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RESEARCH ARTICLE

Discrete vortices on anisotropic lattices

Gui-Hua Chen[†], **Hong-Cheng Wang**, **Zi-Fa Chen**

Department of Electronic Engineering, Dongguan University of Technology, Dongguan 523808, China

Corresponding author. E-mail: [†]*cghphys@gmail.com*

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We consider the effects of anisotropy on two types of localized states with topological charges equal to 1 in two-dimensional nonlinear lattices, using the discrete nonlinear Schrödinger equation as a paradigm model. We find that on-site-centered vortices with different propagation constants are not globally stable, and that upper and lower boundaries of the propagation constant exist. The region between these two boundaries is the domain outside of which the on-site-centered vortices are unstable. This region decreases in size as the anisotropy parameter is gradually increased. We also consider off-site-centered vortices on anisotropic lattices, which are unstable on this lattice type and either transform into stable quadrupoles or collapse. We find that the transformation of off-site-centered vortices into quadrupoles, which occurs on anisotropic lattices, cannot occur on isotropic lattices. In the quadrupole case, a propagation-constant region also exists, outside of which the localized states cannot stably exist. The influence of anisotropy on this region is almost identical to its effects on the on-site-centered vortex case.

Keywords anisotropy, discrete vortex, quadrupole, localized state

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1 Introduction

The nonlinear propagation of an optical wave in a periodic system can lead to the formation of a variety of localized states [1–18]. Discrete vortices on two-dimensional (2D) nonlinear lattices, which are localized states of an optical wave with an embedded nonzero phase circulation over a closed lattice contour, have attracted considerable attention over the past decade [19–30]. Discrete vortices have been observed separately by two independent groups [22, 23], who experimentally created and detected fundamental vortices with topological charges equal to 1 in photorefractive crystals. Two main examples of charge-one discrete vortices include the vortex cross (an on-site-centered vortex) and the vortex cell (an off-site-centered vortex). These discoveries have stimulated further theoretical studies and numerical-computation-based research. Discrete vortices of charge two have been shown to be unstable [19], whereas discrete vortices of charge three have been demonstrated to be stable [24]. Based on the theoretical predictions of Ref. [24], further experiments on localized structures have been undertaken so as to uncover other interesting structures, such as the discrete soliton necklace, which is more globally stable than the charge-three vortex [31]. Studies of discrete vortices have also been extended to incorporate three-dimensional discrete models [32]. Furthermore, asymmetric vortices have been predicted in the two-dimensional lattices in Ref. [25].

The above findings and studies clearly indicate the importance of discrete vortices on 2D nonlinear lattices. However, the majority of the abovementioned works is based on isotropic lattices, while the effects of anisotropy on discrete vortices on 2D nonlinear lattices have been discussed only minimally [33, 34]. Kevrekidis *et al.* have considered this problem, and have shown that anisotropy either expands or reduces the stability areas of on-site-centered vortices, depending on its strength. Further, anisotropy destabilizes off-site-centered vortex configurations [33]. However, as regards on-site-centered vortices, that study only considered cases for which the propagation constant was rescaled to be equal to 1. The range of propagation-constant values for which on-site-centered vortices stably exist is meaningful; however, this was not investigated by Kevrekidis *et al.* Moreover, for off-site-centered vortices, interesting information may be obtained from unstable configurations of these vortices on anisotropic lattices. Obviously, detailed research on the effects of anisotropy on discrete vortices on 2D nonlinear lattices is very necessary.

The objective of this work is to determine the effects of anisotropy on discrete vortices on 2D nonlinear lattices. We focus on on-site-centered and off-site-centered vortices with topological charges equal to 1. Some of the findings reported below are surprising, and demonstrate that the effects of anisotropy are not straightforward. The intuitive expectation might be that weak anisotropy is a small perturbation that possibly alters details of the parametric dependences of the observed phenomenology, but does not result in a "structural" change. We find that this is indeed the case for the on-site-centered vortex structure, but different behavior occurs in the off-site-centered case. Off-site-centered vortices are truly unstable on anisotropic lattices, and either transform into stable quadrupoles or collapse. We also find that the transformation of off-site-centered vortices into quadrupoles, which occurs on anisotropic lattices, cannot occur on isotropic lattices. This paper is structured as follows: The model is introduced in Section 2; the results and discussion are given in Section 3; and, finally, Section 4 concludes the paper with a summary of our results.

2 The model

Some 2D nonlinear lattices are inherently anisotropic, e.g., photorefractive crystals [21, 31, 35–42], while others (in particular, fragmented BECs trapped in strong optical lattices [43–45]) can be easily rendered anisotropic through slight variations of the control parameters. One example of such a scenario is variations to the intensities of laser beams that create two sublattices, which together constitute a 2D optical lattice. In this work, we consider a 2D anisotropic lattice, in which the couplings in the horizontal and vertical directions may differ. The nonlinear propagation of an optical beam in this system is described by the self-focusing discrete nonlinear Schrödinger (DNLS) equation for complex variables $\Psi_{m,n}$,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\Psi_{m,n} = -\frac{C}{2}\Delta_\eta \Psi_{m,n} - |\Psi_{m,n}|^2\Psi_{m,n}, \tag{1}$$

where $\Psi_{m,n}$ is the complex 2D lattice field, (m,n) are discrete coordinates in the transverse plane, z is the propagation distance, C is the real coupling constant of the lattice, and

$$\Delta_\eta \Psi_{m,n} = \eta(\Psi_{m+1,n} + \Psi_{m-1,n}) + \Psi_{m,n+1} + \Psi_{m,n-1} - 2(1+\eta)\Psi_{m,n}, \tag{2}$$

is the anisotropic discrete Laplacian, which becomes isotropic with anisotropy parameter $\eta = 1$. The localized states produced by Eq. (1) are characterized by the total power

$$P = \sum_{m,n} |\Psi_{m,n}|^2. \tag{3}$$

As has been previously demonstrated [46–48], the stationary modes with real propagation constant k are determined as solutions to Eq. (1). Here, $\Psi_{m,n}(z) = e^{ikz}u_{m,n}$, where the stationary amplitude $u_{m,n}$ obeys

$$\frac{C}{2}\left[\eta(u_{m+1,n} + u_{m-1,n}) + u_{m,n+1} + u_{m,n-1} - 2(1+\eta)u_{m,n}\right] + |u_{m,n}|^2 u_{m,n} = k u_{m,n}. \tag{4}$$

In the numerical simulations, it is convenient to combine $u_{m,n}$ into a single "long" vector U of length N^2, where $N \times N$ is the size of the numerical domain. Thus, Eq. (4) can be rewritten in vectorial form as

$$\hat{C}U + \mathrm{diag}(|U|^2)U = kU, \tag{5}$$

where $\mathrm{diag}(|U|^2)$ indicates a diagonal matrix with the respective elements. The other matrix $\hat{C}$ is a linear-coupling matrix with elements

$$C_{j,j'} = \frac{C}{2}\left[\delta_{j,j'-N} + \eta\delta_{j,j'-1} + \eta\delta_{j,j'+1} + \delta_{j,j'+N} - 2(1+\eta)\delta_{j,j'}\right]. \tag{6}$$

The stability of the stationary modes is investigated using a numerical approach, by computing the eigenvalues λ for small perturbations and verifying the results using direct simulations. Linearized equations for the perturbation eigenmodes are produced by expressing the perturbed solution in terms of U. Hence, $\Psi_{m,n}(z) = \mathrm{e}^{\mathrm{i}kz}(U + \alpha\mathrm{e}^{-\mathrm{i}\lambda z} + \beta^*\mathrm{e}^{\mathrm{i}\lambda^* z})$, where α and β are the respective vectors combining the eigenmode amplitudes and $*$ indicates the complex conjugation. Thus,

$$\begin{pmatrix} -\hat{C} + k - 2\mathrm{diag}(|U|^2) & -\mathrm{diag}(U^2) \\ \mathrm{diag}(U^{*2}) & \hat{C} - k + 2\mathrm{diag}(|U|^2) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{7}$$

The underlying solution, U, is stable if all λ are real.

It is well known that two types of vortices are possible on discrete systems, viz., on- and off-site-centered vortices [49]. In what follows, we consider these two types of vortices with topological charges equal to 1. Both can be produced by application of Newton's method to Eq.

(4), beginning with the following respective inputs:

$$\{u_{m,n}\}_{on} = \begin{cases} u_0, & \text{at } (m,n) = (0,1), \\ \mathrm{i}u_0, & \text{at } (m,n) = (1,0), \\ -u_0, & \text{at } (m,n) = (0,-1), \\ -\mathrm{i}u_0, & \text{at } (m,n) = (-1,0), \\ 0, & \text{at all others } (m,n), \end{cases} \tag{8}$$

and

$$\{u_{m,n}\}_{off} = \begin{cases} u_0, & \text{at } (m,n) = (0,0), \\ \mathrm{i}u_0, & \text{at } (m,n) = (1,0), \\ -u_0, & \text{at } (m,n) = (1,1), \\ -\mathrm{i}u_0, & \text{at } (m,n) = (0,1), \\ 0, & \text{at all others } (m,n), \end{cases} \tag{9}$$

where u_0 is a real constant.

3 Results and discussion

On-site-centered vortices on 2D anisotropic lattices are first considered. The results show that the vortices can stably exist for a wide degree of anisotropy. A typical example of stable on-site-centered vortices is shown in Fig. 1. Figures 1(a) and (d) show the real part, imaginary part, intensity, and phase structure of the stable vortex, respectively. The stability of the vortex is investigated by computing λ for small perturbations [shown in Fig. 1(e)] and by direct evolution [shown in Fig. 1(f)].

Through detailed examination, we find that the on-site-centered vortices with different propagation constants are not globally stable for fixed values of η and C. The propagation constant of the stable states does not even exist within a continuous region. However, upper and lower propagation-constant boundaries exist. When the propagation constant of the on-site-centered vortex is below or above the lower and upper boundaries, respectively, the vortex is definitely unstable. Since the region between these two boundaries is the domain outside of which the on-site-centered vortices are unstable, we examine the effects of anisotropy on this region. The results are shown in Fig. 2. From Figs. 2(a)–(d), it can be seen that the size of this region decreases remarkably as the coupling constant increases. This means that strong coupling is detrimental to the stability of the on-site-centered vortices. Furthermore, the region also decreases in size in response to a gradual increase in η. Finally, the influence of the anisotropy is obviously enhanced as the coupling constant increases. This simulated result coincides with the laws of physics as regards the effect of anisotropy. Lattice anisotropy strengthens one direction (e.g., the vertical direction) and weakens the other (e.g., the horizontal direction), which causes disequilibrium in the lattice couplings and destabilizes the localized states (such as the on-site-centered vortex and quadrupole mentioned below), whose structures have some symmetry. Consequently, the region becomes increasingly small as η increases ($\eta > 1$).

The on-site-centered vortices produced by Eq. (1) are characterized by P, and the boundary states have corresponding powers. In terms of energy, some interesting results are revealed concerning the influence of anisotropy on the cases shown in Fig. 2. The respective two-parameter diagrams (P, η) are shown in Fig. 3. In comparison to Fig. 2, a distinct difference in the boundary variation can be observed. The lower boundary in the (P, η) diagrams ascends depending on the strength of the anisotropy, while the upper boundary remains almost fixed. That is to say, the influence of the anisotropy on the power of the lower boundary is quite apparent, but

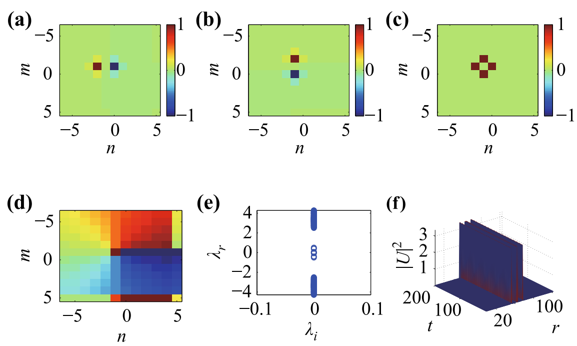

Fig. 1 Real part **(a)**, imaginary part **(b)**, intensity **(c)** and phase structure **(d)** of field $u_{m,n}$ of a typical stable on-site-centered vortex with $k = 2.5$, $C = 0.5$, and $\eta = 0.8$. **(e)** Eigenvalues λ of infinitesimal perturbations around the stable vortex. **(f)** The evolution of the stable vortex.

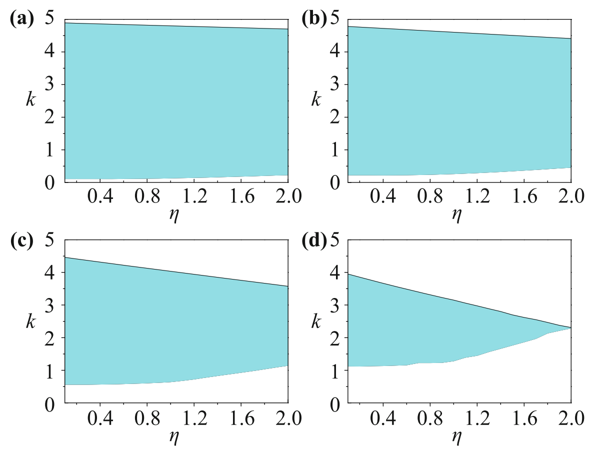

Fig. 2 The effects of anisotropy (indicated by anisotropy parameter η) on the region between the lower and upper boundaries. When propagation constant k of the on-site-centered vortex is below the lower boundary or above the upper one, the on-site-centered vortex cannot exist stably. **(a)** $C = 0.1$; **(b)** $C = 0.2$; **(c)** $C = 0.5$; **(d)** $C = 1.0$.

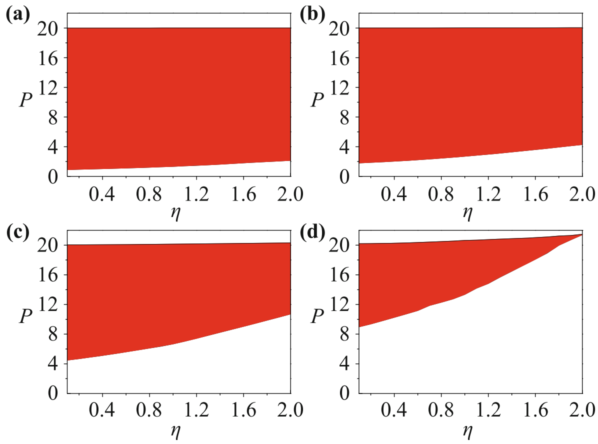

Fig. 3 The influence of anisotropy from the viewpoint of energy. **(a)** $C = 0.1$; **(b)** $C = 0.2$; **(c)** $C = 0.5$; **(d)** $C = 1.0$.

it is almost invisible in the upper-boundary case.

The rules governing the behavior of the on-site-centered vortices examined so far are in accordance with the observations of Ref. [33]. That study also predicted that off-site-centered vortices are unstable on anisotropic lattices. Further, the information that can be obtained from examination of unstable configurations of off-site-centered vortices may prove interesting. Next, the numerical examination of off-site-centered vortices with topological charges equal to one is discussed.

We find that the off-site-centered vortices are indeed unstable on anisotropic lattices, and either transform into stable quadrupole-like localized states or collapse. A typical example of quadrupole-like localized states is shown in Fig. 4. Note that this stable localized state is not a regular off-site-centered vortex, which can be observed in Fig. 4(d). Note also that the transformation of off-site-centered vortices into quadrupoles that occurs on anisotropic lattices cannot occur on isotropic lattices. This means that anisotropy is most likely responsible for this transformation.

In the case of quadrupole-like localized states, a specific propagation-constant region also appears, outside of which the localized states cannot stably exist. The

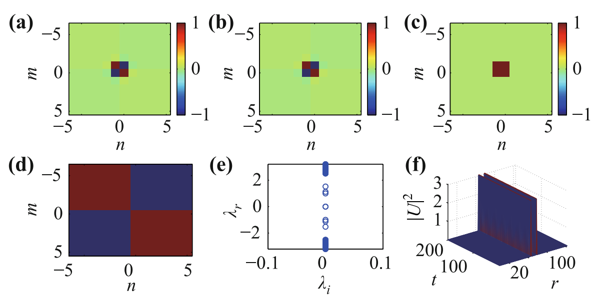

Fig. 4 A typical example of stable localized states analogue to quadrupoles with $k = 2.5$, $C = 0.2$, and $\eta = 0.8$. The subgraphs show the same quantities as in Fig. 1.

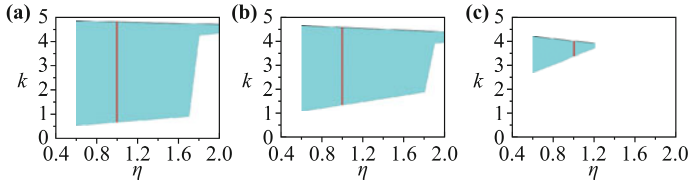

Fig. 5 The effects of anisotropy on the region between the lower and upper boundaries. When propagation constant of the quadrupoles is below the lower boundary or above the upper one, they cannot exist stably. **(a)** $C = 0.1$; **(b)** $C = 0.2$; **(c)** $C = 0.5$.

effect of anisotropy on this region is considered, and the results are shown in Fig. 5. Note that the red lines in the diagrams highlight the unachievable transformation of the off-site-centered vortices into quadrupoles on the isotropic lattice. An obvious reduction in the region occurs as η is gradually increased, and the influence of the anisotropy is enhanced as the coupling constant increases. These findings are almost identical to those obtained in the case of the on-site-centered vortices. However, there are two extreme distinctions between the two types of localized states. The first is that the lower boundary of the propagation-constant region is raised significantly when the coupling constant is small, and the second is that the region vanishes when the coupling constant becomes sufficiently large (e.g., equal to 1.0). That is, no quadrupole-like localized state can stably exist under these conditions.

4 Conclusion

In this work, we examined the effects of anisotropy on nonlinear periodic systems supporting discrete vortices. The two-dimensional discrete nonlinear Schrödinger equation was used as a paradigm model, and our study focused on on-site-centered and off-site-centered vortices with topological charges equal to 1. Through a detailed examination, we found that on-site-centered vortices with different propagation constants are not globally stable for fixed values of the anisotropy parameter η and the coupling constant C. Upper and lower boundaries of the propagation constant exist, and the region between these boundaries is the domain outside of which the on-site-centered vortices are unstable. As η is gradually increased, this region decreases in size. Further, the influence of anisotropy is obviously enhanced as the coupling constant is increased. The responses of the two boundaries to anisotropy differ significantly in terms of energy. The power of the lower boundary markedly increases depending on the strength of the anisotropy, but the power of the upper boundary remains almost constant. We also investigated off-site-centered vortices on anisotropic lattices. These are indeed unstable on anisotropic lattices, and transform into stable quadrupole-like localized states or, alternatively, collapse. We found that it is impossible for off-site-centered vortices on isotropic lattices to transform into quadrupoles, as is the case for anisotropic lattices. As regards the quadrupole-like localized states, a propagation-constant region was also determined, outside of which the localized states cannot stably exist. The influence of anisotropy on this region is almost identical to that of the on-site-centered vortex case. However, there are two extreme distinctions between the two localized-state categories examined here. The first is that a jump occurs in the lower boundary of the region when the coupling constant is small, and the second is that the propagation-constant region vanishes when the coupling constant becomes sufficiently large.

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