the metric used in problems of gravitational contraction and expansion [1–4], which we write as

$$ds^2 = e^{2\Phi/c^2} c^2 dt^2 - e^\lambda dr^2 - R^2 d\Omega^2, \quad (4)$$

where, for simplicity, we have introduced the notation $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Here, Φ , λ , and R are each functions of r and t , and are to be determined by solving Einstein’s equations,

$$G_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R} = -\frac{8\pi G}{c^4}T_{\alpha\beta}, \quad (5)$$

where $\mathcal{R}_{\alpha\beta}$ and \mathcal{R} are the Ricci tensor and scalar, respectively.

These equations have been solved many times in the literature, so we will simply borrow the principal results, especially those of Refs. [3, 4]. Throughout this paper, an overdot signifies differentiation with respect to t , and a prime indicates differentiation with respect to r . In addition, we will introduce the so-called Misner–Sharp mass $m(r, t)$, defined as

$$e^{\lambda(r,t)} = g_{rr} = \left[1 + U^2 - \frac{2Gm(r, t)}{c^2 R}\right]^{-1} (R')^2, \quad (6)$$

where the quantity

$$U \equiv \frac{e^{-\Phi/c^2} \dot{R}}{c} \quad (7)$$

gives the relative velocity $U d\theta$ (in units of c) of adjacent fluid particles on the same sphere of constant r [3, 4, 17, 18].

We emphasize that we have chosen to work with a system of coordinates moving at each point with the fluid at that point, a condition first highlighted in Ref. [3]. In this co-moving (or Lagrangian) frame, the four-velocity components are

$$u_t = ce^{-\Phi/c^2}, \quad u_i = 0 \quad (i = r, \theta, \phi). \quad (8)$$

Therefore, the coordinate t must be the time in this co-moving frame, a situation that contrasts with the more typical approach in which the coordinates are chosen arbitrarily to simplify the metric before Einstein’s equations are invoked to determine its coefficients. In such cases, the coordinates are often interpreted after the solution has been found. However, we are not free to do this here, because the physical meaning of t has already been employed to write Eq. (8). It is therefore straightforward (if somewhat tedious) to confirm that Einstein’s equations result in the following relations for $m(r, t)$:

$$\dot{m}c^2 = -4\pi R^2 \dot{R} p, \quad (9)$$

and

$$m'c^2 = 4\pi\rho R^2 R'. \quad (10)$$

It follows, therefore, that the quantity

$$m(r, t) = \int_0^r \frac{4\pi}{c^2} \rho R^2 R' dr \quad (11)$$

denotes the mass from the origin (where the observer is located) out to r at time t . Eq. (9) is the energy equation for the rate of work due to the pressure.²⁾ In situations where $m(r_s, t)$ represents the mass of a body undergoing gravitational collapse or expansion, r_s would be the radius at its surface, where an important boundary condition is $p = 0$ [3]. In the cosmological context, Birkhoff’s theorem and its corollary allow us to consider $m(r, t)$ to be the mass-energy bounded by a shell of radius r anywhere in the medium, and to view this $m(r, t)$ (and its associated pressure; more on this below) as being solely responsible for any gravitational influence between the origin and a particle at that radius [5, 7, 8].

Three more equations are critical to the discussion in this paper. The first two come from the conservation equation $T^{\alpha\beta}_{;\beta} = 0$, which yields the Euler equation

$$\frac{\partial\Phi}{\partial r} = -\frac{p'c^2}{\rho + p} \quad (12)$$

and the conservation of energy

$$\dot{\rho} = -3\left(\frac{\dot{R}}{R}\right)(\rho + p). \quad (13)$$

The dynamical equation may be written as

$$e^{-\Phi/c^2} \frac{\partial}{\partial t} \left(e^{-\Phi/c^2} \dot{R} \right) = -c^2 \left[\frac{1 + U^2 - 2Gm/(c^2 R)}{\rho + p} \right] \times \left(\frac{\partial p}{\partial R} \right)_t - \frac{Gmc^2 + 4\pi GR^3 p}{c^2 R^2}. \quad (14)$$

It should be emphasized that these expressions are completely general for any spherically symmetric distribution of mass-energy described as a perfect fluid. We have not yet introduced the key symmetries leading to the metric given in Eq. (1). In the following section, we will examine what happens in the cosmological expansion and stellar collapse scenarios. Specifically, we will see what is required to reduce the general metric in Eq. (4) to the more streamlined FRW formulation of Eq. (1).

²⁾ Incidentally, $m(r, t)$ is also the mass used to define the gravitational horizon $R_h \equiv 2Gm/c^2$ associated with the FRW metric [8], and it is not difficult to show from this that $R_h = c/H$, where $H \equiv \dot{a}/a$ is the Hubble constant. In other words, the gravitational horizon defined in terms of the Misner–Sharp mass is in fact the Hubble radius.

3 Discussion

3.1 Gravitational collapse

Let us now begin to introduce some of the principal symmetries. Suppose the medium is static, so $\dot{R} = 0$. Then Eq. (14) gives the pressure gradient $\partial p/\partial R$ required to maintain equilibrium against gravitational collapse. In the special case when ρ is uniform throughout the sphere, Einstein's equations show that $\partial/\partial R = e^{-\lambda/2}(\partial/\partial r)_t$, so we may combine Eqs. (12) and (14) to arrive at the well-known Tolman–Oppenheimer–Volkoff equation [19, 20] describing the hydrostatic interior of a star:

$$\frac{\partial \Phi}{\partial R} = \frac{Gm(R) + 4\pi GR^3 p/c^2}{R^2 - 2Gm(R)R/c^2}. \quad (15)$$

In this application of the metric in Eq. (4), gravity is clearly present because it counteracts the pressure gradient, and we see that $\Phi' \neq 0$, which means that $e^{2\Phi/c^2}$ cannot be absorbed into a rescaled time coordinate that would have allowed us to write $g_{tt} = 1$. We would arrive at similar conclusions should the star be undergoing gravitational collapse, except that in this case \dot{R} and U are not zero, so the pressure gradient would be insufficient to prevent at least partial conversion of gravitational energy into kinetic energy during the infall.

3.2 The lapse function in FRW

Turning now to cosmology, we see that in order to convert the general metric of Eq. (4) into the standard FRW form shown in Eq. (1), it is necessary to force the condition

$$\frac{\partial \Phi(r, t)}{\partial r} = 0. \quad (16)$$

To do this, however, we must have $p' = 0$, which confirms that the pressure is homogeneous (as well as isotropic), and the dynamical Eq. (14) reduces to

$$e^{-\Phi/c^2} \frac{\partial}{\partial t} \left(e^{-\Phi/c^2} \dot{R} \right) = -\frac{Gmc^2 + 4\pi GR^3 p}{c^2 R^2}, \quad (17)$$

or

$$e^{-\Phi/c^2} \dot{R} \frac{\partial}{\partial t} \left(e^{-\Phi/c^2} \dot{R} \right) = -\frac{Gm\dot{R}}{R^2} - \frac{4\pi G}{c^2} R p \dot{R}. \quad (18)$$

From Eq. (9), the last term on the right-hand side is just $G\dot{m}/R$, and changing the variable to $u \equiv e^{-\Phi/c^2} \dot{R}$ then gives

$$u \frac{\partial u}{\partial t} = \frac{d}{dt} \frac{Gm}{R}, \quad (19)$$

whose solution is

$$\frac{1}{2} u^2 - \frac{Gm}{R} = K(r), \quad (20)$$

where $K(r)$ is an arbitrary function of r only. Anticipating the meaning of this function in relation to the spatial curvature constant in FRW, we define $K(r) \equiv -(c^2/2)\tilde{k}(r)r^2$, where \tilde{k} is possibly a function of r , but not of t . Thus, the solution to the dynamics equation in the cosmological context may be written as

$$\frac{1}{2} \dot{R}^2 e^{-2\Phi/c^2} - \frac{Gm}{R} = -\frac{c^2}{2} \tilde{k}(r)r^2. \quad (21)$$

Those familiar with FRW dynamics will immediately recognize that this expression reduces to the Friedmann equation if we impose the constraint $e^{-2\Phi/c^2} = 1$ and the final required symmetry—that ρ should be uniform throughout the medium.³⁾ The G_{tr} component of the Einstein equations, together with Eq. (12), would then force R to have the form $a(t)f(r)$ [4], and the arbitrary function $f(r)$ could be used to rescale the coordinate r and allow us to recover Eq. (1). Therefore, the factor \tilde{k} is indeed the spatial curvature constant k in the FRW metric, affirming the view that it merely represents the local energy of the expanding cosmic fluid. Note also that the factor r^2 is common to all the terms in this expression, so Φ depends only on the co-moving time t , which is consistent with Eq. (16). Equation (21) couples Φ to Gm/R . This is not surprising in light of Birkhoff's theorem and its corollary, which indicate that Gm/R should represent the gravitational potential on a sphere at R relative to the origin. The lapse function exists specifically because of the time dilation effects due to the curvature of space-time. We are therefore not permitted to *arbitrarily* set $e^{2\Phi/c^2}$ equal to 1. We will now formally derive the general expression for $e^{2\Phi/c^2}$ and show that it is constant only when $\rho + 3p = 0$.

Because the density ρ and pressure p are uniform, g_{rr} may be written as $e^\lambda = a(t)^2$, where the expansion factor $a(t)$ is a function only of time [4]. Moreover, we showed above that under these conditions, $\sqrt{g_{\theta\theta}} = R(r, t) = a(t)f(r)$. [As is well known, the precise form of the function $f(r)$ depends on the value of the spatial curvature constant k in the FRW metric, which we define below.] Equation (21) may therefore be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \rho e^{2\Phi/c^2} - \frac{kc^2}{a^2} e^{2\Phi/c^2}, \quad (22)$$

which, as we have already noted, is simply the familiar Friedmann equation, except for the factor $e^{2\Phi/c^2}$, and we have also defined $k \equiv \tilde{k}(r)(r/f[r])^2$. From the rr compo-

³⁾ Note also that the inclusion of Eq. (19) in Eq. (6) quite trivially reproduces the FRW form of the metric coefficient g_{rr} .

ment of the spherically symmetric perfect-fluid field equations [21] [or even just simply from Eq. (17)], one can also easily derive the corresponding acceleration equation:

$$\left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right) \frac{\dot{\Phi}}{c^2} = -\frac{4\pi G}{3c^2} e^{2\Phi/c^2} (\rho + 3p) . \tag{23}$$

We will examine the impact of Φ on these two equations by first finding its limiting form when $\rho + 3p \rightarrow 0$; i.e., we seek a solution to Eqs. (22) and (23) for

$$\rho + 3p = \epsilon\rho , \tag{24}$$

where $\epsilon \ll 1$. For this simplified approach, we shall also set $k = 0$ (though we relax this condition for the more general derivation that follows). A straightforward manipulation of Eq. (23) then produces the expression

$$\frac{1}{a\dot{a}} \frac{\partial}{\partial t} \left(\dot{a}^2 e^{-2\Phi/c^2} \right) = -\frac{8\pi G}{3c^2} \epsilon\rho , \tag{25}$$

and combining this with Eq. (22) yields

$$\frac{\partial}{\partial t} \left\{ \ln \left(\dot{a}^2 e^{-2\Phi/c^2} \right) \right\} = -\epsilon \frac{\partial}{\partial t} \ln a . \tag{26}$$

The solution to this equation is therefore

$$e^{2\Phi/c^2} = h\dot{a}^2 a^\epsilon , \tag{27}$$

where h is an integration constant. Clearly, because $a(t)$ is a function of t , Φ cannot in general be equal to zero. To obtain a time-independent Φ , we must take the limit $\epsilon \rightarrow 0$, in which case Φ would be constant as long as \dot{a} is also constant.

The more general form of Eq. (26), without the use of Eq. (24) (and without setting $k = 0$), is

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \ln \left(\dot{a}^2 e^{-2\Phi/c^2} \right) \right\} &= -\frac{kc^2}{a\dot{a}} \left(1 + \frac{3p}{\rho} \right) e^{2\Phi/c^2} \\ &\quad - \frac{\dot{a}}{a} \left(1 + \frac{3p}{\rho} \right) . \end{aligned} \tag{28}$$

The solution to this equation may be written as

$$e^{2\Phi(t)/c^2} = h\dot{a}^2 e^{\mathcal{I}(t)} , \tag{29}$$

with

$$\mathcal{I}(t) \equiv \int_0^t dt' \frac{8\pi G}{3c^2 H} e^{\Phi/c^2} (\rho + 3p) , \tag{30}$$

where $H \equiv e^{-\Phi/c^2} (\dot{a}/a)$ is the Hubble constant, and the integrand is a function of t' only. This expression is more complicated than Eq. (27), but the result is the same. To achieve a constant Φ , we must have $\mathcal{I} \rightarrow 0$, which is guaranteed only when $\rho + 3p \rightarrow 0$. Then Φ is constant as long as \dot{a} is constant [which is ensured by Eq. (23)], and the lapse function may be set equal to 1 with an appropriate choice of the initial condition h .

3.3 Uniqueness of the co-moving, free-falling frame

Suppose we were to choose a cosmic equation of state such that $\rho + 3p \neq 0$. In that case, we know that $\Phi(t)$ cannot be constant, though it is a function only of t , not of the spatial coordinates. The reason for this is clear. In other spherically symmetric spacetimes, such as the Schwarzschild spacetime, where the curvature depends on the position, the time dilation is itself a function of r (though for that particular spacetime the curvature is static, so Φ is independent of time). This results in a spatially dependent lapse function to which we are accustomed. In the FRW metric, however, the Universe is homogeneous and isotropic throughout each time slice, so the lapse function $e^{2\Phi/c^2}$ must be independent of (r, θ, ϕ) ; it can change only from slice to slice if the spacetime curvature is evolving with time.

The fact that Φ is a function only of t could be viewed as an inconsequential ‘‘gauge’’ freedom. In other words, the interpretation of $e^{2\Phi/c^2}$ as a true lapse function representing a spatially uniform time dilation in this metric would not be recognized as such. However, now that we have formally derived the metric coefficients in FRW from the general form of the spherically symmetric metric, we can demonstrate that the supposed change in gauge, $g_{tt} \rightarrow 1$, is actually a transformation of the coordinates into the free-falling frame. It is only in this frame that dt can reduce to the usual (local) proper time $d\tau \equiv ds/c$, so $g_{tt} = 1$ (corresponding to an acceleration-free environment).

The required coordinate transformation to eliminate the lapse function in Eq. (4) is

$$d\tilde{t} \equiv e^{\Phi(t)/c^2} dt , \tag{31}$$

so

$$\tilde{t} = \int_0^\tau e^{\Phi(t')/c^2} dt' . \tag{32}$$

The coordinate \tilde{t} therefore subsumes the accumulated (spatially uniform) dilation of t when $\Phi \neq 0$. Correspondingly, if we were to also define $\dot{\tilde{a}} \equiv da(\tilde{t})/d\tilde{t}$, then

$$\dot{\tilde{a}} = e^{-\Phi(t)/c^2} \dot{a} , \tag{33}$$

and rewriting Eqs. (22) and (23) in terms of \tilde{a} , $\dot{\tilde{a}}$, and $\ddot{\tilde{a}}$ would then recover the familiar Friedmann and acceleration equations, though with the derivatives now written in terms of \tilde{t} rather than t . As expected, this transformation has placed us in the free-falling frame, corresponding to the FRW metric in Eq. (1), with the time coordinate now given as \tilde{t} .

However, we already selected our set of coordinates to be those in the co-moving (i.e., Lagrangian) frame from

the beginning. This was necessary to derive our equations, starting with the four-velocity in Eq. (8), which allowed us to move with the fluid at each spacetime point. The transformation in Eq. (31) to put us in the free-falling frame, where $d\tilde{t} \rightarrow d\tau$, therefore does not represent a true gauge freedom at all, because in the context of FRW, the free-falling and co-moving frames are one and the same. If we want to recover the FRW metric in Eq. (1), the choice of gauge is not free because the uniqueness of the co-moving and free-falling frames forces t and \tilde{t} ($= \tau$) to be the same coordinate. $\Phi(t)$ must always be identically zero, so, from Eq. (30), we must have $\rho + 3p \rightarrow 0$.

As a concrete example, consider what happens to an Einstein–de Sitter spacetime under such a transformation. Written in terms of \tilde{t} , the expansion factor in a Universe containing only matter (with corresponding zero pressure) has the well-known solution $a(\tilde{t}) = \tilde{t}^{2/3}$. However, because $\rho + 3p \neq 0$ in this case, $a(t) \neq t^{2/3}$. Conceptually, a gauge transformation is supposed to leave the equations of motion unchanged, yet here, the geodesics written in t are different from those written in \tilde{t} , even though we are using the same ρ and p .

The FRW metric is special among spherically symmetric spacetimes because of its elegance and simplicity. However, its attractiveness and practicality come at a cost—they are valid only for a perfect fluid whose equation of state is uniquely given by the expression $p = -\rho/3$ and whose expansion rate is therefore constant, with $\ddot{a} = 0$.

4 Conclusion

Current cosmological observations are precise enough to test whether this conclusion is confirmed in reality. Despite the perception that this result may be in conflict with these measurements, quite the opposite is true. The zero active mass condition gives rise to what we have been calling the $R_h = ct$ Universe in the literature [8, 22, 23]. As the quality of the observations continues to improve, we increasingly see that optimization of the parameters in Λ CDM/ w CDM brings the overall expansion history in this model ever closer to that expected in a Universe with $p = -\rho/3$. The evidence comes from cosmic chronometers [24, 25], gamma-ray bursts [26, 29], high-redshift quasars [30], Type Ia SNe [31], and, most recently, an application of the Alcock–Paczynski test using model-independent baryon acoustic oscillation (BAO) data [32–34], among others. The BAO measurements are particularly noteworthy because, with their $\sim 4\%$ accuracy, they now rule out the standard model

when the zero active mass condition is ignored at better than the 99.34% C.L. Instead, they strongly favor the $R_h = ct$ model, with its equation of state $p = -\rho/3$. The conclusion from comparative studies such as these is that, although Λ CDM is a parameter-rich cosmology, ultimately, when its parameters are optimized to fit the data, its predictions fall in line with the expansion history we would have expected all along in an FRW spacetime for a fluid with zero active mass.

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