

# 3D entangled fractional squeezing transformation and its quantum mechanical correspondence

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A new type of entangled fractional squeezing transformation (EFrST) has been theoretically proposed for 2D entanglement [Front. Phys. 10, 100302 (2015)]. In this paper, we shall extend this case to that of 3D entanglement by introducing a type of three-mode entangled state representation, which is not the product of three 1D cases. Using the technique of integration within an ordered product of operators, we derive a compact unitary operator corresponding to the 3D fractional entangling transformation, which is an entangling operator that presents a clear transformation relation. We also verified that the additivity property of the novel 3D EFrST is of a Fourier character by using its quantum mechanical description. As an application of this representation, the EFrST of the three-mode number state is calculated using the quantum description of the EFrST.

**Keywords** entangled fractional squeezing transformation, entangled state representation

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## 1 Introduction

The fractional Fourier transform (FrFT) is a generalized Fourier transform. It has various applications in Fourier optics and information processing techniques such as optical information processing, beam synthesis, phase retrieval, perspective projections, shift-variant filtering, image restoration, and pattern recognition. The FrFT of a function was first introduced by Condon [1], through a solution of Green's function for phase-space rotations, and also by Namias as a mathematical tool for solving theoretical physics problems [2]. Mendlovic and Ozakatas *et al.* then explored its applications in optics by redefining the FrFT as the change in the field caused by propagation along a quadratic graded index (GRIN) medium [3–9]. The FrFT provides a comprehensive and widely accessible account of the subject covering both theory and applications.

There is, in fact, a correspondence between classical optical transforms such as the optical Fresnel transform, the Hankel transform, and the quantum optical transform, or quantum mechanical unitary operator. In a recent review paper [10], we constructed this correspondence relation using the IWOP (integration within an

ordered product of operators) technique [11, 12]. In this way, we have not only obtained the mechanical operator correspondence of the classical transforms, but also proposed a new optical transformation. For example, by converting the wavelet transform into its quantum mechanical version, we discovered new mother wavelets that satisfy the normalization condition [13]. In addition, we have extended the FrFT to the two-dimensional (2D) entangled case. The  $\alpha$ -order one-dimensional (1D) FrFT is given by

$$\mathcal{F}_\alpha[f(x)] = \int_{-\infty}^{\infty} K_\alpha(x, y) f(x) dx, \quad (1)$$

$$K_\alpha(x, y) = \sqrt{\frac{e^{i(\frac{\pi}{2}-\alpha)}}{2\pi \sin \alpha}} e^{i \frac{x^2+y^2}{2 \tan \alpha} - \frac{ixy}{\sin \alpha}}. \quad (2)$$

By defining  $f(x) \equiv \langle x|f\rangle$  and  $K_\alpha(x, y) = \langle y|S_\alpha|x\rangle$ , one finally obtains the corresponding unitary operator  $S_\alpha = \exp\{i\alpha a^\dagger a\}$  and we may rewrite the FrFT as

$$\mathcal{F}_\alpha[f(x)] = \langle y|\exp\{i\alpha a^\dagger a\}|f\rangle, \quad (3)$$

$$K_\alpha(x, y) = \langle y|\exp\{i\alpha a^\dagger a\}|x\rangle, \quad (4)$$

where  $\exp\{i\alpha a^\dagger a\}$  is actually a rotation operator and  $|y\rangle$  and  $|x\rangle$  are the coordinate eigenvectors,  $X|x\rangle = x|x\rangle$ . From Eqs. (3) and (4), one can clearly see that the ma-

trix element  $\langle y|S_\alpha|f\rangle$  is just the quantum mechanical description of the FrFT and the integral kernel  $K_\alpha(x, y)$  is just the matrix element of  $S_\alpha$  in the coordinate state representation. These two descriptions allow an easy and direct discussion of certain properties of the FrFT, and the extension to the multi-dimensional case by introducing a multi-mode entangled representation. For the two-mode entangled state  $|\eta\rangle$  [see Eq. (11)] [14, 15], we can construct an integral kernel [16]

$$\begin{aligned} \mathcal{K}_C(\eta', \eta) &= \langle \eta' | \exp[-i\alpha(a^\dagger a + b^\dagger b)] | \eta \rangle \\ &= \frac{e^{i(\alpha - \frac{\pi}{2})}}{2 \sin \alpha} \exp \left[ \frac{i(|\eta'|^2 + |\eta|^2)}{2 \tan \alpha} - \frac{i(\eta'^* \eta + \eta^* \eta')}{2 \sin \alpha} \right], \end{aligned} \quad (5)$$

and then the complex FrFT can be defined as  $\mathcal{F}_\alpha[f](\eta') = \langle \eta' | \exp[-i\alpha(a^\dagger a + b^\dagger b)] | f \rangle$ , which is not the product of two 1D FrFTs, but is in fact entangled.

Recently, a new type of (entangled) fractional squeezing transformation ((E)FrST) has been proposed [17–19]. In Ref. [19] the authors introduced the so-called fractional squeezing transformation through a conversion of the triangular functions in the integration kernel of the FrFT to hyperbolic functions, i.e.,  $\tan \alpha \rightarrow \tanh \alpha$ , and  $\sin \alpha \rightarrow \sinh \alpha$ . It has been shown that the quantum mechanical fractional squeezing transformation satisfies additivity and that the corresponding unitary operator is simply the single-mode squeezing operator with a pure imaginary number as the squeezing parameter. As a natural and non-trivial extension, the EFrST is then also proposed using two mutually conjugate entangled state representations with a two-mode squeezing operator [18]. Based on the 2D complex FrFT (CFrFT) [20], we introduced a new fractional entangling transformation (FrET), whose quantum mechanical operator corresponds to an entangling operator, which is very different from the corresponding operators of both the CFrFT and the EFrST [21].

These results show that the quantum mechanical representation plays an important role in terms of constructing FrFTs, while there is a corresponding relation between the quantum mechanical unitary operator and the classical transformation. Thus, one can propose new classical-optical transforms from the point of view of quantum mechanics and find the correspondence between these using the unitary operator and the IWOP technique [22]. At this point we question whether or not there exists a multi-dimensional EFrST that is not the product of three 1D cases. The answer is affirmative. In this paper, we introduce a 3D EFrST using the three-mode entangled state representation, and then drive the corresponding operator using the IWOP technique. Based on the compact exponential expression and

the entangled representation, it is easy to prove that this is additive.

This paper is arranged as follows. In Section 2, we will give a brief review of certain newly proposed FrFTs. In Section 3, through the introduction of a three-mode entangled representation, we propose a novel 3D entangled FrFT. In Section 4, following the method used in Refs. [17–19], we then construct another 3D EFrST and derive its corresponding operator using the IWOP technique. Section 5 is devoted to proving the additivity of this new 3D EFrST. We also consider its application to calculations in Section 6. Our conclusions and discussions are contained in the last section.

## 2 Brief review of new FrFTs

In this section, we provide a brief review of various new FrFTs. In the 1D case, the standard FrFT is defined in Eq. (1). Replacing  $\tan \alpha \rightarrow \tanh \alpha$ , and  $\sin \alpha \rightarrow \sinh \alpha$  in the integral kernel  $K_\alpha(x, y)$ , we obtain

$$\mathcal{K}_\alpha(x, y) = \frac{1}{\sqrt{2\pi i \sinh \alpha}} e^{i \frac{x^2 + y^2}{2 \tanh \alpha} - \frac{ixy}{\sinh \alpha}}. \quad (6)$$

Taking  $f(x) \equiv \langle x | f \rangle$  and  $\mathcal{K}_\alpha(x, y) = \langle y | U_1 | x \rangle$  where  $|y\rangle$  and  $|x\rangle$  are the coordinate eigenvectors,  $X|x\rangle = x|x\rangle$ , then multiplying the function  $\mathcal{K}_\alpha(x, y)$  by the ket  $\int_{-\infty}^{\infty} dy |y\rangle$  and bra  $\int_{-\infty}^{\infty} dx \langle x|$  from left and right, and using the IWOP technique in order to perform the integration, we finally obtain

$$U_1 = \exp[i\alpha(a^2 + a^{\dagger 2})/2]. \quad (7)$$

This is simply the single-mode squeezing operator with  $i\alpha$  as the squeezing parameter, which is different from the corresponding quantum mechanical description for the standard FrFT with a unitary operator  $\exp\{i\alpha a^\dagger a\}$ . Thus, the new squeezing FrFT can be rewritten as

$$\begin{aligned} \mathcal{F}_\alpha[f(x)] &= \int_{-\infty}^{\infty} \mathcal{K}_\alpha(x, y) f(x) dx \\ &= \langle y | U_1 | f \rangle, \end{aligned} \quad (8)$$

where we have used the completeness relation of coordinate representation,  $\int_{-\infty}^{\infty} dx |x\rangle \langle x| = 1$ .

In the 2D case, the EFrST is introduced and defined in a similar manner [20]

$$\mathcal{F}_\alpha[f(\eta)] = \int_{-\infty}^{\infty} \frac{d^2 \eta}{\pi} \mathcal{K}_C(\eta', \eta) f(\eta), \quad (9)$$

where the integral kernel function is given by replacing  $\tan \alpha \rightarrow \tanh \alpha$  and  $\sin \alpha \rightarrow \sinh \alpha$  in the integral kernel (5), i.e.,

$$\mathcal{K}_C(\eta', \eta) = \frac{1}{2i \sinh \alpha} \exp \left[ \frac{i(|\eta'|^2 + |\eta|^2)}{2 \tanh \alpha} - \frac{i(\eta'^* \eta + \eta^* \eta')}{2 \sinh \alpha} \right]. \quad (10)$$

If we define  $\mathcal{K}_C(\eta', \eta) = \langle \eta' | U_2 | \eta \rangle$ , where  $|\eta\rangle$  and  $|\eta'\rangle$  are the eigenvectors of the commutable operators for relative position  $X_1 - X_2$  and total momentum  $P_1 + P_2$ , i.e.,  $(X_1 - X_2) |\eta_{=\eta_1+i\eta_2}\rangle = \sqrt{2}\eta_1 |\eta\rangle$ ,  $(P_1 + P_2) |\eta_{=\eta_1+i\eta_2}\rangle = \sqrt{2}\eta_2 |\eta\rangle$ ,  $X_j = (a_j + a_j^\dagger)/\sqrt{2}$ ,  $P_j = (a_j - a_j^\dagger)/(i\sqrt{2})$  and the explicit form of the entangled state representation is

$$|\eta\rangle = \exp \left[ -\frac{1}{2} |\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger \right] |00\rangle. \quad (11)$$

Then, multiplying the function  $\mathcal{K}_C(\eta', \eta)$  by the ket  $\int_{-\infty}^{\infty} d^2\eta' |\eta'\rangle / \pi$  and bra  $\int_{-\infty}^{\infty} d^2\eta \langle \eta | / \pi$  from left and right, we obtain

$$U_2 = \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \frac{d^2\eta'}{\pi} |\eta'\rangle \mathcal{K}_C(\eta', \eta) \langle \eta | \\ = \exp [-i\alpha(a^\dagger b^\dagger + ab)], \quad (12)$$

which corresponds to the two-mode squeezed operator. Thus, the new EFrST can be rewritten as

$$\mathcal{F}_\alpha [f(\eta)] = \langle \eta' | U_2 | f \rangle, \quad (13)$$

where we define  $f(\eta) = \langle \eta | f \rangle$  and use the completeness relation  $\int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1$ . For further details as to applications of the entangled state representation, one can refer to Refs. [23, 24].

### 3 Standard 3D entangled FrFT

Now, we extend the 2D entangled FrFT to the case of 3D entanglement. For this purpose, we first introduce a standard 3D entangled state. The three operators  $X_1 - X_3$ ,  $X_1 + X_2 + X_3$ , and  $P_1 - 2P_2 + P_3$  are compatible with each other, and the common eigenvector is given by [25]

$$|\bar{\rho}\rangle = \frac{1}{\sqrt{6\pi^{3/4}}} \exp \left[ -\frac{1}{12} (\rho^2 + 3\chi_1^2 + 2\chi_2^2) + \frac{\sqrt{2}}{6} i\rho(a_1^\dagger - 2a_2^\dagger + a_3^\dagger) + \frac{\sqrt{2}}{2} \chi_1 (a_1^\dagger - a_3^\dagger) + \frac{\sqrt{2}}{3} \chi_2 \sum_{l=1}^3 a_l^\dagger - \frac{1}{6} (2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}) - \frac{1}{3} (2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger) \right] |000\rangle, \quad (14)$$

which satisfies the following eigenequations

$$(X_1 - X_3) |\bar{\rho}\rangle = \chi_1 |\bar{\rho}\rangle, \\ (X_1 + X_2 + X_3) |\bar{\rho}\rangle = \chi_2 |\bar{\rho}\rangle,$$

$$(P_1 - 2P_2 + P_3) |\bar{\rho}\rangle = \rho |\bar{\rho}\rangle, \quad (15)$$

and the completeness and orthogonality relation

$$\int d\rho d\chi_1 d\chi_2 |\bar{\rho}\rangle \langle \bar{\rho}| \\ =: \exp \left[ -\frac{1}{6} (2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}) - \frac{1}{6} (2a_1^2 - a_2^2 + 2a_3^2) - \frac{1}{3} (2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger) - \frac{1}{3} (2a_1 a_2 - a_1 a_3 + 2a_2 a_3) + \frac{1}{6} (a_1^\dagger + a_2^\dagger + a_3^\dagger + a_1 + a_2 + a_3)^2 + \frac{1}{4} (a_1^\dagger - a_3^\dagger + a_1 - a_3)^2 - a_1^\dagger a_2 - a_2^\dagger a_2 - a_3^\dagger a_3 - \frac{1}{12} (a_1^\dagger - 2a_2^\dagger + a_3^\dagger - a_1 + 2a_2 - a_3)^2 \right]: \\ = \int \frac{d\rho d\chi_1 d\chi_2}{6\pi^{3/2}} : \exp \left\{ -\frac{1}{6} [\rho - (P_1 - 2P_2 + P_3)]^2 - \frac{1}{2} [\chi_1 - (X_1 - X_3)]^2 - \frac{1}{3} [\chi_2 - (X_1 + X_2 + X_3)]^2 \right\} : \\ = 1, \quad (16)$$

and

$$\langle \bar{\rho}' | \bar{\rho} \rangle = \delta(\rho - \rho') \delta(\chi_1 - \chi_1') \delta(\chi_2 - \chi_2'). \quad (17)$$

By using  $|\bar{\rho}\rangle$ , we can construct the squeezing operator

$$S = \int \frac{d\rho d\chi_1 d\chi_2}{\mu^{\frac{3}{2}}} \left| \frac{\bar{\rho}}{\mu} \right\rangle \langle \bar{\rho}| \\ = (\text{sech} \lambda)^{\frac{3}{2}} : \exp \left\{ -\left[ \frac{1}{6} (2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}) + \frac{1}{3} (2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger) \right] \tanh \lambda + (\text{sech} \lambda - 1) (a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3) + \frac{1}{6} (2a_1^2 - a_2^2 + 2a_3^2) \tanh \lambda + \frac{1}{3} (2a_1 a_2 - a_1 a_3 + 2a_2 a_3) \tanh \lambda \right\} :, \quad (18)$$

where

$$\mu = e^\lambda, \quad \tanh \lambda = \frac{\mu^2 - 1}{\mu^2 + 1}, \quad \text{sech} \lambda = \frac{2\mu}{\mu^2 + 1}. \quad (19)$$

In a method similar to that used in introducing the standard 1D or 2D FrFTs [see Eq. (5)], we can calculate the following matrix element

$$\langle \bar{\rho}' | e^{-i\alpha \sum_{l=1}^3 a_l^\dagger a_l} | \bar{\rho} \rangle \\ = \frac{\sqrt{e^{3i\alpha}}}{6(2\pi i \sin \alpha)^{3/2}} \exp \left[ \frac{i}{12 \tan \alpha} (\rho^2 + 3\chi_1^2 + 2\chi_2^2 + \rho'^2 + 3\chi_1'^2 + 2\chi_2'^2) - \frac{i(\rho\rho' + 3\chi_1\chi_1' + 2\chi_2\chi_2')}{6 \sin \alpha} \right]$$

$$\equiv K_E(\bar{\rho}', \bar{\rho}), \tag{20}$$

where we used the formula  $e^{-i\alpha a_l^\dagger a_l} a_l^\dagger e^{i\alpha a_l^\dagger a_l} = e^{-i\alpha} a_l^\dagger$ , and the completeness relation of the coherent state  $\int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{\pi^3} |z_1, z_2, z_3\rangle \langle z_1, z_2, z_3| = 1$ . Thus, we can construct a type of 3D entangled FrFT by performing the following integration

$$\begin{aligned} \mathcal{F}_\alpha[f(\bar{\rho}')] &= \int d^3 \bar{\rho} K_E(\bar{\rho}', \bar{\rho}) f(\bar{\rho}) \\ &= \langle \bar{\rho}' | e^{-i\alpha \sum_{l=1}^3 a_l^\dagger a_l} | f \rangle. \end{aligned} \tag{21}$$

Comparing Eq. (20) with the kernels of the 1D and 2D entangled FrFTs in Eqs. (3) and (5), it is of interest to note that Eq. (20) can be considered to be the transformation kernel of the  $\alpha$ -th entangled FrFT, and that the kernel of the entangled FrFT is undecomposable and different from that of the 3D FrFT, which was the product of three 1D FrFTs.

#### 4 Novel 3D EFrST

Following the method proposed in Refs. [17–19], we convert the trigonometric functions into hyperbolic functions, i.e.,  $\tan \alpha \rightarrow \tanh \alpha$  and  $\sin \alpha \rightarrow \sinh \alpha$  in the integral kernels  $K_E(\bar{\rho}', \bar{\rho})$  and  $K_E \rightarrow \mathcal{K}_E(\bar{\rho}', \bar{\rho})$ ,

$$\begin{aligned} \mathcal{K}_E &= \frac{1}{6(2\pi i \sinh \alpha)^{3/2}} \exp \left[ \frac{i}{12 \tanh \alpha} (\rho^2 + 3\chi_1^2 + 2\chi_2^2 \right. \\ &\quad \left. + \rho'^2 + 3\chi_1'^2 + 2\chi_2'^2) - \frac{i(\rho\rho' + 3\chi_1\chi_1' + 2\chi_2\chi_2')}{6 \sinh \alpha} \right]. \end{aligned} \tag{22}$$

This prompts a discussion related to the type of transformation that can be obtained and whether it satisfies the additivity of the 3D EFrST.

Next, we shall investigate the transform kernel  $\mathcal{K}_E$  in the introduced 3D entangled state representation  $|\rho, \chi_1, \chi_2\rangle$ . For this purpose, allowing  $\mathcal{K}_E = \langle \bar{\rho}' | U_3 | \bar{\rho} \rangle$ , and using the completeness relation of the entangled state representation (16), we can express the operator  $U_3$  as

$$\begin{aligned} U_3 &= \int_{-\infty}^{\infty} d^3 \bar{\rho} d^3 \bar{\rho}' | \bar{\rho}' \rangle \mathcal{K}_E \langle \bar{\rho} |, \\ (d^3 \bar{\rho} &= d\rho d\chi_1 d\chi_2, d^3 \bar{\rho}' = d\rho' d\chi_1' d\chi_2'). \end{aligned} \tag{23}$$

Furthermore, by using Eqs. (14) and (22), and the normally ordered form of the vacuum projection operator  $|000\rangle \langle 000| = \exp\{-\sum_{l=0}^3 a_l^\dagger a_l\}$ : together with the IWOP technique, we finally derive

$$U_3 = \int_{-\infty}^{\infty} d^3 \bar{\rho} d^3 \bar{\rho}' | \bar{\rho}' \rangle \mathcal{K}_E \langle \bar{\rho} |$$

$$\begin{aligned} &= (\text{sech} \alpha)^{\frac{3}{2}} : \exp \left[ \frac{i}{6} (2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}) \tanh \alpha \right. \\ &\quad \left. + \frac{i}{3} (2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger) \tanh \alpha \right. \\ &\quad \left. + (\text{sech} \alpha - 1) (a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3) \right. \\ &\quad \left. + \frac{i}{6} (2a_1^2 - a_2^2 + 2a_3^2) \tanh \alpha \right. \\ &\quad \left. + \frac{i}{3} (2a_1 a_2 - a_1 a_3 + 2a_2 a_3) \tanh \alpha \right] : \\ &= \text{sech}^{3/2} \alpha : e^{i(\hat{O}^\dagger + \hat{O}) \tanh \alpha + (\text{sech} \alpha - 1)(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3)} : \\ &= e^{i\hat{O}^\dagger \tanh \alpha} e^{\hat{B} \ln \text{sech} \alpha} e^{i\hat{O} \tanh \alpha}, \end{aligned} \tag{24}$$

where we have set

$$\begin{aligned} \hat{O}^\dagger &= \frac{1}{6} (2a_1^{\dagger 2} - a_2^{\dagger 2} + 2a_3^{\dagger 2}) \\ &\quad + \frac{1}{3} (2a_1^\dagger a_2^\dagger - a_1^\dagger a_3^\dagger + 2a_2^\dagger a_3^\dagger) \\ \hat{B} &= \sum_{l=1}^3 (a_l^\dagger a_l) + \frac{3}{2}, \end{aligned} \tag{25}$$

and we have used the integration formula

$$\begin{aligned} &\int \frac{d^2 z}{\pi} \exp(-\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left(\frac{\zeta \xi \eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right). \end{aligned} \tag{26}$$

Eq. (24) is the normally ordered form of the operator  $U_3$  that corresponds to the new 3D EFrST that was defined in Eq. (22).

In order to further elucidate the quantum mechanical correspondence of Eq. (24), we note that the three operators  $\hat{O}^\dagger, \hat{O}$ , and  $\hat{B}$  obey a closed SU(1,1) Lie algebra, i.e.,

$$[\hat{O}, \hat{O}^\dagger] = \hat{B}, [\frac{\hat{B}}{2}, \hat{O}^\dagger] = \hat{O}^\dagger, [\frac{\hat{B}}{2}, \hat{O}] = -\hat{O}. \tag{27}$$

Thus, using the following decomposition formula of SU(1,1) Lie algebra, [26]

$$\exp(\xi \hat{O}^\dagger + \xi^* \hat{O}) = \exp(\tau \hat{O}^\dagger) \exp(B \ln \text{sech} \alpha) \exp(\tau^* \hat{O}), \tag{28}$$

where  $\xi = \alpha e^{i\varphi}$  and  $\tau = e^{i\varphi} \tanh \alpha$ , we can obtain the compact exponential form of  $U_3$  by using  $\varphi = \pi/2$ ,

$$U_3 = \exp[i\alpha(\hat{O}^\dagger + \hat{O})], \tag{29}$$

which is a compact exponential form. It is clear that  $U_3$  is a unitary operator ( $U_3^\dagger = U_3^{-1}$ ). Again, we observe a correspondence between the optical transformation and the quantum mechanical unitary operator. It is of note

that  $U_3$  is actually a three-mode squeezing operator (or an entangling operator). Thus, we can name the unitary operator  $U_3$  the fractional entangling operator.

The new 3D EFrST corresponding to the kernel in Eq. (22) can then be presented as

$$\mathcal{F}_\alpha [f] (\bar{\rho}') = \int_{-\infty}^{\infty} d^3\bar{\rho} \mathcal{K}_E (\bar{\rho}', \bar{\rho}) f (\bar{\rho}). \tag{30}$$

### 5 The additivity of the 3D EFrST

The essence of FrFT, denoted as  $\mathcal{F}_\alpha [f]$ , is its additive or compositional nature, i.e.,  $\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$ , which gives rise to the name “fractional” transformation. In this section, we shall examine the additivity of the new 3D EFrST. When a transformation is written in terms of quantum mechanics, additivity can be easily verified. For example, in the case of the 1D FrFT in Eq. (3), using the completeness relation  $\int dx |x\rangle \langle x| = 1$  and Eq. (4), we obtain the additivity for the 1D FrFT:

$$\begin{aligned} \mathcal{F}_{\alpha+\beta} [f(x)] &= \langle y | \exp \{i(\alpha + \beta)a^\dagger a\} | f \rangle \\ &= \langle y | e^{i\beta a^\dagger a} \int dx |x\rangle \langle x| e^{i\alpha a^\dagger a} | f \rangle \\ &= \int dx \langle y | e^{i\beta a^\dagger a} | x \rangle \mathcal{F}_\alpha [f(x)] \\ &= \mathcal{F}_\beta \mathcal{F}_\alpha [f(x)]. \end{aligned} \tag{31}$$

Thus, in order to verify the additive property of the fractional entangling transform, using Eq. (22), we first convert the transform corresponding to the kernel (22) into a quantum mechanical description, i.e.,

$$\begin{aligned} \mathcal{F}_\alpha [f] (\bar{\rho}') &= \int_{-\infty}^{\infty} d^3\bar{\rho} \mathcal{K}_E (\bar{\rho}', \bar{\rho}) f (\bar{\rho}) \\ &= \int_{-\infty}^{\infty} d^3\bar{\rho} \langle \bar{\rho}' | U_3 | \bar{\rho} \rangle \langle \bar{\rho} | f \rangle \\ &= \langle \bar{\rho}' | U_3 | f \rangle, \end{aligned} \tag{32}$$

which is simply the quantum version of the new 3D EFrST. This indicates that the EFrST corresponds to the matrix element of the unitary operator  $U_3$  in the entangled state representation  $\langle \bar{\rho}' |$  with the signal vector  $|f\rangle$ .

Using the quantum mechanical version of the EFrST and the completeness relation for the entangled state (16), we may observe that

$$\begin{aligned} \mathcal{F}_{\beta+\alpha} [f (\bar{\rho})] &= \langle \bar{\rho}' | \exp [i(\beta + \alpha)(\hat{O}^\dagger + \hat{O})] | f \rangle \\ &= \int_{-\infty}^{\infty} d^3\bar{\rho}'' \langle \bar{\rho}' | \exp [i\beta(\hat{O}^\dagger + \hat{O})] | \bar{\rho}'' \rangle \end{aligned}$$

$$\begin{aligned} &\times \int_{-\infty}^{\infty} d^3\bar{\rho} \langle \bar{\rho}'' | \exp [i\alpha(\hat{O}^\dagger + \hat{O})] | \bar{\rho} \rangle f (\bar{\rho}) \\ &= \int_{-\infty}^{\infty} d^3\bar{\rho}'' \langle \bar{\rho}' | \exp [i\beta(\hat{O}^\dagger + \hat{O})] | \bar{\rho}'' \rangle \mathcal{F}_\alpha [f (\bar{\rho})] \\ &= \mathcal{F}_\beta \mathcal{F}_\alpha [f (\bar{\rho})]. \end{aligned} \tag{33}$$

Thus, the 3D EFrST satisfies fractional additivity. This proof is clear and concise when using the representation  $\langle \bar{\rho} |$  and the quantum mechanical version of the EFrST.

### 6 Applications

Next, we consider applications of the 3D entangled FrFT and EFrST. In Eq. (21), if we take  $|f\rangle = |m, n, q\rangle$ , which is a three-mode number state, then we obtain

$$\mathcal{F}_\alpha [f (\bar{\rho}')] = e^{-i\alpha(m+n+q)} f (\bar{\rho}'), \tag{34}$$

which indicates that the function  $f (\bar{\rho}') \equiv \langle \bar{\rho}' | m, n, q \rangle$  is the eigenfunction of the 3D entangled FrFT, and the corresponding eigenvalue is  $e^{-i\alpha(m+n+q)}$ . Using  $|m, n, q\rangle = a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger q} / \sqrt{m!n!q!} |000\rangle$  and Eq. (14), together with the coherent state representation of the number state:  $|m\rangle = 1/\sqrt{m!} \frac{\partial^m}{\partial z^m} |z\rangle|_{z=0}$ , where  $|z\rangle = \exp (za^\dagger) |0\rangle$  is the unnormalized coherent state ( $\langle 0 | z\rangle = 1$ ), we obtain

$$\langle \bar{\rho}' | m, n, q \rangle = \frac{\exp \left( -\sum_{l=1}^3 |y_l|^2 / 4 \right)}{\sqrt{6m!n!q!} \pi^{3/4}} H_{mnq}^{\{M\}} \{y_1, y_2, y_3\}, \tag{35}$$

where we have set

$$M = \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \tag{36}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{\sqrt{2}}{6} \begin{pmatrix} 1 & 3i & 2i \\ -2 & 0 & 2i \\ 1 & -3i & 2i \end{pmatrix} \begin{pmatrix} \rho \\ \chi_1 \\ \chi_2 \end{pmatrix}, \tag{37}$$

and used the polynomials  $H_{mnq}^{\{M\}} (y_1, y_2, y_3)$ , defined as [25]

$$\begin{aligned} H_{mnq}^{\{M\}} (y_1, y_2, y_3) &= \frac{\partial^{m+n+q}}{\partial s^m \partial \tau^n \partial t^q} e^{-\frac{1}{2}SMS^T + SMY} \Big|_{s=\tau=t=0}, \end{aligned} \tag{38}$$

where both  $S$  and  $Y$  can be any complex number and  $M$  is a symmetric matrix. Thus, the state  $\langle \bar{\rho} |$  can be described in the following form

$$\langle \bar{\rho} | = \frac{1}{\sqrt{6}\pi^{3/4}} \langle mnq | \exp \left\{ - \sum_{l=1}^3 |y_l|^2 / 4 \right\} \times \sum_{m,n,q=0}^{\infty} \frac{i^{m+n+q}}{\sqrt{m!n!q!}} H_{mnq}^{\{M\}} \{y_1, y_2, y_3\}. \quad (39)$$

Combining Eqs. (34) and (35), it then follows that

$$\mathcal{F}_\alpha \left\{ \frac{\exp \left[ - \sum_{l=1}^3 |y_l|^2 / 4 \right]}{\sqrt{6m!n!q!}\pi^{3/4}} H_{mnq}^{\{M\}} [y_1, y_2, y_3] \right\} = e^{-i\alpha(m+n+q)} \frac{\exp \left[ - \sum_{l=1}^3 |y_l|^2 / 4 \right]}{\sqrt{6m!n!q!}\pi^{3/4}} H_{mnq}^{\{M\}} [y_1, y_2, y_3]. \quad (40)$$

This result implies that we can obtain the eigenequation of the 3D entangled FrFT with an eigenvalue of  $e^{-i\alpha(m+n+q)}$ . It should be noted that the polynomial  $H_{mnq}^{\{M\}}$  is not the product of three 1D Hermite polynomials, and thus this eigenfunction is not the trivial product of three 1D cases.

In addition, by using the quantum version (32) of the new 3D EFrST, it is convenient for us to derive the new EFrST of certain wavefunctions. For the vector  $|f\rangle = |m, n, q\rangle$ , the new EFrST is given by

$$\mathcal{F}_\alpha \{ \langle \bar{\rho} | m, n, q \rangle \} = \langle \bar{\rho}' | U_3 | m, n, q \rangle. \quad (41)$$

The representation  $\langle \bar{\rho}' |$  is in Fock space (39), and therefore we only calculate the matrix element  $\langle \bar{m}, \bar{n}, \bar{q} | U_3 | m, n, q \rangle$ . Using the coherent-state representation of the number state and the normally ordered form of the unitary operator  $U_3$  in Eq.(24), we obtain

$$\langle \bar{m}, \bar{n}, \bar{q} | U_3 | m, n, q \rangle = \frac{1}{\sqrt{m!n!q!}\bar{m}!\bar{n}!\bar{q}!} \frac{\partial^{m+n+q}}{\partial z_1^m \partial z_2^n \partial z_3^q} \times \left. \frac{\partial^{\bar{m}+\bar{n}+\bar{q}}}{\partial \bar{z}_1^{\bar{m}} \partial \bar{z}_2^{\bar{n}} \partial \bar{z}_3^{\bar{q}}} U_3(z_j, \bar{z}_j) \right|_{z_j=\bar{z}_j=0}, \quad (42)$$

where  $U_3(z_j, \bar{z}_j) \equiv \langle \bar{z}_1, \bar{z}_2, \bar{z}_3 | U_3 | z_1, z_2, z_3 \rangle$ . Therefore, substituting Eqs. (39) and (42) into Eq. (41) yields the transformed function.

## 7 Conclusions and discussion

In this paper, we proposed a 3D FrFT in an entangled form through the introduction of a type of three-mode entangled state representation. Then, following the approach proposed by Fan *et al.*, i.e., by replacing the integration kernel with a new kernel, we successively extend the novel 2D EFrST to a 3D case that is not simply the

product of three 1D cases. In order to obtain the correspondence to quantum mechanical description of this EFrST, we derived a compact exponential form for the unitary operator corresponding to the 3D EFrST, which is simply an entangling operator that presents a clear transformation relation for the Bose creation operators. In addition, we verified the additivity of the novel 3D EFrST, which is an important fractional feature, using a matrix element expression and the completeness relation of the 3D entangled state representation within the framework of quantum mechanics. The IWOP technique plays an important role in our derivation.

In general, each optical setup corresponds to an optical transformation, for example, the description of a thick lens as a fractional Fourier transformer. We expect that the novel 3D EFrST can be implemented experimentally. In addition, it is also interesting to extend this EFrST to the multi-dimensional case. With this purpose in mind, it might be convenient to consider the ideal preparation protocol for this type of entangled state representation and to search for the relationship between the corresponding unitary operator and the ideal protocol.

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