

Three-dimensional imaging of the nucleon and semi-inclusive high-energy reactions

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We present a short overview of studies of the transverse-momentum-dependent parton distribution functions of the nucleon. The aim of such studies is to provide three-dimensional imaging of the nucleon and a comprehensive description of semi-inclusive high-energy reactions. By summarizing what we have done in constructing the theoretical framework for inclusive deep inelastic lepton–nucleon scattering and one-dimensional imaging of the nucleon, we try to sketch out an outline of what we need to do to construct such a comprehensive theoretical framework for semi-inclusive processes in terms of three-dimensional gauge-invariant parton distributions. Next, we present an overview of what we have already achieved, with an emphasis on the theoretical framework for semi-inclusive reactions in leading-order perturbative quantum chromodynamics but with leading and higher twist contributions. We summarize in particular the results for the differential cross section and azimuthal spin asymmetries in terms of the gauge-invariant transverse-momentum-dependent parton distribution functions. We also briefly summarize the available experimental results on semi-inclusive reactions and the parameterizations of transverse-momentum-dependent parton distributions extracted from them and present an outlook for future studies.

Keywords transverse momentum dependence, parton distribution function, fragmentation function, collinear expansion, higher twists

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1 Introduction

In the ongoing intensive study of the nucleon structure, three-dimensional imaging has become the frontier and has been a hot topic in recent years. It is commonly

recognized that three-dimensional imaging reveals much more abundant physics of the nucleon structure and the properties of quantum chromodynamics (QCD). The study of three-dimensional imaging was initially triggered by the experimental finding of striking single-spin asymmetries (SSAs) in inclusive hadron production in hadron–hadron collisions with transversely polarized hadrons [1]. Gradually it grew into a field aiming at a comprehensive three-dimensional description of the nucleon structure, including the spin and transverse momentum dependence.

One-dimensional imaging of the nucleon is provided by the parton distribution functions (PDFs) such as the number density $q(x)$, helicity distribution $\Delta q(x)$, and transversity $\delta q(x)$ for quarks of different flavors in the nucleon. These one-dimensional PDFs can be studied in inclusive high-energy reactions and are necessary for the description of such inclusive processes. The three-dimensional case, in which the parton transverse momen-

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tum is also considered, involves not only direct extensions of these distribution functions to include the transverse momentum dependence, but also many other correlation functions that describe in particular the correlations between the transverse momenta and spins, such as the Sivers function, Boer–Mulders function, and pretzelosity. They are generally called transverse-momentum-dependent (TMD) PDFs. Moreover, higher twist effects also become important and need to be considered consistently. The content of the studies is therefore much more abundant and more interesting. These TMD PDFs can be studied in semi-inclusive reactions and are necessary for the description of such processes.

The study of three-dimensional imaging of the nucleon is in the rapidly developing stage, and it is not easy to provide a comprehensive overview of all the different aspects of the studies. Here, we choose to arrange the review as follows: First, we will briefly review what we have done in constructing the theoretical framework in one-dimensional case with inclusive deep inelastic lepton–nucleon scattering (DIS). In this way, we hope that we can sketch out the main line of what we need to do in the three-dimensional case. Next, we will try to follow this line and summarize the progress already achieved along this direction and what we need to do next. This brief review of the one-dimensional case will be presented in Section 2. In Section 3, we will summarize the TMDs defined via the quark–quark correlator. In Section 4, we will present a brief overview of the available information for constructing the theoretical framework of semi-inclusive processes. In Section 5, we will summarize the available experimental results and TMD parameterizations extracted from them. Finally, we will summarize this review in Section 6.

This overview article is an extended version of a plenary talk at the 21st International Symposium on Spin Physics (Spin2014) [2]. It is clear that the simplest and most basic picture is at the leading order (LO) in perturbative QCD (pQCD) and at the leading twist. Hence, there are also two major directions in theoretical developments toward a comprehensive description of semi-inclusive processes: taking higher-order pQCD into account and considering higher twist contributions. These contributions are important not only for higher accuracy but also for consistency. The major advances that have been made in recent years have also followed one of these two directions, i.e., either at the leading twist but leading and higher order in pQCD or in the LO in pQCD but leading and higher twists. The talk [2] concentrated mainly on the second direction. For higher-order pQCD contributions involving the evolution of PDFs, an overview talk was presented by Daniel Boer at the same

conference [3]. There are also many other reviews and monographs (e.g., [4, 6, 7]). The study of higher orders in pQCD and higher twists seems to be rather difficult, and even the factorization properties are unclear [5]. In this article, we follow the same line as in the talk [2] but briefly summarize progress in the studies on QCD evolution and refer interested readers to those reviews.

2 Inclusive DIS & one-dimensional imaging of the nucleon

Our studies on the structure of a fast-moving nucleon started with inclusive DIS such as $e^- + N \rightarrow e^- + X$. We recall that, under the one-photon exchange approximation, the differential cross section is given by the Lorentz contraction of the well-known leptonic tensor $L^{\mu\nu}(l, l', \lambda_l)$ and the hadronic tensor $W_{\mu\nu}(q, p, S)$, i.e.,

$$d\sigma = \frac{2\alpha_{\text{em}}^2}{sQ^4} L^{\mu\nu}(l, l', \lambda_l) W_{\mu\nu}(q, p, S) \frac{d^3l'}{2E_{l'}}. \quad (2.1)$$

The leptonic tensor is calculable and is given by

$$L_{\mu\nu}(l, l', \lambda_l) = 2(l_\mu l'_\nu + l_\nu l'_\mu - g_{\mu\nu} l \cdot l') + i2\lambda_l \epsilon_{\mu\nu\rho\sigma} l^\rho q^\sigma. \quad (2.2)$$

Information on the structure of the nucleon is contained in the hadronic tensor, which is defined as

$$W_{\mu\nu}(q, p, S) = \frac{1}{2\pi} \sum_X \langle p, S | j_\mu(0) | X \rangle \langle X | j_\nu(0) | p, S \rangle \times (2\pi)^4 \delta^4(p + q - p_X). \quad (2.3)$$

Here, l and p denote the four-momenta of the lepton and nucleon, respectively, and primes indicate the final states; λ stands for the helicity, and S is the polarization vector of the nucleon. We use light-cone coordinates and define the light-cone unit vectors as $\bar{n} = (1, 0, \vec{0}_\perp)$, $n = (0, 1, \vec{0}_\perp)$, and $n_\perp = (0, 0, \vec{n}_\perp)$, so a general four-vector can be decomposed as $A^\mu = A^+ \bar{n}^\mu + A^- n^\mu + A_\perp^\mu$, where $A^\pm = (A^0 \pm A^3)/\sqrt{2}$, and $A_\perp = (0, 0, \vec{A}_\perp)$. We work in the center-of-mass frame of $\gamma^* N$ and choose the nucleon's momentum as being in the z direction, so p and S are decomposed as

$$p^\mu = p^+ \bar{n}^\mu + \frac{M^2}{2p^+} n^\mu, \quad (2.4)$$

$$S^\mu = \lambda \frac{p^+}{M} \bar{n}^\mu + S_T^\mu - \lambda \frac{M}{2p^+} n^\mu. \quad (2.5)$$

The Bjorken variable is defined as $x_B = Q^2/(2p \cdot q)$, $q = -x_B p^+ \bar{n} + n Q^2/(2x_B p^+)$; we also define $y = (p \cdot q)/(p \cdot l)$.

The theoretical framework for inclusive DIS has been constructed in the following steps. First, we studied the

kinematics and obtained the general form of the hadronic tensor by applying the basic constraints from the general symmetry requirements such as Lorentz covariance, gauge invariance, parity conservation, and Hermiticity, e.g.,

$$q^\mu W_{\mu\nu}(q, p, S) = 0, \tag{2.6}$$

$$W_{\mu\nu}(\tilde{q}, \tilde{p}, -\tilde{S}) = W^{\mu\nu}(q, p, S), \tag{2.7}$$

$$W_{\mu\nu}^*(q, p, S) = W_{\nu\mu}(q, p, S), \tag{2.8}$$

where \tilde{A} denotes the results of A after space reflection, i.e., $\tilde{A}^\mu = A_\mu$. The general form of the hadronic tensor is given by the sum of a symmetric part and an antisymmetric part,

$$W_{\mu\nu}(q, p, S) = W_{\mu\nu}^{(S)}(q, p) + iW_{\mu\nu}^{(A)}(q, p, S), \tag{2.9}$$

where $W_{\mu\nu}^{(S)}(q, p)$ and $W_{\mu\nu}^{(A)}(q, p, S)$ are given by

$$W_{\mu\nu}^{(S)}(q, p) = 2(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2})F_1(x, Q^2) + \frac{1}{xQ^2}(q_\mu + 2xp_\mu)(q_\nu + 2xp_\nu)F_2(x, Q^2), \tag{2.10}$$

$$W_{\mu\nu}^{(A)}(q, p, S) = \frac{2M\varepsilon_{\mu\nu\rho\sigma}q^\rho}{p \cdot q} \times [S^\sigma g_1(x, Q^2) + (S^\sigma - \frac{S \cdot q}{p \cdot q}p^\sigma)g_2(x, Q^2)], \tag{2.11}$$

respectively. We found that the hadronic tensor is determined by four independent structure functions, F_1 , F_2 , g_1 , and g_2 , where the first two describe the unpolarized case, and the latter two are needed for polarized cases.

Our knowledge of one-dimensional imaging of the nucleon starts with the “intuitive parton model”, which is very nicely formulated, e.g., in Ref. [8]. Here, it was argued that, in a fast-moving frame, because of time dilation, quantum fluctuations such as vacuum polarizations can exist for quite a long time. In the infinite momentum frame, such fluctuations exist forever. In this case, a fast-moving nucleon can be viewed as a beam of free “partons”. The probability of the scattering of an electron with a nucleon is taken as the incoherent sum of that of the scattering with each individual parton, more precisely, as a convolution of the number density of partons in the nucleon with the probability of scattering with the parton, i.e.,

$$|\mathcal{M}(eN \rightarrow eX)|^2 = \sum_q \int dx f_q(x) |\hat{\mathcal{M}}(eq \rightarrow eq)|^2, \tag{2.12}$$

where $f_q(x)$ is the number density of partons of flavor q in the nucleon. In this way, we obtained the famous results [8]

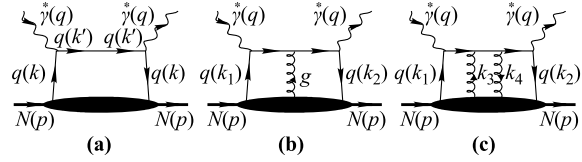


Fig. 1 Examples of the Feynman diagram series with multiple gluon scattering considered for $\gamma^* + N \rightarrow q + X$ with (a) $j = 0$, (b) $j = 1$ and (c) $j = 2$ gluons exchanged.

$$F_2(x, Q^2) = 2xF_1(x, Q^2) = \sum_q e_q^2 x f_q(x), \tag{2.13}$$

$$g_1(x, Q^2) = \sum_q e_q^2 \Delta f_q(x), \tag{2.14}$$

$$g_1(x, Q^2) + g_2(x, Q^2) = \sum_q e_q^2 x \delta f_q(x). \tag{2.15}$$

Here, we would like to point out that, with this intuitive parton model, we are doing nothing else but the impulse approximation that we often use in describing a collision process, where we make the following approximations:

- during the interaction of the electron with the parton, interactions between the partons are neglected;
- the electron interacts with only one single parton each time;
- the scatterings of the electron with different partons are added incoherently.

Although the physical picture of the intuitive model is very clear, and the model is elegant and practical, we are not satisfied with the formulation because it is partly qualitative or semiclassical; hence, it is not easy to control the accuracy. A proper formulation should be based on quantum field theory (QFT) and is obtained by starting with the Feynman diagram, as shown in Fig. 1(a). From this diagram, we obtain immediately that

$$W_{\mu\nu}^{(0)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \text{Tr}[\hat{H}_{\mu\nu}^{(0)}(k, q)\hat{\phi}^{(0)}(k, p, S)], \tag{2.16}$$

where k is the four-momentum of the parton.

$$\hat{H}_{\mu\nu}^{(0)}(q, k) = \gamma_\mu(\not{k} + \not{q})\gamma_\nu(2\pi)\delta_+((k + q)^2) \tag{2.17}$$

is a calculable hard part. The matrix element

$$\hat{\phi}^{(0)}(k, p, S) = \int d^4z e^{ik \cdot z} \langle p, S | \bar{\psi}(0)\psi(z) | p, S \rangle \tag{2.18}$$

is known as the quark–quark correlator and describes the structure of the nucleon. By taking the collinear approximation, i.e., taking $k \approx xp$, and neglecting the power-suppressed contributions, i.e., the $o(M/Q)$ terms, we obtain

$$W_{\mu\nu}^{(0)}(q, p) \approx \left[(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) + \frac{(q + 2xp)_\mu (q + 2xp)_\nu}{2xp \cdot q} \right] \times f_q(x). \quad (2.19)$$

This is exactly the same result as that obtained from Eq. (2.12) on the basis of the intuitive parton model. At the same time, we obtain the QFT operator expression of $f_q(x)$, defined via the quark–quark correlator given by Eq. (2.18), as

$$f_q(x) = \int \frac{dz^-}{2\pi} e^{ixp^+ z^-} \langle p | \bar{\psi}(0) \frac{\gamma^+}{2} \psi(z) | p \rangle. \quad (2.20)$$

By inserting the expanded expression for the field operator $\psi(z)$ in terms of the plane wave and the creation and/or annihilation operators, we see clearly that $f_q(x)$ is indeed the number density of partons in the nucleon. However, from this expression, we also immediately see a severe problem; i.e., this expression is *not* (local) gauge invariant! We understand that the physical quantity has to be gauge invariant and therefore have to find a solu-

tion for this.

The gauge-invariant formulation is obtained by taking into account the multiple gluon scattering shown by the diagram series in Figs. 1(a)–(c). This is clear because (local) gauge invariance implies the existence of a gauge interaction that needs to be taken into account. In this way, we obtain

$$W_{\mu\nu}(q, p, S) = \sum_{j=0}^{\infty} W_{\mu\nu}^{(j)}(q, p, S), \quad (2.21)$$

where $W_{\mu\nu}^{(j)}(q, p, S)$ represents the contribution from the diagram with exchange of j gluon(s). They are all expressed as a trace of a calculable hard part and a matrix element depending on the structure of the nucleon. E.g., corresponding to Fig. 1(b), we have $j = 1$, and $W_{\mu\nu}^{(1)}(q, p, S)$ is given by

$$W_{\mu\nu}^{(1)}(q, p, S) = \sum_{c=L,R} W_{\mu\nu}^{(1,c)}(q, p, S), \quad (2.22)$$

$$W_{\mu\nu}^{(1,c)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \text{Tr}[\hat{H}_{\mu\nu}^{(1,c)}(k_1, k_2, q) \hat{\phi}_\rho^{(1)}(k_1, k_2, p, S)], \quad (2.23)$$

$$\hat{\phi}_\rho^{(1)}(k_1, k_2, p, S) = \int d^4 z d^4 y e^{ik_1 z + (k_2 - k_1)y} \langle p, S | \bar{\psi}(0) A_\rho(y) \psi(z) | p, S \rangle, \quad (2.24)$$

where c in the superscript represents different cuts (left or right) in the diagram. Similarly, corresponding to Fig. 1(c), we have

$$W_{\mu\nu}^{(2)}(q, p, S) = \sum_{c=L,M,R} W_{\mu\nu}^{(2,c)}(q, p, S), \quad (2.25)$$

$$W_{\mu\nu}^{(2,c)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \text{Tr}[\hat{H}_{\mu\nu}^{(2,c)\rho\sigma}(k_1, k_2, k, q) \hat{\phi}_{\rho\sigma}^{(2)}(k_1, k_2, k, p, S)], \quad (2.26)$$

$$\hat{\phi}_{\rho\sigma}^{(2)}(k_1, k_2, k, p, S) = \int d^4 y d^4 y' d^4 z e^{ik_1 y + ik(y' - y) + ik_2(z - y')} \langle p, S | \bar{\psi}(0) g A_\rho(y) g A_\sigma(y') \psi(z) | p, S \rangle. \quad (2.27)$$

The matrix element is now a quark– j -gluon(s)–quark correlator. We also immediately see that none of these quark– j -gluon(s)–quark correlators is gauge invariant.

To obtain the gauge-invariant form, we need to ap-

ply the collinear expansion proposed in [9–11], which is carried out in the following four steps.

(1) Make Taylor expansions of all the hard parts at $k_i = x_i p$, e.g.,

$$\hat{H}_{\mu\nu}^{(0)}(k, q) = \hat{H}_{\mu\nu}^{(0)}(x) + \frac{\partial \hat{H}_{\mu\nu}^{(0)}(x)}{\partial k_\rho} \omega_\rho^{\rho'} k_{\rho'} + \frac{1}{2} \frac{\partial^2 \hat{H}_{\mu\nu}^{(0)}(x)}{\partial k_\rho \partial k_\sigma} \omega_\rho^{\rho'} k_{\rho'} \omega_\sigma^{\sigma'} k_{\sigma'} + \dots, \quad (2.28)$$

$$\hat{H}_{\mu\nu}^{(1,L)\rho}(k_1, k_2, q) = \hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2) + \frac{\partial \hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2)}{\partial k_{1\sigma}} \omega_\sigma^{\sigma'} k_{1\sigma'} + \frac{\partial \hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2)}{\partial k_{2\sigma}} \omega_\sigma^{\sigma'} k_{2\sigma'} + \dots, \quad (2.29)$$

and so on, where $\omega_\rho^{\rho'}$ is a projection operator defined by $\omega_\rho^{\rho'} \equiv g_\rho^{\rho'} - \bar{n}_\rho n^{\rho'}$.

(2) Decompose the gluon field into longitudinal and transverse components, i.e.,

$$A_\rho(y) = A^+(y) \bar{n}_\rho + \omega_\rho^{\rho'} A_{\rho'}(y). \quad (2.30)$$

(3) Apply the Ward identities, such as

$$\frac{\partial \hat{H}_{\mu\nu}^{(0)}(x)}{\partial k_\rho} = -\hat{H}_{\mu\nu}^{(1)\rho}(x, x), \quad (2.31)$$

$$\frac{\partial \hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2)}{\partial k_{1,\sigma}} = -\hat{H}_{\mu\nu}^{(2,L)\rho\sigma}(x_1, x_1, x_2)$$

$$-\hat{H}_{\mu\nu}^{(2,M)\rho\sigma}(x_1, x_1, x_2), \quad (2.32)$$

$$p_\rho \hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2) = \frac{H_{\mu\nu}^{(0)}(x_1)}{x_2 - x_1 - i\epsilon}, \quad (2.33)$$

$$p_\rho \hat{H}_{\mu\nu}^{(2,L)\rho\sigma}(x_1, x, x_2) = \frac{1}{x - x_1 - i\epsilon} H_{\mu\nu}^{(1,L)\sigma}(x_1, x_2), \quad (2.34)$$

$$p_\rho \hat{H}_{\mu\nu}^{(2,M)\rho\sigma}(x_1, x, x_2) = -\frac{1}{x_2 - x_1 - i\epsilon} H_{\mu\nu}^{(1,L)\sigma}(x_1, x_2)$$

$$-\frac{1}{x_1 - x + i\epsilon} H_{\mu\nu}^{(1,R)\sigma}(x, x_2). \quad (2.35)$$

(4) Sum up all the terms with the same hard part, and we obtain

$$W_{\mu\nu}(q, p, S) = \sum_j \tilde{W}_{\mu\nu}^{(j)}(q, p, S), \quad (2.36)$$

$$\tilde{W}_{\mu\nu}^{(0)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \text{Tr}[\hat{H}_{\mu\nu}^{(0)}(x) \hat{\Phi}^{(0)}(k, p, S)], \quad (2.37)$$

$$\tilde{W}_{\mu\nu}^{(1)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \sum_{c=L,R} \text{Tr}[\hat{H}_{\mu\nu}^{(1,c)\rho}(x_1, x_2) \omega_\rho^{\rho'} \hat{\Phi}_{\rho'}^{(1)}(k_1, k_2, p, S)], \quad (2.38)$$

$$\tilde{W}_{\mu\nu}^{(2)}(q, p, S) = \frac{1}{2\pi} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \sum_{c=L,R,M} \text{Tr}[\hat{H}_{\mu\nu}^{(2,c)\rho\sigma}(x_1, x_2, x) \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \hat{\Phi}_{\rho'\sigma'}^{(2)}(k_1, k_2, k, p, S)], \quad (2.39)$$

where $\hat{\Phi}^{(j)}$'s are the gauge-invariant unintegrated quark–quark and quark– j -gluon(s)–quark correlators given by

$$\hat{\Phi}^{(0)}(k, p, S) = \int d^4y e^{iky} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; y) \psi(y) | p, S \rangle, \quad (2.40)$$

$$\hat{\Phi}_\rho^{(1)}(k_1, k_2, p, S) = \int d^4y d^4z e^{ik_2z + ik_1(y-z)} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; z) D_\rho(z) \mathcal{L}(z; y) \psi(y) | p, S \rangle, \quad (2.41)$$

$$\begin{aligned} \hat{\Phi}_{\rho\sigma}^{(2)}(k_1, k_2, k, p, S) \\ = \int d^4y d^4y' d^4z e^{ik_1y + ik(y'-y) + ik_2(z-y')} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; y) D_\rho(y) \mathcal{L}(y; y') D_\sigma(y') \mathcal{L}(y'; z) \psi(z) | p, S \rangle. \end{aligned} \quad (2.42)$$

$D(y)$ is the covariant derivative and is defined as $D_\rho(y) = -i\partial_\rho + gA_\rho(y)$. The factor $\mathcal{L}(0; y)$ is obtained during the summation of different contributions with the same hard part and is given by

$$\mathcal{L}(0; y) = \mathcal{L}^\dagger(\infty; 0) \mathcal{L}(\infty; y), \quad (2.43)$$

$$\begin{aligned} \mathcal{L}(\infty; y) &= P e^{-ig \int_{y^-}^{\infty} d\xi^- A^+(y^+, \xi^-, \vec{y}_\perp)} \\ &= 1 - ig \int_{y^-}^{\infty} d\xi^- A^+(y^+, \xi^-, \vec{y}_\perp) + (-ig)^2 \int_{y^-}^{\infty} d\xi^- \int_{\xi^-}^{\infty} d\eta^- A^+(y^+, \xi^-, \vec{y}_\perp) A^+(y^+, \eta^-, \vec{y}_\perp) + \dots, \end{aligned} \quad (2.44)$$

where P stands for the path-ordered integral. $\mathcal{L}(0; y)$ is nothing else but the well-known gauge link, which makes the quark–quark or quark– j -gluon(s)–quark correlator, and thus the PDFs defined using them, gauge invariant.

In this way, we have constructed a theoretical framework for systematically calculating the contributions to the hadronic tensor at the leading order (LO) in pQCD but at leading as well as higher twists. The results are given in terms of the gauge-invariant parton distribution and correlation functions (generally referred to as PDFs).

We emphasize the following two further points derived

directly from these expressions.

First, we note that after collinear expansion, the hard parts contained in the expressions for $\tilde{W}_{\mu\nu}^{(j)}$'s, such as those given by Eqs. (2.37)–(2.39), are only functions of the longitudinal component x . They are independent of the other components of the parton momentum k . We can integrate over these components of k and simplify them to

$$\tilde{W}_{\mu\nu}^{(0)}(q, p, S) = \frac{1}{2\pi} \int p^+ dx \text{Tr}[\hat{H}_{\mu\nu}^{(0)}(x) \hat{\Phi}^{(0)}(x, p, S)], \quad (2.45)$$

$$\tilde{W}_{\mu\nu}^{(1)}(q, p, S) = \frac{1}{2\pi} \int p^+ dx_1 p^+ dx_2 \sum_{c=L,R} \text{Tr}[\hat{H}_{\mu\nu}^{(1,c)\rho}(x_1, x_2) \omega_\rho^{\rho'} \hat{\Phi}_{\rho'}^{(1)}(x_1, x_2, p, S)], \quad (2.46)$$

$$\tilde{W}_{\mu\nu}^{(2)}(q, p, S) = \frac{1}{2\pi} \int p^+ dx_1 p^+ dx_2 p^+ dx \sum_{c=L,R,M} \text{Tr}[\hat{H}_{\mu\nu}^{(2,c)\rho\sigma}(x_1, x_2, x) \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \hat{\Phi}_{\rho'\sigma'}^{(2)}(x_1, x_2, x, p, S)], \quad (2.47)$$

where the matrix elements $\hat{\Phi}$'s are given by

$$\hat{\Phi}^{(0)}(x, p, S) \equiv \int \frac{d^4k}{(2\pi)^4} \delta(k^+ - xp^+) \hat{\Phi}^{(0)}(k, p, S) = \int \frac{dy^-}{2\pi} e^{ixp^+y^-} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; y^-) \psi(y^-) | p, S \rangle, \quad (2.48)$$

$$\begin{aligned} \hat{\Phi}_\rho^{(1)}(x_1, x_2, p, S) &\equiv \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \delta(k_1^+ - x_1p^+) \delta(k_2^+ - x_2p^+) \hat{\Phi}^{(1)}(k_1, k_2, p, S) \\ &= \int \frac{dy^-}{2\pi} \frac{dz^-}{2\pi} e^{ix_2p^+z^- + ix_1p^+(y^- - z^-)} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; z^-) D_\rho(z^-) \mathcal{L}(z^-; y^-) \psi(y^-) | p, S \rangle, \end{aligned} \quad (2.49)$$

$$\begin{aligned} \hat{\Phi}_{\rho\sigma}^{(2)}(x_1, x_2, x, p, S) &\equiv \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \delta(k_1^+ - x_1p^+) \delta(k_2^+ - x_2p^+) \delta(k^+ - xp^+) \hat{\Phi}^{(2)}(k_1, k_2, k, p, S) \\ &= \int \frac{dy^-}{2\pi} \frac{dy'^-}{2\pi} \frac{dz^-}{2\pi} e^{ix_1p^+y^- + ix_2p^+(y'^- - y^-) + ix_2p^+(z^- - y'^-)} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; y^-) D_\rho(y^-) \mathcal{L}(y^-; y'^-) \\ &\quad \times D_\sigma(y'^-) \mathcal{L}(y'^-; z^-) \psi(z^-) | p, S \rangle. \end{aligned} \quad (2.50)$$

From these expressions, we see explicitly that only x_i dependences of the quark–quark and/or quark– j -gluon–quark correlators are involved. This means that only one-dimensional imaging of the nucleon is relevant in inclusive DIS.

Second, owing to the existence of the projection operator $\omega_\rho^{\rho'}$, the hard parts can be further simplified a great deal. They are given by

$$\hat{H}_{\mu\nu}^{(0)}(x) = \pi \hat{h}_{\mu\nu}^{(0)} \delta(x - x_B), \quad (2.51)$$

$$\hat{H}_{\mu\nu}^{(1,L)\rho}(x_1, x_2) \omega_\rho^{\rho'} = \frac{\pi}{2q \cdot p} \hat{h}_{\mu\nu}^{(1)\rho} \omega_\rho^{\rho'} \delta(x_1 - x_B), \quad (2.52)$$

$$\hat{H}_{\mu\nu}^{(2,L)\rho\sigma}(x_1, x_2, x) \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} = \frac{2\pi}{(2q \cdot p)^2} [\bar{n}^\rho \hat{h}_{\mu\nu}^{(1)\sigma} + \frac{\hat{N}_{\mu\nu}^{(2)\rho\sigma}}{x_2 - x_B - i\epsilon}] \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \delta(x_1 - x_B), \quad (2.53)$$

$$\hat{H}_{\mu\nu}^{(2,M)\rho\sigma}(x_1, x_2, x) \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} = \frac{2\pi}{(2q \cdot p)^2} \hat{h}_{\mu\nu}^{(2)\rho\sigma} \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \delta(x - x_B), \quad (2.54)$$

where $\hat{h}_{\mu\nu}^{(0)} = \gamma_\mu \not{x} \gamma_\nu / p^+$, $\hat{h}_{\mu\nu}^{(1)\rho} = \gamma_\mu \not{x} \gamma^\rho \not{x} \gamma_\nu$, $\hat{h}_{\mu\nu}^{(2)\rho\sigma} = p^+ \gamma_\mu \not{x} \gamma^\rho \not{x} \gamma^\sigma \not{x} \gamma_\nu / 2$, and $\hat{N}_{\mu\nu}^{(2)\rho\sigma} = q^- \gamma_\mu \gamma^\rho \not{x} \gamma^\sigma \gamma_\nu$ are matrices independent of x_i . We insert them into Eqs. (2.45)–(2.47) and obtain the simplified expressions for the hadronic tensor as

$$\tilde{W}_{\mu\nu}^{(0)}(q, p, S) = \frac{1}{2} \text{Tr}[\hat{h}_{\mu\nu}^{(0)} \hat{\Phi}^{(0)}(x_B)], \quad (2.55)$$

$$\tilde{W}_{\mu\nu}^{(1,L)}(q, p, S) = \frac{1}{4q \cdot p} \text{Tr}[\hat{h}_{\mu\nu}^{(1)\rho} \omega_\rho^{\rho'} \hat{\phi}_{\rho'}^{(1)}(x_B)], \quad (2.56)$$

$$\tilde{W}_{\mu\nu}^{(2,L)}(q, p, S) = \frac{1}{(2q \cdot p)^2} \left\{ \text{Tr}[\hat{h}_{\mu\nu}^{(1)\rho} \omega_\rho^{\rho'} \hat{\phi}_{\rho'}^{(2L)}(x_B)] + \text{Tr}[\hat{N}_{\mu\nu}^{(2)\rho\sigma} \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \hat{\phi}_{\rho'\sigma'}^{(2L)}(x_B)] \right\}, \quad (2.57)$$

$$\tilde{W}_{\mu\nu}^{(2,M)}(q, p, S) = \frac{1}{(2q \cdot p)^2} \text{Tr}[\hat{h}_{\mu\nu}^{(2)\rho\sigma} \omega_\rho^{\rho'} \omega_\sigma^{\sigma'} \hat{\phi}_{\rho'\sigma'}^{(2M)}(x_B)], \quad (2.58)$$

where, for explicitness, we omit p, S in the arguments of the correlators. These correlators are defined as

$$\hat{\phi}_\rho^{(1)}(x_1) \equiv \int dx_2 \hat{\Phi}_\rho^{(1)}(x_1, x_2, p, S) = \int \frac{p^+ dy^-}{2\pi} e^{ixp^+y^-} \langle p, S | \bar{\psi}(0) D_\rho(0) \mathcal{L}(0; y^-) \psi(y^-) | p, S \rangle, \quad (2.59)$$

$$\hat{\phi}_{\rho\sigma}^{(2M)}(x) \equiv \int dx_1 dx_2 \hat{\Phi}_{\rho\sigma}^{(2M)}(x_1, x_2, x, p, S) = \int \frac{p^+ dy^-}{2\pi} e^{ixp^+y^-} \langle p, S | \bar{\psi}(0) D_\rho(0) \mathcal{L}(0; y^-) D_\sigma(y^-) \psi(y^-) | p, S \rangle, \quad (2.60)$$

$$\begin{aligned} \hat{\phi}_{\rho\sigma}^{(2L)}(x_1) &\equiv \int \frac{dx dx_2}{x_2 - x_1 - i\epsilon} \hat{\Phi}_{\rho\sigma}^{(2L)}(x_1, x_2, x, p, S) \\ &= \int \frac{dx_2}{x_2 - x - i\epsilon} \frac{p^+ dy^-}{2\pi} \frac{p^+ dz^-}{2\pi} e^{ixp^+y^- + i(x_2 - x)p^+z^-} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; z^-) D_\rho(z^-) D_\sigma(z^-) \mathcal{L}(z^-; y^-) \psi(y^-) | p, S \rangle, \end{aligned} \quad (2.61)$$

$$\hat{\phi}_\sigma^{(2L)}(x_1) \equiv \int dx dx_2 \bar{n}^\rho \hat{\Phi}_{\rho\sigma}^{(2L)}(x_1, x_2, x, p, S) = \int \frac{p^+ dy^-}{2\pi} e^{ixp^+y^-} \langle p, S | \bar{\psi}(0) D^-(0) D_\sigma(0) \mathcal{L}(0; y^-) \psi(y^-) | p, S \rangle. \quad (2.62)$$

We see explicitly that all the relevant components of the quark– j -gluon–quark correlators depend only on one single parton momentum. This means that only quark– j -gluon–quark correlators that depend on one single parton momentum are relevant in inclusive DIS.

We emphasize that the results given by Eqs. (2.37)–(2.39) and their simplified forms given by Eqs. (2.55)–(2.62), including the gauge links, are derived in the collinear expansion. They are just the sum of the contributions from the diagram series shown in Fig. 1. This formalism provides a basic theoretical framework for describing inclusive DIS at LO pQCD but at leading and higher twist contributions in terms of gauge-invariant PDFs.

The PDFs are defined in terms of QFT operators via these quark–quark correlators by expanding them in terms of γ matrices and basic Lorentz covariants. For example, for $\hat{\Phi}^{(0)}(x, p, S)$, we have

$$\hat{\Phi}^{(0)}(x) = \frac{1}{2} \left[\Phi_S^{(0)}(x) + i\gamma_5 \Phi_{PS}^{(0)}(x) + \gamma^\alpha \Phi_\alpha^{(0)}(x) + \gamma_5 \gamma^\alpha \tilde{\Phi}_\alpha^{(0)}(x) + i\sigma^{\alpha\beta} \gamma_5 \Phi_{T\alpha\beta}^{(0)}(x) \right]. \quad (2.63)$$

The basic Lorentz covariants are constructed from p_α , n_α , S_α , and $\varepsilon_{\alpha\beta\rho\sigma}$. We obtain the following general results:

$$\Phi_S^{(0)}(x) = M e(x), \quad (2.64)$$

$$\Phi_{PS}^{(0)}(x) = \lambda M e_L(x), \quad (2.65)$$

$$\Phi_\alpha^{(0)}(x) = p^+ \bar{n}_\alpha f_1(x) + M \varepsilon_{\perp\alpha\rho} S_T^\rho f_T(x) + \frac{M^2}{p^+} n_\alpha f_3(x), \quad (2.66)$$

$$\tilde{\Phi}_\alpha^{(0)}(x) = \lambda p^+ \bar{n}_\alpha g_{1L}(x) + M S_{T\alpha} g_T(x) + \lambda \frac{M^2}{p^+} n_\alpha g_{3L}(x), \quad (2.67)$$

$$\Phi_{T\rho\alpha}^{(0)}(x) = p^+ \bar{n}_{[\rho} S_{T\alpha]} h_{1T}(x) - M \varepsilon_{\perp\rho\alpha} h(x) + \lambda M \bar{n}_{[\rho} n_{\alpha]} h_L(x) + \frac{M^2}{p^+} n_{[\rho} S_{T\alpha]} h_{3T}(x), \quad (2.68)$$

where $\varepsilon_{\perp\rho\sigma} \equiv \varepsilon_{\alpha\beta\rho\sigma} \bar{n}^\alpha n^\beta$, and the anticommutation symbol $A^{[\rho} B^{\sigma]} \equiv A^\rho B^\sigma - A^\sigma B^\rho$. The scalar functions $f(x)$, $g(x)$, and $h(x)$ are the corresponding PDFs. There are a total of 12 such functions; 3 of them, i.e., $f_1(x)$, $g_{1L}(x)$, and $h_{1T}(x)$, contribute at leading twist and have clear probability interpretations, whereas 6 of them contribute at twist-3, and the other 3 contribute at twist-4. We further note that in fact the three time-reversal odd terms $e_L(x)$, $f_T(x)$, and $h(x)$ vanish in the one-dimensional case. We keep them in Eqs. (2.65)–(2.68) for later comparison with fragmentation functions.

We also see that the PDFs involved here are all scale

independent. This is because we have so far considered only the LO pQCD contributions, i.e., the tree diagrams. To go to higher orders of pQCD, we take the loop diagrams, gluon radiation, and so on into account. After proper handling of these contributions, we obtain the factorized form [6], in which the PDFs acquire the scale (Q) dependence governed by the QCD evolution equations. In practice, PDFs are parameterized and are given in the PDF library (PDFlib).

In summary, to study one-dimensional imaging of the nucleon with inclusive DIS, we take the following steps.

- General symmetry analysis leads to the general form of the hadronic tensor and/or the cross section in terms of four independent structure functions.
- The parton model without QCD interaction leads to LO in pQCD and leading twist results for the structure functions in terms of Q -independent PDFs without (local) gauge invariance.
- The parton model with QCD multiple gluon scattering after collinear expansion leads to LO in pQCD and leading and higher twist contributions in terms of Q -independent but gauge-invariant PDFs.
- The parton model with QCD multiple gluon scattering and loop diagram contributions after collinear approximation, regularization, and renormalization leads to leading and higher-order pQCD and leading twist contributions in factorized forms in terms of Q -evolved and gauge-invariant PDFs.

In the following, we will follow these four steps and summarize what we have achieved in the three-dimensional case. As in Ref. [2], we will focus mainly on the theoretical framework at LO pQCD but consistently take leading and higher twist contributions into account. Before that, we emphasize the following two historical developments that may be helpful in constructing the theoretical framework for the TMD case.

First, as mentioned, the study of three-dimensional imaging of the nucleon was triggered by the experimental observation of single-spin left–right asymmetries (SSAs) in the inclusive hadron–hadron collision with a transversely polarized projectile or target. It was known that pQCD leads to negligibly small asymmetry for the hard part [12], but the observed asymmetry can be as large as 40% [13]. The hunt for such large asymmetries has lasted for decades, with the following milestones:

- In 1991, Sivers introduced [14] the asymmetric quark distribution in a transversely polarized nucleon, which is now known as the Sivers function.

- In 1993, Boros, Liang, and Meng proposed [15] a phenomenological model that provides an intuitive physical picture showing that the asymmetry arises from the orbital angular momenta of quarks and what they called the surface effect caused by the initial- or final-state interactions.
- In 1993, Collins published [16] his proof that the Sivers function has to vanish because of parity and time-reversal invariance.
- In 2002, Brodsky, Hwang, and Schmidt calculated [17] the SSA for semi-inclusive deep inelastic scattering (SIDIS) using an explicit example in which they took the orbital angular momentum of quarks and multiple gluon scattering into account.
- In 2002, immediately after [17], Collins pointed out [18] that multiple gluon scattering is contained in the gauge link and that the conclusion of his proof in 1993 was incorrect because he did not consider the gauge link. He further showed that by taking the gauge link into account, the same proof leads to the conclusion that the Sivers function for DIS and that for the Drell–Yan process have opposite signs. Belitsky, Ji, and Yuan resolved [19, 20] the problem of defining the gauge link for a TMD parton density in the light-cone gauge where the gauge potential does not vanish asymptotically.

The second historical development concerns the study of azimuthal asymmetry in SIDIS. Georgi and Politzer showed in 1977 [21] that final-state gluon radiation leads

to azimuthal asymmetries and could be used as a “clean test” of pQCD. However, soon after, in 1978, Cahn showed [22] that similar asymmetries can also be obtained if one includes the intrinsic transverse momenta of partons. The latter (now called the Cahn effect), although power suppressed at higher twist, can be quite significant and cannot be neglected, as the values of the asymmetries themselves are usually not very large.

The following two points are particularly valuable lessons that we learned from these historical developments: when studying TMDs,

- it is important to take the gauge link into account;
- higher twist effects can be important.

Both of these points demand that, to describe SIDIS in terms of TMDs, we need the proper QFT formulation rather than the intuitive parton model.

3 TMDs defined via quark–quark correlator

The TMD PDFs of quarks are defined via the TMD quark–quark correlator $\Phi^{(0)}(x, k_{\perp}; p, S)$ given by Eq. (2.40) (after integration over k^{-}). A systematical study is given in Ref. [23], and a very comprehensive treatment can also be found in Ref. [24]. Here, we first expand it in terms of γ matrices and obtain a scalar, a pseudoscalar, a vector, an axial vector, and an antisymmetric and space-reflection odd tensor part, i.e.,

$$\hat{\Phi}^{(0)}(x, k_{\perp}; p, S) = \frac{1}{2} [\Phi_S^{(0)}(x, k_{\perp}; p, S) + i\gamma_5 \Phi_{PS}^{(0)}(x, k_{\perp}; p, S) + \gamma^{\alpha} \Phi_{\alpha}^{(0)}(x, k_{\perp}; p, S) + \gamma_5 \gamma^{\alpha} \tilde{\Phi}_{\alpha}^{(0)}(x, k_{\perp}; p, S) + i\sigma^{\alpha\beta} \gamma_5 \Phi_{T\alpha\beta}^{(0)}(x, k_{\perp}; p, S)]. \quad (3.1)$$

The operator expressions of these coefficients are given by the traces of the quark–quark correlator with the corresponding Dirac matrices. For example, for the vector component, we have

$$\Phi_{\alpha}^{(0)}(x, k_{\perp}; p, S) = \frac{1}{2} \text{Tr} [\gamma_{\alpha} \hat{\Phi}^{(0)}(x, k_{\perp}; p, S)] = \int dz^{-} d^2 z_{\perp} e^{i(xp^{+}z^{-} - \vec{k}_{\perp} \cdot \vec{z}_{\perp})} \langle p, S | \bar{\psi}(0) \mathcal{L}(0; z) \frac{\gamma_{\alpha}}{2} \psi(z) | p, S \rangle. \quad (3.2)$$

We then analyze the Lorentz structure of each part by expressing it in terms of possible “basic Lorentz covariants” and scalar functions. From $\hat{\Phi}^{(0)}(x, k_{\perp}; p, S)$, we obtain the results as [23]

$$\Phi_S^{(0)}(x, k_{\perp}; p, S) = M \left[e(x, k_{\perp}) - \frac{\varepsilon_{\perp\rho\sigma} k_{\perp}^{\rho} S_T^{\sigma}}{M} e_T^{\perp}(x, k_{\perp}) \right], \quad (3.3)$$

$$\Phi_{PS}^{(0)}(x, k_{\perp}; p, S) = M \left[\lambda e_L(x, k_{\perp}) - \frac{k_{\perp} \cdot S_T}{M} e_T(x, k_{\perp}) \right], \quad (3.4)$$

$$\Phi_{\alpha}^{(0)}(x, k_{\perp}; p, S) = p^{+} \bar{n}_{\alpha} \left[f_1(x, k_{\perp}) - \frac{\varepsilon_{\perp\rho\sigma} k_{\perp}^{\rho} S_T^{\sigma}}{M} f_{1T}^{\perp}(x, k_{\perp}) \right] + k_{\perp\alpha} \left[f^{\perp}(x, k_{\perp}) - \frac{\varepsilon_{\perp\rho\sigma} k_{\perp}^{\rho} S_T^{\sigma}}{M} f_T^{\perp 1}(x, k_{\perp}) \right] + \varepsilon_{\perp\alpha\rho} k_{\perp}^{\rho} \left[\lambda f_L^{\perp}(x, k_{\perp}) - \frac{k_{\perp} \cdot S_T}{M} f_T^{\perp 2}(x, k_{\perp}) \right] + \frac{M^2}{p^{+}} n_{\alpha} \left[f_3(x, k_{\perp}) - \frac{\varepsilon_{\perp\rho\sigma} k_{\perp}^{\rho} S_T^{\sigma}}{M} f_{3T}^{\perp}(x, k_{\perp}) \right], \quad (3.5)$$

$$\tilde{\Phi}_{\alpha}^{(0)}(x, k_{\perp}; p, S) = p^{+} \bar{n}_{\alpha} \left[\lambda g_{1L}(x, k_{\perp}) - \frac{k_{\perp} \cdot S_T}{M} g_{1T}^{\perp}(x, k_{\perp}) \right] + M S_{T\alpha} g_T^{\prime}(x, k_{\perp}) - \varepsilon_{\perp\alpha\beta} k_{\perp}^{\beta} g^{\perp}(x, k_{\perp})$$

$$+k_{\perp\alpha} \left[\lambda g_L^\perp(x, k_\perp) - \frac{k_\perp \cdot S_T}{M} g_T^\perp(x, k_\perp) \right] + \frac{M^2}{p^+} n_\alpha \left[\lambda g_{3L}(x, k_\perp) - \frac{k_\perp \cdot S_T}{M} g_{3T}(x, k_\perp) \right], \quad (3.6)$$

$$\begin{aligned} \Phi_{T\rho\alpha}^{(0)}(x, k_\perp; p, S) &= p^+ \bar{n}_{[\rho} S_{T\alpha]} h_{1T}(x, k_\perp) - \frac{p^+ \bar{n}_{[\rho} \varepsilon_{\perp\alpha]\beta} k_\perp^\beta}{M} h_1^\perp(x, k_\perp) + \frac{p^+ \bar{n}_{[\rho} k_{\perp\alpha]}}{M} \left[\lambda h_{1L}^\perp(x, k_\perp) - \frac{k_\perp \cdot S_T}{M} h_{1T}^\perp(x, k_\perp) \right] \\ &+ S_{T[\rho} k_{\perp\alpha]} h_T^\perp(x, k_\perp) - M \varepsilon_{\perp\rho\alpha} h(x, k_\perp) + M \bar{n}_{[\rho} n_{\alpha]} \left[\lambda h_L(x, k_\perp) - \frac{k_\perp \cdot S_T}{M} h_T(x, k_\perp) \right] \\ &+ \frac{M^2}{p^+} \left\{ n_{[\rho} S_{T\alpha]} h_{3T}(x, k_\perp) + \frac{n_{[\rho} k_{\perp\alpha]}}{M} \left[\lambda h_{3L}^\perp(x, k_\perp) - \frac{k_\perp \cdot S_T}{M} h_{3T}^\perp(x, k_\perp) \right] - \frac{n_{[\rho} \varepsilon_{\perp\alpha]\beta} k_\perp^\beta}{M} h_3^\perp(x, k_\perp) \right\}. \end{aligned} \quad (3.7)$$

These scalar functions are known as TMD PDFs. There are a total of 32 such TMD PDFs. Among them, 8 contribute at leading twist, and they all have clear probability interpretations such as the number density $f_1(x, k_\perp)$, helicity distribution $g_{1L}(x, k_\perp)$, transversity $h_{1T}(x, k_\perp)$, Sivers function $f_{1T}^\perp(x, k_\perp)$, and Boer–Mulders function $h_1^\perp(x, k_\perp)$; 16 contribute at twist-3, and the other 8 contribute at twist-4. We emphasize that they are all scalar functions of x and k_\perp ; i.e., they depend on x and k_\perp^2 .

If we integrate over d^2k_\perp , terms in which the basic Lorentz covariants are odd in k_\perp vanish. Eqs. (3.3)–(3.7) just reduce to the corresponding Eqs. (2.64)–(2.68). At the leading twist, only 3 of the 8 survive: the number density $f_1(x)$, helicity distribution $g_{1L}(x)$, and transversity $h_{1T}(x)$.

We show the leading twist and twist-3 TMD PDFs in Tables 1 and 2, respectively. In these tables, we show also the results for $\mathcal{L} = 1$, i.e., if we neglect multiple gluon scattering and simply take a nucleon as an ideal gas system consisting of quarks and antiquarks (see, e.g., [24]). We also note that the conventions used here have the following systematics: f , g , and h represent unpolarized, longitudinally polarized, and transversely polarized quarks, respectively; the subscript L or T stands for longitudinally or transversely polarized nucleons, respectively; the subscript 1 indicates leading twist, no number indicates twist-3, and subscript 3 indicates twist-4; the symbol \perp in the superscript denotes that the correspond-

ing basic Lorentz covariant is k_\perp dependent.

Higher twist TMD PDFs are also defined via quark– j -gluon(s)–quark correlators such as those given by Eqs. (2.59)–(2.62). Many of them, however, are not independent, as they are related to those defined via the quark–quark correlator through the QCD equation of motion, $\gamma \cdot D(z)\psi(z) = 0$. We can obtain relations such as

$$\begin{aligned} x\Phi_{\perp\rho}^{(0)}(x, k_\perp; p, S) &= -\frac{n^\alpha}{p^+} \left[\text{Re}\varphi_{\alpha\rho}^{(1)}(x, k_\perp; p, S) \right. \\ &\quad \left. + \varepsilon_{\perp\rho}{}^\sigma \text{Im}\tilde{\varphi}_{\alpha\sigma}^{(1)}(x, k_\perp; p, S) \right], \end{aligned} \quad (3.8)$$

$$\begin{aligned} x\tilde{\Phi}_{\perp\rho}^{(0)}(x, k_\perp; p, S) &= -\frac{n^\alpha}{p^+} \left[\text{Re}\tilde{\varphi}_{\alpha\rho}^{(1)}(x, k_\perp; p, S) \right. \\ &\quad \left. + \varepsilon_{\perp\rho}{}^\sigma \text{Im}\varphi_{\alpha\sigma}^{(1)}(x, k_\perp; p, S) \right]. \end{aligned} \quad (3.9)$$

It is interesting to see that [35], although it is not generally proved, all the twist-3 TMD PDFs that are defined via the quark–gluon–quark correlator $\varphi_\rho^{(1)}$ and involved in SIDIS are replaced by those defined via the quark–quark correlator $\Phi^{(0)}$.

We emphasize that fragmentation is just conjugate to parton distribution. A systematic study of the general structure of the fragmentation function (FF) defined via the corresponding quark–quark correlator is presented in Ref. [26]. We should have one-to-one correspondence between TMD PDFs and TMD FFs. E.g., corresponding to the quark–quark correlator $\Phi^{(0)}(k, p, S)$ given by Eq. (2.40) and the expanded form in Eq. (3.1), we have

Table 1 The 8 leading twist TMD PDFs defined via the quark–quark correlator. A \times means that the corresponding term disappears upon integrating the quark–quark correlator over d^2k_\perp .

Quark polarization	Nucleon polarization	TMD PDFs	if $\mathcal{L} = 1$	integrated over \vec{k}_\perp	Name
U	U	$f_1(x, k_\perp)$		$f_1(x)$	Number density
	T	$f_{1T}^\perp(x, k_\perp)$	0	\times	Sivers function
L	L	$g_{1L}(x, k_\perp)$		$g_{1L}(x)$	Helicity distribution
	T	$g_{1T}^\perp(x, k_\perp)$		\times	Worm-gear/Trans-helicity distribution
T	U	$h_1^\perp(x, k_\perp)$	0	\times	Boer–Mulders function
	$T(\parallel)$	$h_{1T}(x, k_\perp)$		$h_{1T}(x)$	Transversity distribution
	$T(\perp)$	$h_{1T}^\perp(x, k_\perp)$			Pretzelocity
	L	$h_{1L}^\perp(x, k_\perp)$		\times	Worm-gear/longi-transversity

Table 2 The 16 twist-3 TMD PDFs defined via the quark-quark correlator. A \times means that the corresponding term disappears upon integrating the quark-quark correlator over d^2k_\perp .

Quark polarization	Nucleon polarization	TMD PDFs			if $\mathcal{L} = 1$	Integrated over \vec{k}_\perp		
U	U	$e(x, k_\perp)$, $f^\perp(x, k_\perp)$	0 , $f_1(x, k_\perp)/x$	0 , $f_1(x, k_\perp)/x$	$e(x)$, \times	\times	\times	\times
	T	$e_T^\perp(x, k_\perp)$, $f_T^{\perp 1}(x, k_\perp)$, $f_T^{\perp 2}(x, k_\perp)$	0 , 0 , 0	0 , 0 , 0	\times , \times , \times	\times	\times	\times
L	L	$e_L(x, k_\perp)$, $g_L^\perp(x, k_\perp)$	0 , $g_1(x, k_\perp)/x$	0 , $g_1(x, k_\perp)/x$	\times , \times	\times	\times	\times
	T	$e_T(x, k_\perp)$, $g_T^\perp(x, k_\perp)$, $g_T^\perp(x, k_\perp)$	0 , 0 , $g_{1T}(x, k_\perp)/x$	0 , 0 , $g_{1T}(x, k_\perp)/x$	\times , $g_T(x)$	\times	$g_T(x)$	\times
T	U	$h(x, k_\perp)$	0	0	\times	\times	\times	\times
	$T(\parallel)$	$h_T^\perp(x, k_\perp)$	$h_{1T}^\perp(x, k_\perp)/x$	$h_{1T}^\perp(x, k_\perp)/x$	\times	\times	\times	\times
	$T(\perp)$	$h_T(x, k_\perp)$	$h_{1T}(x, k_\perp)/x + k_\perp^2 h_{1T}^\perp(x, k_\perp)/M^2 x$	$h_{1T}(x, k_\perp)/x + k_\perp^2 h_{1T}^\perp(x, k_\perp)/M^2 x$	\times	\times	\times	\times
	L	$h_L(x, k_\perp)$	$k_\perp^2 h_{1L}^\perp(x, k_\perp)/M^2 x$	$k_\perp^2 h_{1L}^\perp(x, k_\perp)/M^2 x$	$h_L(x)$	\times	\times	\times
U	L	$f_L^\perp(x, k_\perp)$	0	0	\times	\times	\times	\times
L	U	$g^\perp(x, k_\perp)$	0	0	\times	\times	\times	\times

$$\hat{\Xi}^{(0)}(k_F, p, S) = \frac{1}{2\pi} \sum_X \int d^4\xi e^{-ik_F\xi} \langle 0 | \mathcal{L}^\dagger(0, \infty) \psi(0) | hX \rangle \langle hX | \bar{\psi}(\xi) \mathcal{L}(\xi, \infty) | 0 \rangle, \quad (3.10)$$

$$\begin{aligned} \hat{\Xi}^{(0)}(z, k_{F\perp}; p, S) = & \frac{1}{2} [\Xi_S^{(0)}(z, k_{F\perp}; p, S) + i\gamma_5 \Xi_{PS}^{(0)}(z, k_{F\perp}; p, S) + \gamma^\alpha \Xi_\alpha^{(0)}(z, k_{F\perp}; p, S) \\ & + \gamma_5 \gamma^\alpha \tilde{\Xi}_\alpha^{(0)}(z, k_{F\perp}; p, S) + i\sigma^{\alpha\beta} \gamma_5 \Xi_{T\alpha\beta}^{(0)}(z, k_{F\perp}; p, S)]. \end{aligned} \quad (3.11)$$

For a spin-1/2 hadron, we have perfect one-to-one correspondence to those given by Eqs. (3.3)–(3.7) for parton distributions in the nucleon, i.e.,

$$z\Xi_S^{(0)}(z, k_{F\perp}; p, S) = M \left[E(z, k_{F\perp}) + \frac{\varepsilon_{\perp\rho\sigma} k_{F\perp}^\rho S_T^\sigma}{M} E_T^\perp(z, k_{F\perp}) \right], \quad (3.12)$$

$$z\Xi_{PS}^{(0)}(z, k_{F\perp}; p, S) = M \left[\lambda E_L(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} E_T(z, k_{F\perp}) \right], \quad (3.13)$$

$$\begin{aligned} z\Xi_\alpha^{(0)}(z, k_{F\perp}; p, S) = & p^+ \bar{n}_\alpha \left[D_1(z, k_{F\perp}) + \frac{\varepsilon_{\perp\rho\sigma} k_{F\perp}^\rho S_T^\sigma}{M} D_{1T}^\perp(z, k_{F\perp}) \right] + k_{F\perp\alpha} D^\perp(z, k_{F\perp}) + M \varepsilon_{\perp\alpha\rho} S_T^\rho D_T(z, k_{F\perp}), \\ & + \varepsilon_{\perp\alpha\rho} k_{F\perp}^\rho \left[\lambda D_L^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} D_T^\perp(z, k_{F\perp}) \right] + \frac{M^2}{p^+} n_\alpha \left[D_3(z, k_{F\perp}) + \frac{\varepsilon_{\perp\rho\sigma} k_{F\perp}^\rho S_T^\sigma}{M} D_{3T}^\perp(z, k_{F\perp}) \right], \end{aligned} \quad (3.14)$$

$$\begin{aligned} z\tilde{\Xi}_\alpha^{(0)}(z, k_{F\perp}; p, S) = & p^+ \bar{n}_\alpha \left[\lambda G_{1L}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} G_{1T}^\perp(z, k_{F\perp}) \right] + M S_{T\alpha} G_T(z, k_{F\perp}) + \varepsilon_{\perp\alpha\beta} k_{F\perp}^\beta G^\perp(z, k_{F\perp}) \\ & + k_{F\perp\alpha} \left[\lambda G_L^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} G_T^\perp(z, k_{F\perp}) \right] + \frac{M^2}{p^+} n_\alpha \left[\lambda G_{3L}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} G_{3T}^\perp(z, k_{F\perp}) \right], \end{aligned} \quad (3.15)$$

$$\begin{aligned} z\Xi_{T\rho\alpha}^{(0)}(z, k_{F\perp}; p, S) = & p^+ \bar{n}_{[\rho} S_{T\alpha]} H_{1T}(z, k_{F\perp}) + \frac{p^+ \bar{n}_{[\rho} \varepsilon_{\perp\alpha]\beta} k_{F\perp}^\beta}{M} H_1^\perp(z, k_{F\perp}) \\ & + \frac{p^+ \bar{n}_{[\rho} k_{F\perp\alpha]}}{M} \left[\lambda H_{1L}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} H_{1T}^\perp(z, k_{F\perp}) \right] + S_{T[\rho} k_{F\perp\alpha]} H_T^\perp(z, k_{F\perp}) + M \varepsilon_{\perp\rho\alpha} H(z, k_{F\perp}) \\ & + \bar{n}_{[\rho} n_{\alpha]} \left[M \lambda H_L(z, k_{F\perp}) + k_{F\perp} \cdot S_T H_T^{\perp 1}(z, k_{F\perp}) \right] + \frac{M^2}{p^+} \left\{ n_{[\rho} S_{T\alpha]} H_{3T}(z, k_{F\perp}) + \frac{n_{[\rho} \varepsilon_{\perp\alpha]\beta} k_{F\perp}^\beta}{M} H_3^\perp(z, k_{F\perp}) \right. \\ & \left. + \frac{n_{[\rho} k_{F\perp\alpha]}}{M} \left[\lambda H_{3L}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_T}{M} H_{3T}^\perp(z, k_{F\perp}) \right] \right\}. \end{aligned} \quad (3.16)$$

Comparing them with the results given by Eqs. (3.3)–(3.7), we see clearly the one-to-one correspondence between the FFs and PDFs. As an example, we show the eight leading twist components in Table 3. We do not show the results for $\mathcal{L} = 1$ for FFs. This is because even

if we neglect the multiple gluon scattering that leads to the gauge link, final-state interactions can still exist between h and X . In this case, time-reversal invariance does not lead to zero results for the T-odd amplitudes.

For spin-1 hadrons, the polarization is described by

the polarization vector S and also the polarization tensor T (see, e.g., [25] and [26]). The tensor polarization part has five independent components. They are given by a Lorentz scalar S_{LL} , a Lorentz vector $S_{LT}^\mu = (0, S_{LT}^x, S_{LT}^y, 0)$, and a Lorentz tensor $S_{TT}^{\mu\nu}$, which has two independent nonzero components, S_{TT}^{xx} and S_{TT}^{xy} , in the rest frame of the hadron. These polarization parameters can be related to the probabilities for particles in different spin states [25]. In this case, the TMD quark-quark correlator $\hat{\Xi}^{(0)}(z, k_{F\perp}; p, S)$ is decomposed

into a spin-independent part, a vector-polarization-dependent part, and a tensor-polarization-dependent part; i.e., $\hat{\Xi}^{(0)}(z, k_{F\perp}; p, S) = \hat{\Xi}^{U(0)}(z, k_{F\perp}; p, S) + \hat{\Xi}^{V(0)}(z, k_{F\perp}; p, S) + \hat{\Xi}^{T(0)}(z, k_{F\perp}; p, S)$. The spin-independent and vector-polarization-dependent part $\hat{\Xi}^{U+V(0)}(z, k_{F\perp}; p, S)$ takes exactly the same decomposition as that for the spin-1/2 hadron given by Eqs. (3.12)–(3.16). The tensor-polarization-dependent part is presented in Ref. [26] and is given by

Table 3 The 8 leading twist TMD FFs for spin-1/2 hadrons defined via the quark-quark correlator. A \times means that the corresponding term disappears upon integrating the quark-quark correlator over $d^2k_{F\perp}$.

Quark polarization	Hadron polarization	TMD FFs	Integrated over $\vec{k}_{F\perp}$	Name
U	U	$D_1(z, k_{F\perp})$	$D_1(z)$	Number density
	T	$D_{1T}^\perp(z, k_{F\perp})$	\times	
L	L	$G_{1L\alpha}(z, k_{F\perp})$	$G_{1L}(z)$	Spin transfer (longitudinal)
	T	$G_{1T}^\perp(z, k_{F\perp})$	\times	
T	U	$H_1^\perp(z, k_{F\perp})$	\times	Collins function
	$T(\parallel)$	$H_{1T}(z, k_{F\perp})$	$H_{1T}(z)$	Spin transfer (transverse)
	$T(\perp)$	$H_{1T}^\perp(z, k_{F\perp})$	\times	
	L	$H_{1L}^\perp(z, k_{F\perp})$	\times	

$$z\Xi_S^{T(0)}(z, k_{F\perp}; p, S) = M \left[S_{LL} E_{LL}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} E_{LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} E_{TT}^\perp(z, k_{F\perp}) \right], \quad (3.17)$$

$$z\Xi_{PS}^{T(0)}(z, k_{F\perp}; p, S) = M \left[\frac{\epsilon^{k_F S_{LT}}}{M} E_{LT}^{\prime\perp}(z, k_{F\perp}) + \frac{\epsilon_{\perp k_F \alpha} k_{F\beta} S_{TT}^{\alpha\beta}}{M^2} E_{TT}^{\prime\perp}(z, k_{F\perp}) \right], \quad (3.18)$$

$$\begin{aligned} z\Xi_\alpha^{T(0)}(z, k_{F\perp}; p, S) &= p^+ \bar{n}_\alpha \left[S_{LL} D_{1LL}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} D_{1LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} D_{1TT}^\perp(z, k_{F\perp}) \right] \\ &+ k_{F\perp\alpha} \left[S_{LL} D_{LL}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} D_{LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} D_{TT}^\perp(z, k_{F\perp}) \right] \\ &+ M S_{LT\alpha} D_{LT}(z, k_{F\perp}) + k_{F\perp}^\rho S_{TT\rho\alpha} D_{TT}^{\prime\perp}(z, k_{F\perp}) \\ &+ \frac{M^2}{p^+} n_\alpha \left[S_{LL} D_{3LL}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} D_{3LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} D_{3TT}^\perp(z, k_{F\perp}) \right], \end{aligned} \quad (3.19)$$

$$\begin{aligned} z\tilde{\Xi}_\alpha^{T(0)}(z, k_{F\perp}; p, S) &= p^+ \bar{n}_\alpha \left[\frac{\epsilon_{\perp}^{k_F S_{LT}}}{M} G_{1LT}^\perp(z, k_{F\perp}) + \frac{\epsilon_{\perp k_F \perp \rho} k_{F\perp \sigma} S_{TT}^{\rho\sigma}}{M^2} G_{1TT}^\perp(z, k_{F\perp}) \right] \\ &+ \epsilon_{\perp\rho\alpha} k_{F\perp}^\rho \left[S_{LL} G_{LL}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} G_{LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} G_{TT}^\perp(z, k_{F\perp}) \right] \\ &+ M \epsilon_{\perp\rho\alpha} S_{LT}^\rho G_{LT}(z, k_{F\perp}) + \epsilon_{\perp\alpha\rho} k_{F\perp\sigma} S_{TT}^{\rho\sigma} G_{TT}^{\prime\perp}(z, k_{F\perp}) \\ &+ \frac{M^2}{p^+} n_\alpha \left[\frac{\epsilon_{\perp}^{k_F S_{LT}}}{M} G_{3LT}^\perp(z, k_{F\perp}) + \frac{\epsilon_{\perp k_F \perp \rho} k_{F\perp \sigma} S_{TT}^{\rho\sigma}}{M^2} G_{3TT}^\perp(z, k_{F\perp}) \right], \end{aligned} \quad (3.20)$$

$$\begin{aligned} z\Xi_{T\rho\alpha}^{T(0)}(z, k_{F\perp}; p, S) &= \frac{p^+ \bar{n}_{[\rho} \epsilon_{\perp\alpha]} k_{F\perp}^\sigma}{M} \left[S_{LL} H_{1LL}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} H_{1LT}^\perp(z, k_{F\perp}) \right. \\ &+ \left. \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} H_{1TT}^\perp(z, k_{F\perp}) \right] + p^+ \bar{n}_{[\rho} \epsilon_{\perp\alpha]} \sigma_{LT}^\sigma H_{1LT}(z, k_{F\perp}) + \frac{p^+ \bar{n}_{[\rho} \epsilon_{\perp\alpha]} k_{F\perp \delta} S_{TT}^{\sigma\delta}}{M} H_{1TT}^{\prime\perp}(z, k_{F\perp}) \\ &+ M \epsilon_{\perp\rho\alpha} \left[S_{LL} H_{LL}(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} H_{LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} H_{TT}^\perp(z, k_{F\perp}) \right] \\ &+ \bar{n}_{[\rho} n_{\alpha]} \left[\epsilon_{\perp}^{k_F S_{LT}} H_{LT}^{\prime\perp}(z, k_{F\perp}) + \frac{\epsilon_{\perp k_F \perp \sigma} k_{F\perp \delta} S_{TT}^{\sigma\delta}}{M} H_{TT}^{\prime\perp}(z, k_{F\perp}) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{M^2}{p^+} \left\{ \frac{n_{[\rho\varepsilon\perp\alpha]}\sigma k_{F\perp}^\sigma}{M} \left[S_{LL} H_{3LL}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{LT}}{M} H_{3LT}^\perp(z, k_{F\perp}) + \frac{k_{F\perp} \cdot S_{TT} \cdot k_{F\perp}}{M^2} H_{3TT}^\perp(z, k_{F\perp}) \right] \right. \\
 & \left. + n_{[\rho\varepsilon\perp\alpha]}\sigma S_{LT}^\sigma H_{3LT}(z, k_{F\perp}) + \frac{n_{[\rho\varepsilon\perp\alpha]}\sigma k_{F\perp\delta} S_{TT}^{\sigma\delta}}{M} H_{3TT}^{\prime\perp}(z, k_{F\perp}) \right\}. \tag{3.21}
 \end{aligned}$$

We see that, for the vector-polarization-dependent part, similar to case of the nucleon TMD PDFs, there are a total of 32 components; 8 contribute at leading twist, 16 contribute at twist-3, and the other 8 contribute at twist-4. For the tensor-polarization-dependent part, there are a total of 40 components, where 10 contribute at leading twist, 20 contribute at twist-3, and the other 10 contribute at twist-4. In Table 4, we list the twist-2 components for the tensor-polarization-dependent part.

If we integrate over $d^2k_{F\perp}$, we have the following, corresponding to Eqs. (3.12)–(3.16), for the spin-independent and vector-polarization-dependent parts:

$$z\Xi_S^{U+V(0)}(z; p, S) = ME(z), \tag{3.22}$$

$$z\Xi_{PS}^{U+V(0)}(z; p, S) = \lambda ME_L(z), \tag{3.23}$$

$$z\Xi_\alpha^{U+V(0)}(z; p, S) = p^+ \bar{n}_\alpha D_1(z) + M\varepsilon_{\perp\alpha\rho} S_T^\rho D_T(z) + \frac{M^2}{p^+} n_\alpha D_3(z), \tag{3.24}$$

$$z\tilde{\Xi}_\alpha^{U+V(0)}(z; p, S) = \lambda p^+ \bar{n}_\alpha G_{1L}(z) + M S_{T\alpha} G_T(z) + \lambda \frac{M^2}{p^+} n_\alpha G_{3L}(z), \tag{3.25}$$

$$z\Xi_{T\rho\alpha}^{U+V(0)}(z; p, S) = p^+ \bar{n}_{[\rho} S_{T\alpha]} H_{1T}(z) + M\varepsilon_{\perp\rho\alpha} H(z) + \lambda M \bar{n}_{[\rho} n_{\alpha]} H_L(z) + \frac{M^2}{p^+} n_{[\rho} S_{T\alpha]} H_{3T}(z). \tag{3.26}$$

For the tensor-polarization-dependent part, we have

$$z\Xi_S^{T(0)}(z; p, S) = MS_{LL} E_{LL}(z), \tag{3.27}$$

$$z\Xi_{PS}^{T(0)}(z; p, S) = 0, \tag{3.28}$$

$$z\Xi_\alpha^{T(0)}(z; p, S) = p^+ \bar{n}_\alpha S_{LL} D_{1LL}(z) + M S_{LT\alpha} D_{LT}(z, k_{F\perp}) + \frac{M^2}{p^+} n_\alpha S_{LL} D_{3LL}(z), \tag{3.29}$$

$$z\tilde{\Xi}_\alpha^{T(0)}(z; p, S) = M\varepsilon_{\perp\rho\alpha} S_{LT}^\rho G_{LT}(z), \tag{3.30}$$

$$z\Xi_{T\rho\alpha}^{T(0)}(z; p, S) = p^+ \bar{n}_{[\rho} \varepsilon_{\perp\alpha]\sigma} S_{LT}^\sigma H_{1LT}(z) + M\varepsilon_{\perp\rho\alpha} S_{LL} H_{LL}(z) + \frac{M^2}{p^+} n_{[\rho} \varepsilon_{\perp\alpha]\sigma} S_{LT}^\sigma H_{3LT}(z). \tag{3.31}$$

Table 4 The 10 tensor polarization dependent TMD FFs for spin-1 hadrons defined via the quark-quark correlator. A \times means that the corresponding term disappears upon integrating the quark-quark correlator over $d^2k_{F\perp}$.

Parameters	b_{11}	b_{22}	b_{12}	
Quark polarization	Hadron polarization	TMD FFs	Integrated over $\vec{k}_{F\perp}$	Name
U	LL	$D_{1LL}(z, k_{F\perp})$	$D_{1LL}(z)$	Spin alignment
	LT	$D_{1LT}^\perp(z, k_{F\perp})$	\times	
	TT	$D_{1TT}^\perp(z, k_{F\perp})$	\times	
L	LT	$G_{1LT}^\perp(z, k_{F\perp})$	\times	
	TT	$G_{1TT}^\perp(z, k_{F\perp})$	\times	
T	LL	$H_{1LL}^\perp(z, k_{F\perp})$	\times	
	LT	$H_{1LT}(z, k_{F\perp}), H_{1LT}^\perp(z, k_{F\perp})$	$H_{1LT}(z)$	
	TT	$H_{1TT}^\perp(z, k_{F\perp}), H_{1TT}^{\prime\perp}(z, k_{F\perp})$	\times	

We see that, for the spin-independent and vector-polarization-dependent parts, 12 components survive; 3 of them contribute at twist-2, 6 contribute at twist-3, and the other 3 contribute at twist-4. This is exactly the same as the result for the PDFs for nucleons, and we have exact one-to-one correspondence between the results given by Eqs. (3.22)–(3.26) and those given by Eqs. (2.64)–

(2.68). For the tensor-polarization-dependent part, only 8 components survive; 2 of them contribute at twist-2, 4 contribute at twist-3, and the other 2 contribute at twist-4. This corresponds to the situation for the PDFs for vector mesons. We should have a one-to-one correspondence between the tensor-polarization-dependent FFs for production of spin-1 hadrons and those PDFs for spin-1

hadrons. We also list the twist-2 components in Table 4.

4 Accessing the TMDs in high-energy reactions

The TMDs can be studied in semi-inclusive high-energy reactions such as SIDIS ($e^- + N \rightarrow e^- + h + X$), semi-inclusive Drell–Yan processes ($h + h \rightarrow l^+ + l^- + X$), and semi-inclusive hadron production in e^+e^- -annihilation ($e^+ + e^- \rightarrow h_1 + h_2 + X$). For SIDIS, we study TMD PDFs and TMD FFs, whereas for Drell–Yan processes and e^+e^- annihilation, we study TMD PDFs and TMD FFs separately. We now follow the same steps as those for inclusive DIS and summarize briefly what we have already done in constructing the corresponding theoretical

framework.

(I) The general forms of hadronic tensors: For all three classes of processes, the general forms of hadronic tensors have been studied and obtained. For SIDIS, this has been discussed in Refs. [27–30], and it has been shown that one needs 18 independent structure functions for spinless h . A comprehensive study of Drell–Yan processes was made in Ref. [31], and the number of independent structure functions is 48 for hadrons with spin 1/2. A study of e^+e^- annihilation was presented in Ref. [32], and one needs 72 for spin-1/2 h_1 and h_2 . The results are systematically presented in those papers, and we will not repeat them here. However, we present, as an example, the general form of the differential cross section for $e^- N \rightarrow e^- h X$. It is given by

$$\frac{d\sigma}{dx dy dz d\psi d^2 p_{h\perp}} = \frac{\alpha_{em}^2}{xyQ^2} \left(1 + \frac{\gamma^2}{2x}\right) \left(\mathcal{F}_{UU} + \lambda_l \mathcal{F}_{LU} + \lambda \mathcal{F}_{UL} + \lambda_l \lambda \mathcal{F}_{LL} + S_\perp \mathcal{F}_{UT} + \lambda_l S_\perp \mathcal{F}_{LT}\right), \quad (4.1)$$

$$\mathcal{F}_{UU} = \frac{y^2}{1-\varepsilon} \left(F_{UU,T} + \varepsilon F_{UU,L} + \sqrt{2\varepsilon(1+\varepsilon)} F_{UU}^{\cos\phi_h} \cos\phi_h + \varepsilon F_{UU}^{\cos 2\phi_h} \cos 2\phi_h\right), \quad (4.2)$$

$$\mathcal{F}_{UL} = \frac{y^2}{1-\varepsilon} \left(\sqrt{2\varepsilon(1+\varepsilon)} F_{UL}^{\sin\phi_h} \sin\phi_h + \varepsilon F_{UL}^{\sin 2\phi_h} \sin 2\phi_h\right), \quad (4.3)$$

$$\mathcal{F}_{LU} = \frac{y^2}{1-\varepsilon} \sqrt{2\varepsilon(1-\varepsilon)} F_{LU}^{\sin\phi_h} \sin\phi_h, \quad (4.4)$$

$$\mathcal{F}_{LL} = \frac{y^2}{1-\varepsilon} \left(\sqrt{1-\varepsilon^2} F_{LL} + \sqrt{2\varepsilon(1-\varepsilon)} F_{LL}^{\cos\phi_h} \cos\phi_h\right), \quad (4.5)$$

$$\begin{aligned} \mathcal{F}_{UT} = \frac{y^2}{1-\varepsilon} & \left[\sqrt{2\varepsilon(1+\varepsilon)} F_{UT}^{\sin\phi_S} \sin\phi_S + (F_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon F_{UT,L}^{\sin(\phi_h-\phi_S)}) \sin(\phi_h - \phi_S) \right. \\ & \left. + \varepsilon F_{UT}^{\sin(\phi_h+\phi_S)} \sin(\phi_h + \phi_S) + \sqrt{2\varepsilon(1+\varepsilon)} F_{UT}^{\sin(2\phi_h-\phi_S)} \sin(2\phi_h - \phi_S) + \varepsilon F_{UT}^{\sin(3\phi_h-\phi_S)} \sin(3\phi_h - \phi_S) \right], \quad (4.6) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{LT} = \frac{y^2}{1-\varepsilon} & \left[\sqrt{2\varepsilon(1-\varepsilon)} F_{LT}^{\cos\phi_S} \cos\phi_S + \sqrt{1-\varepsilon^2} F_{LT}^{\cos(\phi_h-\phi_S)} \cos(\phi_h - \phi_S) \right. \\ & \left. + \sqrt{2\varepsilon(1-\varepsilon)} F_{LT}^{\cos(2\phi_h-\phi_S)} \cos(2\phi_h - \phi_S) \right], \quad (4.7) \end{aligned}$$

where $\varepsilon = (1 - y - \frac{1}{4}\gamma^2 y^2)/(1 - y + \frac{1}{2}y^2 + \frac{1}{4}\gamma^2 y^2)$, $\gamma = 2Mx/Q$; the azimuthal angle ψ is that of the outgoing lepton \vec{l}' around the incident lepton beam with respect to an arbitrary fixed direction, which, for a transversely polarized target, is taken as the direction of \vec{S}_T . In the deep inelastic limit, neglecting power-suppressed terms, $d\psi = d\phi_S$.

From Eqs. (4.1)–(4.7), we see explicitly that the 18 structure functions F are determined by the different azimuthal asymmetries in different polarization cases. These different azimuthal asymmetries are just defined by the average value of the corresponding trigonometric functions, for example:

$$\begin{aligned} A_{UT}^{\sin(\phi_h-\phi_S)} & = \langle \sin(\phi_h - \phi_S) \rangle_{UT} \\ & = \frac{F_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon F_{UT,L}^{\sin(\phi_h-\phi_S)}}{2(F_{UU,T} + \varepsilon F_{UU,L})}, \quad (4.8) \end{aligned}$$

$$\begin{aligned} A_{UT}^{\sin(\phi_h+\phi_S)} & = \langle \sin(\phi_h + \phi_S) \rangle_{UT} \\ & = \frac{\varepsilon F_{UT}^{\sin(\phi_h+\phi_S)}}{2(F_{UU,T} + \varepsilon F_{UU,L})}. \quad (4.9) \end{aligned}$$

We also emphasize that they are the general forms independent of the parton model and are valid at leading and higher twist and also leading and higher order in pQCD.

(II) LO in pQCD and leading twist parton model results: These are the simplest parton model results and can be obtained easily. E.g., for SIDIS,

$$\begin{aligned} \frac{d\sigma^{(0)}}{dx dy dz d\phi_S d^2 p_{h\perp}} & = \frac{\alpha_{em}^2}{xyQ^2} \left(\mathcal{F}_{UU}^{(0)} + \lambda_l \mathcal{F}_{LU}^{(0)} + \lambda \mathcal{F}_{UL}^{(0)} \right. \\ & \left. + \lambda_l \lambda \mathcal{F}_{LL}^{(0)} + S_\perp \mathcal{F}_{UT}^{(0)} + \lambda_l S_\perp \mathcal{F}_{LT}^{(0)} \right), \quad (4.10) \end{aligned}$$

$$\mathcal{F}_{UU}^{(0)} = A(y) \mathcal{C}[f_1 D_1] + 2(1-y) \mathcal{C}[w_1 h_1^\perp H_1^\perp] \cos(2\phi_h),$$

$$(4.11) \quad \mathcal{F}_{LU}^{(0)} = 0, \tag{4.13}$$

$$\mathcal{F}_{UL}^{(0)} = 2(1-y)\mathcal{C}[w_1 h_{1L}^\perp H_1^\perp] \sin(2\phi_h), \tag{4.12} \quad \mathcal{F}_{LL}^{(0)} = C(y)\mathcal{C}[g_{1L} D_1], \tag{4.14}$$

$$\mathcal{F}_{UT}^{(0)} = A(y)\mathcal{C}[w_2 f_{1T}^\perp D_1] \sin(\phi_h - \phi_S) + 2(1-y)\mathcal{C}[w_3 h_{1T} H_1^\perp] \sin(\phi_h + \phi_S) + 2(1-y)\mathcal{C}[w_4 h_{1T}^\perp H_1^\perp] \sin(3\phi_h - \phi_S), \tag{4.15}$$

$$\mathcal{F}_{LT}^{(0)} = C(y)\mathcal{C}[-w_2 g_{1T} D_1] \cos(\phi_h - \phi_S), \tag{4.16}$$

where $A(y) = 1 + (1-y)^2$, and $C(y) = y(2-y)$. $\mathcal{C}[w_i f D]$ denotes the convolution of f and D weighted by w_i , i.e.,

$$\mathcal{C}[w_i f D] \equiv x \sum_q e_q^2 \int d^2 k_\perp d^2 k_{F\perp} \delta^{(2)}(k_\perp - k_{F\perp} - p_{hT}/z) w_i(k_\perp, k_{F\perp}, p_{hT}) f^q(x, k_\perp) D^{q \rightarrow hX}(z, k_{F\perp}). \tag{4.17}$$

The weights w_i are given by

$$w_1(\vec{k}_\perp, k_{F\perp}) = \frac{-2(\hat{p}_{hT} \cdot \vec{k}_{F\perp})(\hat{p}_{hT} \cdot \vec{k}_\perp) + (\vec{k}_\perp \cdot \vec{k}_{F\perp})}{MM_h}, \tag{4.18}$$

$$w_2(\vec{k}_\perp, k_{F\perp}) = -\frac{\hat{p}_{hT} \cdot \vec{k}_\perp}{M}, \tag{4.19}$$

$$w_3(\vec{k}_\perp, k_{F\perp}) = -\frac{\hat{p}_{hT} \cdot \vec{k}_{F\perp}}{M_h}, \tag{4.20}$$

$$w_4(\vec{k}_\perp, k_{F\perp}) = \frac{(\hat{p}_{hT} \cdot \vec{k}_\perp)(\vec{k}_\perp \cdot \vec{k}_{F\perp}) + \vec{k}_\perp^2 (\hat{p}_{hT} \cdot \vec{k}_{F\perp})}{M^2 M_h} - \frac{2(\hat{p}_{hT} \cdot \vec{k}_\perp)^2 (\hat{p}_{hT} \cdot \vec{k}_{F\perp})}{M^2 M_h}, \tag{4.21}$$

where $\hat{p}_{hT} = \vec{p}_{hT}/|\vec{p}_{hT}|$ is the corresponding unit vector. The results can be obtained from those given in, e.g., [30] by neglecting all the power-suppressed contributions.

From Eqs. (4.10)–(4.16), we see in particular that, at leading twist, there exist six nonzero azimuthal asymmetries in different polarization cases, i.e.,

$$\langle \cos 2\phi_h \rangle_{UU}^{(0)} = \frac{(1-y)}{A(y)} \frac{\mathcal{C}[w_1 h_{1L}^\perp H_1^\perp]}{\mathcal{C}[f_1 D_1]}, \tag{4.22}$$

$$\langle \sin 2\phi_h \rangle_{UL}^{(0)} = \frac{(1-y)}{A(y)} \frac{\mathcal{C}[w_1 h_{1L}^\perp H_1^\perp]}{\mathcal{C}[f_1 D_1]}, \tag{4.23}$$

$$\langle \sin(\phi_h - \phi_S) \rangle_{UT}^{(0)} = \frac{\mathcal{C}[w_2 f_{1T}^\perp D_1]}{2\mathcal{C}[f_1 D_1]}, \tag{4.24}$$

$$\langle \sin(\phi_h + \phi_S) \rangle_{UT}^{(0)} = \frac{(1-y)}{A(y)} \frac{\mathcal{C}[w_3 h_{1T} H_1^\perp]}{\mathcal{C}[f_1 D_1]}, \tag{4.25}$$

$$\langle \sin(3\phi_h - \phi_S) \rangle_{UT}^{(0)} = \frac{(1-y)}{A(y)} \frac{\mathcal{C}[w_4 h_{1T}^\perp H_1^\perp]}{\mathcal{C}[f_1 D_1]}, \tag{4.26}$$

$$\tilde{W}_{\mu\nu}^{(0,si)}(q, p, S, k_\perp) = \frac{1}{2} \text{Tr}[\hat{h}_{\mu\nu}^{(0)} \hat{\Phi}^{(0)}(x_B, k_\perp)], \tag{4.28}$$

$$\tilde{W}_{\mu\nu}^{(1,L,si)}(q, p, S, k_\perp) = \frac{1}{4q \cdot p} \text{Tr}[\hat{h}_{\mu\nu}^{(1)\rho} \omega_\rho \rho' \hat{\varphi}_{\rho'}^{(1,L)}(x_B, k_\perp)], \tag{4.29}$$

$$\langle \cos(\phi_h - \phi_S) \rangle_{LT}^{(0)} = \frac{C(y)}{2A(y)} \frac{\mathcal{C}[-w_2 g_{1T} D_1]}{\mathcal{C}[f_1 D_1]}. \tag{4.27}$$

They are determined by the Boer–Mulders function h_1^\perp convoluted with the Collins function H_1^\perp , the worm-gear (longi-transversity) h_{1L}^\perp convoluted with the Collins function H_1^\perp , the Sivers function f_{1T}^\perp convoluted with D_1 , the transversity h_{1T} convoluted with the Collins function H_1^\perp , and the worm-gear (trans-helicity distribution) g_{1T}^\perp convoluted with the Collins function H_1^\perp . The azimuthal asymmetries $A_{UT}^{\sin(\phi_h \mp \phi_S)}$ are due to the Sivers and Collins effects and are often referred to as Sivers asymmetry and Collins asymmetry, respectively.

We emphasize that the results given by Eqs. (4.10)–(4.21) give a complete parton model result at LO in pQCD and leading twist. They can be used to extract the TMDs at this order. Any attempt to go beyond LO in pQCD or to consider higher twists needs to go beyond this expression.

(III) LO in pQCD, leading and higher twist results: For semi-inclusive processes in which only one hadron is involved, either in the initial or the final state, it has been shown [33–37] that collinear expansion can be applied. Such processes include SIDIS [$e^- + N \rightarrow e^- + q(\text{jet}) + X$], and e^+e^- -annihilation [$e^+ + e^- \rightarrow h + \bar{q}(\text{jet}) + X$]. By applying collinear expansion, we have constructed the theoretical framework for these processes by which leading as well as higher twist contributions can be calculated systematically to LO in pQCD. The complete results up to twist-3 are presented in Refs. [35–37]. For polarized $e^- + N \rightarrow e^- + q(\text{jet}) + X$, the simplified expressions for the hadronic tensor are very similar to those for the inclusive DIS given by Eqs. (2.55)–(2.58),

$$\tilde{W}_{\mu\nu}^{(2,L,si)}(q, p, S, k_{\perp}) = \frac{1}{(2q \cdot p)^2} \left\{ \text{Tr} [\hat{h}_{\mu\nu}^{(1)\rho} \omega_{\rho}^{\rho'} \hat{\phi}_{\rho'}^{(2L)}(x_B, k_{\perp})] + \text{Tr} [\hat{N}_{\mu\nu}^{(2)\rho\sigma} \omega_{\rho}^{\rho'} \omega_{\sigma}^{\sigma'} \hat{\phi}_{\rho'\sigma'}^{(2L)}(x_B, k_{\perp})] \right\}, \quad (4.30)$$

$$\tilde{W}_{\mu\nu}^{(2,M,si)}(q, p, S, k_{\perp}) = \frac{1}{(2q \cdot p)^2} \text{Tr} [\hat{h}_{\mu\nu}^{(2)\rho\sigma} \omega_{\rho}^{\rho'} \omega_{\sigma}^{\sigma'} \hat{\phi}_{\rho'\sigma'}^{(2M)}(x_B, k_{\perp})]. \quad (4.31)$$

The complete results up to twist-3 are given by

$$\frac{d\sigma}{dx dy d^2 k_{\perp}} = \frac{2\pi\alpha_{em}^2 e_q^2}{Q^2 y} (\mathcal{W}_{UU} + \lambda_l \mathcal{W}_{LU} + S_{\perp} \mathcal{W}_{UT} + \lambda \mathcal{W}_{UL} + \lambda_l \lambda \mathcal{W}_{LL} + \lambda_l S_{\perp} \mathcal{W}_{LT}), \quad (4.32)$$

$$\mathcal{W}_{UU}(x, k_{\perp}, \phi) = A(y) f_q(x, k_{\perp}) - \frac{2x|\vec{k}_{\perp}|}{Q} B(y) f_q^{\perp}(x, k_{\perp}) \cos \phi, \quad (4.33)$$

$$\mathcal{W}_{LU}(x, k_{\perp}, \phi) = -\frac{2x|\vec{k}_{\perp}|}{Q} D(y) g^{\perp}(x, k_{\perp}) \sin \phi, \quad (4.34)$$

$$\begin{aligned} \mathcal{W}_{UT}(x, k_{\perp}, \phi, \phi_S) &= \frac{|\vec{k}_{\perp}|}{M} A(y) f_{1T}^{\perp}(x, k_{\perp}) \sin(\phi - \phi_S) \\ &+ \frac{2xM}{Q} B(y) \left\{ \frac{k_{\perp}^2}{2M^2} f_T^{\perp}(x, k_{\perp}) \sin(2\phi - \phi_S) - f_T(x, k_{\perp}) \sin \phi_S \right\}, \end{aligned} \quad (4.35)$$

$$\mathcal{W}_{UL}(x, k_{\perp}, \phi) = -\frac{2x|\vec{k}_{\perp}|}{Q} B(y) f_L^{\perp}(x, k_{\perp}) \sin \phi, \quad (4.36)$$

$$\mathcal{W}_{LL}(x, k_{\perp}, \phi) = C(y) g_{1L}(x, k_{\perp}) - \frac{2x|\vec{k}_{\perp}|}{Q} D(y) g_L^{\perp}(x, k_{\perp}) \cos \phi, \quad (4.37)$$

$$\begin{aligned} \mathcal{W}_{LT}(x, k_{\perp}, \phi, \phi_S) &= \frac{|\vec{k}_{\perp}|}{M} C(y) g_{1T}^{\perp}(x, k_{\perp}) \cos(\phi - \phi_S) \\ &- \frac{2xM}{Q} D(y) \left[g_T(x, k_{\perp}) \cos \phi_S - \frac{k_{\perp}^2}{2M^2} g_T^{\perp}(x, k_{\perp}) \cos(2\phi - \phi_S) \right], \end{aligned} \quad (4.38)$$

where $B(y) = 2(2-y)\sqrt{1-y}$, and $D(y) = 2y\sqrt{1-y}$. For unpolarized $e^- + N \rightarrow e^- + q(jet) + X$, the results up to twist-4 have also been obtained [34]:

$$\begin{aligned} \frac{d\sigma_{UU}}{dx dy d^2 k_{\perp}} &= \frac{2\pi\alpha_{em}^2 e_q^2}{Q^2 y} \left\{ A(y) f_1(x, k_{\perp}) - 2B(y) \frac{|\vec{k}_{\perp}|}{Q} x f^{\perp}(x, k_{\perp}) \cos \phi \right. \\ &- 4(1-y) \frac{|\vec{k}_{\perp}|^2}{Q^2} x [\varphi_3^{(1)\perp}(x, k_{\perp}) - \tilde{\varphi}_3^{(1)\perp}(x, k_{\perp})] \cos 2\phi \\ &\left. + 8(1-y) \frac{2x^2 M^2}{Q^2} f_3(x, k_{\perp}) - 2A(y) \frac{|\vec{k}_{\perp}|^2}{Q^2} x [\varphi_3^{(2,L)\perp}(x, k_{\perp}) - \tilde{\varphi}_3^{(2,L)\perp}(x, k_{\perp})] \right\}. \end{aligned} \quad (4.39)$$

These results are expressed in terms of the gauge-invariant TMD PDFs or FFs and can be used as the basis for measuring these TMDs via the corresponding process at the LO in pQCD.

We would like in particular to draw attention to the results for $e^+ + e^- \rightarrow h + \bar{q}(jet) + X$ for h with different spins [37]. Here, for the hadronic tensor, we again obtain very similar formulae for this process; e.g., corresponding to Eqs. (4.28)–(4.30), we have

$$\tilde{W}_{\mu\nu}^{(0,si)}(q, p, S, k'_{\perp} | e^+ e^-) = \frac{1}{2} \text{Tr} \left[\hat{h}_{\mu\nu}^{(0)} \hat{\Xi}^{(0)}(z_B, k'_{\perp}) \right], \quad (4.40)$$

$$\tilde{W}_{\mu\nu}^{(1,L,si)}(q, p, S, k'_{\perp} | e^+ e^-) = -\frac{1}{4p \cdot q} \text{Tr} \left[\hat{h}_{\mu\nu}^{(1)\rho} \omega_{\rho}^{\rho'} \hat{\Xi}_{\rho'}^{(1)}(z_B, k'_{\perp}) \right], \quad (4.41)$$

$$\tilde{W}_{\mu\nu}^{(2,L,si)}(q, p, S, k'_{\perp} | e^+ e^-) = \frac{1}{4(p \cdot q)^2} \text{Tr} \left[\hat{h}_{\mu\nu}^{(1)\rho} \omega_{\rho}^{\rho'} \hat{\Xi}_{\rho'}^{(2B)}(z_B, k'_{\perp}) + \hat{N}_{\mu\nu}^{(2)\rho\sigma} \omega_{\rho}^{\rho'} \omega_{\sigma}^{\sigma'} \hat{\Xi}_{\rho'\sigma'}^{(2C)}(z_B, k'_{\perp}) \right], \quad (4.42)$$

$$\tilde{W}_{\mu\nu}^{(2,M,si)}(q, p, S, k'_{\perp} | e^+ e^-) = \frac{1}{4(p \cdot q)^2} \text{Tr} \left[\hat{h}_{\mu\nu}^{(2)\rho\sigma} \omega_{\rho}^{\rho'} \omega_{\sigma}^{\sigma'} \hat{\Xi}_{\rho'\sigma'}^{(2A)}(z_B, k'_{\perp}) \right]. \quad (4.43)$$

Complete twist-3 results for the differential cross sections, azimuthal asymmetries, and polarizations were obtained for hadrons with spin 0, 1/2, and 1 in Ref. [37]. For spin-1 hadrons in particular, we see that tensor polarization is involved, even at the leading twist level; we have, for e^+e^- annihilation at the Z^0 pole,

$$S_{LL}^{(0)}(y, z, p_T) = \frac{\sum_q T_0^q(y) D_{1LL}(z, p_T)}{2 \sum_q T_0^q(y) D_1(z, p_T)}, \quad (4.44)$$

$$S_{LT}^{n(0)}(y, z, p_T) = -\frac{2|\vec{p}_T| \sum_q P_q(y) T_0^q(y) G_{1LT}^\perp(z, p_T)}{3zM \sum_q T_0^q(y) D_1(z, p_T)}, \quad (4.45)$$

$$S_{LT}^{t(0)}(y, z, p_T) = -\frac{2|\vec{p}_T| \sum_q T_0^q(y) D_{1LT}^\perp(z, p_T)}{3zM \sum_q T_0^q(y) D_1(z, p_T)}, \quad (4.46)$$

$$S_{TT}^{nn(0)}(y, z, p_T) = -\frac{2|\vec{p}_T|^2 \sum_q T_0^q(y) D_{1TT}^\perp(z, p_T)}{3M^2 \sum_q T_0^q(y) D_1(y, p_T)}, \quad (4.47)$$

$$S_{TT}^{nt(0)}(y, z, p_T) = \frac{2|\vec{p}_T|^2 \sum_q P_q(y) T_0^q(y) G_{1TT}^\perp(z, p_T)}{3M^2 \sum_q T_0^q(y) D_1(y, p_T)}, \quad (4.48)$$

where n and t denote the two transverse directions of the produced vector meson, normal to and inside the production plane, respectively. The coefficient $T_0^q(y) = c_1^q c_1^e [(1-y)^2 + y^2] - c_3^q c_3^e [1 - 2y]$, where $c_1^e = (c_V^e)^2 + (c_A^e)^2$, and $c_3^e = 2c_V^e c_A^e$; y in this reaction is defined as $y \equiv l_1^+ / k^+$. $P_q(y) = T_1^q(y) / T_0^q(y)$ is the polarization of the quark produced by Z^0 decay, and $T_1^q(y) = -c_3^q c_1^e [(1-y)^2 + y^2] + c_1^q c_3^e [1 - 2y]$. This situation has been much less explored to date and is a worthwhile topic for many further studies.

The three types of semi-inclusive processes mentioned above always involve two hadrons. How to apply collinear expansion to such processes has not been demonstrated. It is unclear how one can systematically calculate leading and higher twist contributions. Nevertheless, practical twist-3 calculations have been performed for these processes [38–41] using the following steps:

- (i) draw Feynman diagrams with multiple gluon scattering to the order of one gluon exchange,
- (ii) insert the gauge link in the correlator wherever needed to make it gauge invariant, and
- (iii) carry out calculations to the order $1/Q$.

Although this method has not been proved, it is interesting to see that the results obtained this way reduce exactly to those obtained in the corresponding simplified cases where collinear expansion is applied if we take the corresponding fragmentation functions as δ functions.

(IV) TMD factorization and evolution: To describe

the semi-inclusive high-energy reactions mentioned above in terms of QCD and the parton model, TMDs are needed, and the factorization theorem has to involve the transverse momentum dependence. A TMD factorization theorem has been established at the leading twist for semi-inclusive processes [42–49]. TMD evolution theory is also developing very rapidly [50–63]. Boer [3] gave an overview at Spin2014, and an annual workshop series dedicated to this topic was established in 2012. We refer interested readers to these talks and overviews.

5 Available data and parameterizations

All three types of semi-inclusive reactions have been investigated experimentally. The results are summarized, e.g., in a number of plenary talks at Spin2014 by Stolarski and Rostomyan [64, 65]. Here, we will just briefly summarize the main data available and then try to sort out the available TMD parameterizations.

For SIDIS, measurements have been made by the HERMES Collaboration [66–70] at DESY, the COMPASS Collaboration [71–78] at CERN, CLAS [79, 80], and the Hall A Collaboration [81–84] at Jefferson Laboratory (JLab). We list these SIDIS experiments in Table 5 and briefly summarize the results as follows.

At DESY, the single-spin asymmetries for SIDIS were first measured with a longitudinally polarized target by HERMES [66] for production of charged pions and then measured for the first time with a transversely polarized target in [67]. They found nonzero Sivers and Collins asymmetries $\langle \sin(\phi_h - \phi_S) \rangle_{UT}$ and $\langle \sin(\phi_h + \phi_S) \rangle_{UT}$. Measurements were then conducted for π^0 and kaons [68, 69] and also for azimuthal asymmetries $\langle \cos \phi_h \rangle_{UU}$ and $\langle \cos(2\phi_h) \rangle_{UU}$ in the unpolarized case [70].

At CERN, COMPASS has measured the Sivers and Collins asymmetries in reactions with deuteron or proton targets for production of charged hadrons, pions, and kaons [71–78], and also $\langle \cos \phi_h \rangle_{UU}$ and $\langle \cos(2\phi_h) \rangle_{UU}$ in the unpolarized case [78].

At JLab, CLAS has measured [79, 80] $\langle \sin(2\phi_h) \rangle_{UL}$ for pions with different charges and $\langle \sin \phi_h \rangle_{LU}$ for π^0 . The Hall A Collaboration has measured [81–84] the Collins and Sivers asymmetries for π^\pm and K^\pm , and $\langle \cos(\phi_h - \phi_s) \rangle_{LT}$ for π^\pm and $\langle \sin(3\phi_h - \phi_s) \rangle_{UT}$. These measurements are all summarized in Table 5.

In addition to the data for SIDIS, we now have measurements of the azimuthal asymmetries in $e^+e^- \rightarrow \pi + \pi + X$ made by Belle [85–87] and the BaBar collaboration [88], and also preliminary results from BES [89]. For Drell–Yan processes, data are available on the azimuthal asymmetries in, e.g., reactions using pion beams [90–93]

and pp or pD collisions [94, 95].

Table 5 Available measurements on azimuthal asymmetries in SIDIS

Collaboration	Reaction	Asymmetries	Ref.'s
HERMES	$e^+N \rightarrow e^+\pi^\pm X$	$A_{UL}^{\sin\phi_h}, A_{UL}^{\sin 2\phi_h}$	[66]
	$e^+N \rightarrow e^+\pi^\pm X$	A_{Siv}, A_{Coll}	[67]
	$e^+N \rightarrow e^+\pi^\pm,0(K^\pm)X$	A_{Siv}	[68]
	$e^+N \rightarrow e^+\pi^\pm,0(K^\pm)X$	A_{Coll}	[69]
	$e^+N \rightarrow e^+\pi^\pm(K^\pm)X$	$A_{UU}^{\cos\phi_h}, A_{UU}^{\cos 2\phi_h}$	[70]
COMPASS	$\mu^-{}^6\text{LiD} \rightarrow \mu^-h^\pm X$	A_{Siv}, A_{Coll}	[71, 72]
	$\mu^-{}^6\text{LiD} \rightarrow \mu^-\pi^\pm(K^\pm,0)X$	A_{Siv}, A_{Coll}	[73]
	$\mu^- \text{NH}_3 \rightarrow \mu^-h^\pm X$	A_{Siv}, A_{Coll}	[74]
	$\mu^- \text{NH}_3 \rightarrow \mu^-h^\pm X$	A_{Coll}	[75]
	$\mu^- \text{NH}_3 \rightarrow \mu^-h^\pm X$	A_{Siv}	[76]
	$\mu^- \text{NH}_3 \rightarrow \mu^-\pi^\pm(K^\pm,0)X$	A_{Siv}, A_{Coll}	[77]
	$\mu^-{}^6\text{LiD} \rightarrow \mu^-h^\pm X$	$A_{UU}^{\cos\phi_h}, A_{UU}^{\cos 2\phi_h}$	[78]
CLAS	$e^-p \rightarrow e^-\pi^\pm,0 X$	$A_{UL}^{\sin 2\phi_h}$	[79]
	$e^-p \rightarrow e^-\pi^0 X$	$A_{LU}^{\sin\phi_h}$	[80]
JLab Hall A	$e^-{}^3\text{He} \rightarrow e^-\pi^\pm X$	A_{Siv}, A_{Coll}	[81]
	$e^-{}^3\text{He} \rightarrow e^-\pi^\pm X$	$A_{LT}^{\cos(\phi_h-\phi_S)}$	[82]
	$e^-{}^3\text{He} \rightarrow e^-\pi^\pm X$	$A_{UT}^{\sin(3\phi_h-\phi_S)}$	[83]
	$e^-{}^3\text{He} \rightarrow e^-K^\pm X$	A_{Siv}, A_{Coll}	[84]

Although the data are still far from sufficiently abundant to give precise control of the TMDs involved, different sets of TMD parameterizations have already been extracted from them. We briefly sort them out as follows.

The first part concerns what people called the first-phase parameterizations, i.e., TMD parameterizations without QCD evolution. Here, the following results in particular are available. We emphasize once more that all the results, including the figures, are taken from Refs. [96–112]. Interested readers are referred to these references for more details.

(1) Transverse momentum dependence: This is usually taken as [96–100] a Gaussian in a factorized form independent of the longitudinal variable z or x , e.g.,

$$f_1(x, k_\perp) = f_1(x) e^{-\vec{k}_\perp^2 / \langle \vec{k}_\perp^2 \rangle} / \pi \langle \vec{k}_\perp^2 \rangle, \quad (5.1)$$

$$D_1(z, k_{F\perp}) = D_1(z) e^{-\vec{k}_{F\perp}^2 / \langle \vec{k}_{F\perp}^2 \rangle} / \pi \langle \vec{k}_{F\perp}^2 \rangle. \quad (5.2)$$

The width has been fitted, and the form and flavor dependence and so on have been tested. The typical values of the fitted widths are, e.g., $\langle \vec{k}_\perp^2 \rangle = 0.25 \text{ GeV}^2$ and $\langle \vec{k}_{F\perp}^2 \rangle = 0.20 \text{ GeV}^2$ [96]. Roughly speaking, this is a quite satisfactory fit. However, it has also been pointed out, e.g., in [99] for the TMD FF, that the Gaussian form seems to depend on the flavor and even on z , which means that it is only a zeroth-order approximation.

(2) Siverson function: All the data available from HERMES [67–69], COMPASS [71–74, 76, 77], and JLab Hall A [81, 82, 84] on the Siverson asymmetries in SIDIS for pi-

ons and kaons have been used for parameterization. The Siverson function is usually parameterized [96, 101–106] in the form of the number density $f_q(x, k_\perp)$ multiplied by an x -dependent factor $\mathcal{N}_q(x)$ and a k_\perp -dependent factor $h(k_\perp)$, i.e.,

$$\Delta^N f_q(x, k_\perp) = 2\mathcal{N}_q(x)h(k_\perp)f_q(x, k_\perp), \quad (5.3)$$

where $\mathcal{N}_q(x)$ is taken as a binomial function of x ,

$$\mathcal{N}_q(x) = \mathcal{N}_q x^{\alpha_q} (1-x)^{\beta_q} (\alpha_q + \beta_q)^{\alpha_q + \beta_q} / \alpha_q^{\alpha_q} \beta_q^{\beta_q}, \quad (5.4)$$

and $h(k_\perp)$ is taken as a Gaussian,

$$h(k_\perp) = \sqrt{2e} (|\vec{k}_\perp| / M_1) e^{-\vec{k}_\perp^2 / M_1^2}. \quad (5.5)$$

Here the Siverson function $\Delta^N f_q(x, k_\perp)$ is defined via

$$f_{q/N^\dagger}(x, k_\perp) = f_{q/N}(x, k_\perp) + \frac{1}{2} \Delta^N f_q(x, k_\perp) \vec{S} \cdot (\hat{p} \times \hat{k}_\perp), \quad (5.6)$$

which is related to the Siverson function $f_{1T}^\perp(x, k_\perp)$ defined in Eq. (3.5) by

$$\Delta^N f_q(x, k_\perp) = -\frac{2|\vec{k}_\perp|}{M} f_{1T}^{\perp q}(x, k_\perp). \quad (5.7)$$

There already exist different sets such as the Bochum [101–103], Torino [96, 104, 106], and Vogelsang–Yuan [105] fits. One thing seems to be clear: the Siverson function is nonzero for protons, and it has different signs for u and d quarks, as shown in Fig. 2.

(3) Transversity and Collins function: A simultaneous extraction of the transversity and Collins function from SIDIS data on Collins asymmetry obtained by the HERMES [67–70] and COMPASS [71–77] collaborations and from e^+e^- data obtained by Belle [85–87] has been carried out by the Torino group [97, 107]. A form similar to that of the Siverson function has been taken, e.g.,

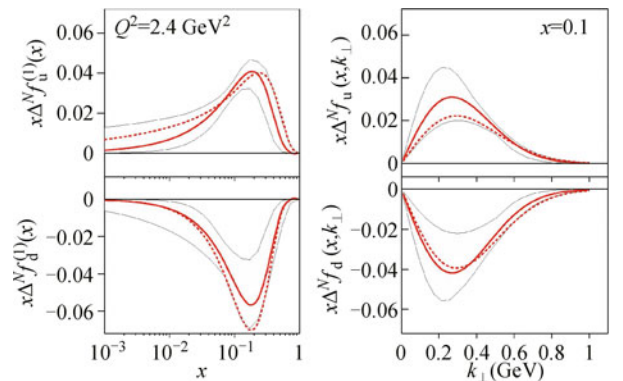


Fig. 2 Example of the parameterizations of the Siverson functions for u and d flavors at $Q^2 = 2.4(\text{GeV}/c)^2$ by the Torino group. The figure is taken from [104].

$$\Delta_T q(x, k_\perp) = \frac{1}{2} \mathcal{N}_q^T(x) [f_q(x) + \Delta q(x)] \times e^{-\vec{k}_\perp^2 / \langle \vec{k}_\perp^2 \rangle_T} / \pi \langle \vec{k}_\perp^2 \rangle, \quad (5.8)$$

$$\Delta^N D_{h/q}(z, p_{hT}) = 2 \mathcal{N}_q^C(z) D_{h/q}(z) h(p_{hT}) \times e^{-\vec{p}_{hT}^2 / \langle \vec{p}_{hT}^2 \rangle} / \pi \langle \vec{p}_{hT}^2 \rangle, \quad (5.9)$$

$$\mathcal{N}_q^T(x) = \mathcal{N}_q^T x^\alpha (1-x)^\beta \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta}, \quad (5.10)$$

$$\mathcal{N}_q^C(z) = \mathcal{N}_q^C z^\gamma (1-z)^\delta \frac{(\gamma + \delta)^{\gamma + \delta}}{\gamma^\gamma \delta^\delta}. \quad (5.11)$$

$$h(p_{hT}) = \sqrt{2} e^{-|\vec{p}_{hT}| / M_h} e^{-\vec{p}_{hT}^2 / M_h^2}, \quad (5.12)$$

and it has been determined that the Collins function is nonzero and has different signs, e.g., for $u \rightarrow \pi^+$ or $d \rightarrow \pi^+$, as shown in Fig. 3. Here, similar to the case for the Sivers function, the Collins function $\Delta^N D_{h/q}(z, k_{F\perp})$ is defined via

$$D_{h/q^\dagger}(z, p_{hT}) = D_{q/N}(z, p_{hT}) + \frac{1}{2} \Delta^N D_{h/q}(z, p_{hT}) \vec{s}_q(\hat{k}_q \times \hat{p}_{hT}), \quad (5.13)$$

which is related to the Collins function $H_1^\perp(z, p_{hT})$ defined in Eq. (3.16) by

$$\Delta^N D_{h/q}(z, p_{hT}) = \frac{2|\vec{p}_{hT}|}{zM_h} H_1^{\perp q}(z, p_{hT}). \quad (5.14)$$

(4) Boer–Mulders function: It was pointed out [111] that the HERMES and COMPASS data on $\langle \cos 2\phi \rangle$ asymmetry [70, 78] provide the first experimental evidence of the Boer–Mulders effect in SIDIS. Studies in this direction have been made in Refs. [110, 111] to extract the Boer–Mulders function from the SIDIS data [70, 78] and in Refs. [108, 109, 112] to extract it from the Drell–Yan data [90–95]. A fit to the first moments of the Boer–Mulders functions of the u and d quarks is shown in Fig. 4. The form is again similar to the Sivers function, being the Sivers function just multiplied by a

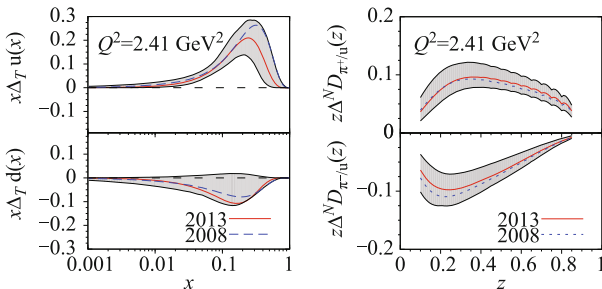


Fig. 3 Example of the Torino parameterizations of the transversity and Collins function. In the left panel, we see the transversities $x\Delta_T q(x) = xh_{1q}(x)$ for $q = u, d$; in the right panel, we see the first moments of the favored and disfavored Collins functions. The figure is taken from Ref. [107].

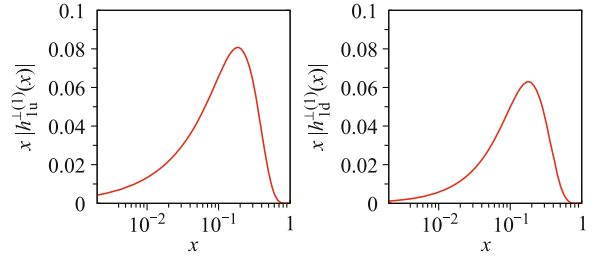


Fig. 4 First extractions of the Boer–Mulders function $h_1^{\perp u}(x)$ and $h_1^{\perp d}(x)$. This figure is taken from Ref. [111].

constant, e.g.,

$$h_1^{\perp q}(x, k_\perp) = \lambda_q f_{1T}^{\perp q}(x, k_\perp). \quad (5.15)$$

However, we note that the $\langle \cos 2\phi \rangle$ asymmetry receives twist-4 contributions because of the Cahn effect [22]. A proper treatment of these twist-4 effects involves twist-4 TMDs, as shown in Eq. (4.39) and in Ref. [34]. Because of the multiple gluon scattering shown in Fig. 1, the twist-4 effects could differ greatly from that given in Ref. [22], the results in which correspond to the case of $\mathcal{L} = 1$. A careful check might change the conclusion obtained in Refs. [108–112].

Attempts to parameterize other TMDs such as the pretzelocity h_{1T}^\perp have also been made [113]. Although there is not enough data to yield high-accuracy constraints, the obtained qualitative features are also interesting.

The second part concerns the QCD evolution of the TMDs. As mentioned earlier, this is a topic that has recently developed very rapidly. Recent dedicated publications include [50–63]. QCD evolution equations have been constructed, in particular for unpolarized TMD PDFs and also for polarized TMDs such as the Sivers function. The numerical results obtained from the evolution equations show explicitly that QCD evolution is

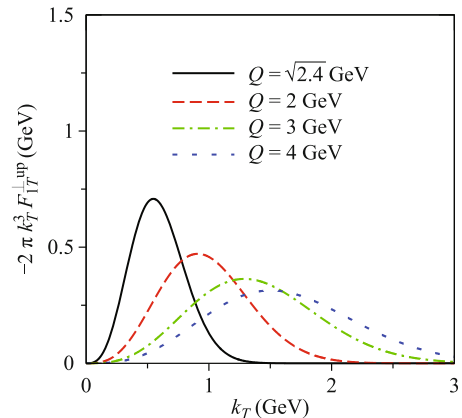


Fig. 5 Example showing the TMD evolution of the Gaussian parameterization in the low k_\perp -region. The curves show the evolved Bochum Gaussian fits to up quark Sivers function at $x = 0.1$. This figure is taken from Ref. [54].

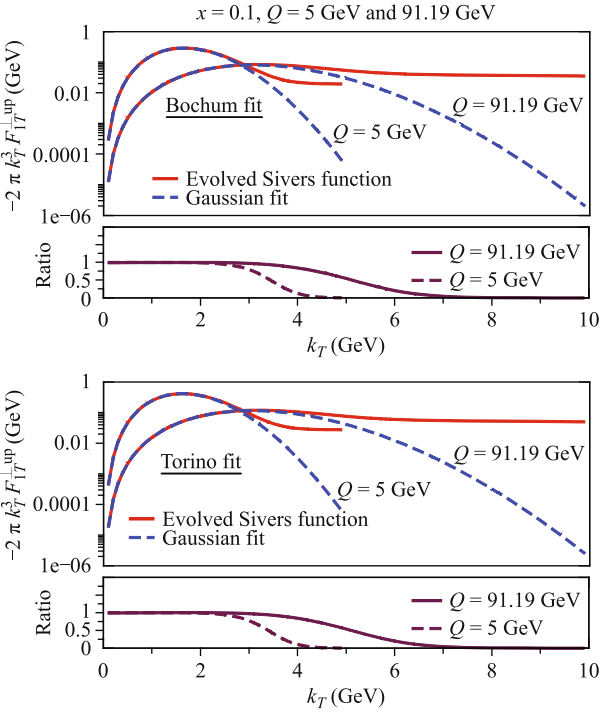


Fig. 6 Example showing the evolved k_{\perp} dependence in the large k_{\perp} region. Here we see the up-quark Siverts function at $Q = 5$ GeV and $Q = 91.19$ compared with the corresponding Gaussian fits at low- k_{\perp} region at $x = 0.1$. This figure is taken from Ref. [54].

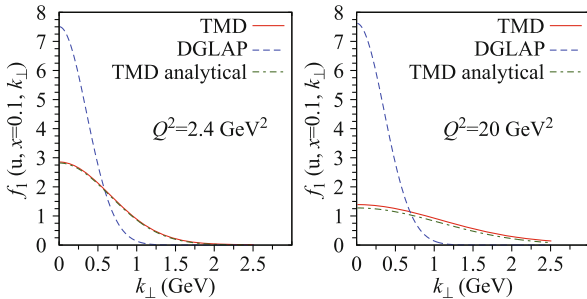


Fig. 7 Example showing the difference between the results of the TMD evolution with a DGLAP evolution for x -dependence only for unpolarized TMD PDF. This figure is taken from Ref. [55].

very significant for the TMDs. Not only the form of the k_{\perp} dependence, but also the width of the Gaussian, evolves with Q . More precisely, at small k_{\perp} , Gaussian parameterization can be used, but the width evolves with Q . At larger k_{\perp} , the form of the k_{\perp} dependence is determined mainly by gluon radiation and deviates greatly from a Gaussian; it also evolve with Q . In Fig. 5, we see an example for the evolution of the Gaussian parameterization at small k_{\perp} ; in Fig. 6, we see the evolution of the shape at large k_{\perp} . It is also important to use the comprehensive TMD evolution rather than a separate evolution of the transverse and longitudinal dependences. We show an example in Fig. 7.

The last thing we would like to mention regarding

TMD parameterizations is the TMD library (TMDlib). We are happy to see that an initial version was created [114] in 2014 and that it was updated recently.

6 Summary and outlook

In summary, by comparing the studies with what we did in constructing the theoretical framework in studying one-dimensional imaging of the nucleon with inclusive DIS, we presented a brief overview of available results on the studies of three-dimensional imaging of the nucleon with SIDIS and other semi-inclusive reactions. We summarized in particular the general form of the TMDs defined via quark-quark correlators for both TMD PDFs and FFs. We emphasized the theoretical framework for semi-inclusive reactions at LO pQCD but with leading and higher twist contributions consistently. This theoretical framework was obtained by applying the collinear expansion technique developed in the 1980s for inclusive DIS to these semi-inclusive processes. We summarized in particular how it applies to all high energy processes involving one hadron. The results obtained in such a framework should be used as starting points for studying TMDs experimentally.

Finally, we emphasize that three-dimensional imaging of the nucleon has been a hot and rapidly developing topic in recent years. Many advances have been made, and many questions are open. We see in particular that frameworks at LO pQCD and leading and higher twists for processes involving one hadron can be constructed using collinear expansions. A factorization theorem for leading twist but with LO and higher-order pQCD contributions and QCD evolution equations for unpolarized TMD PDFs and the Siverts functions have also been established. Especially in view of the operational and planned facilities such as electron-ion colliders, we expect even more rapid development in coming years.

This overview is far from complete. We apologize for omitting many aspects, such as the generalized parton distributions, the Wigner function, model calculations of TMDs, the nuclear dependences, and the hyperon polarization.

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