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RESEARCH ARTICLE

Quantum computation in triangular decoherence-free subdynamic space

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A formalism of quantum computing with 2000 qubits or more in decoherence-free subspaces is presented. The subspace is triangular with respect to the index related to the environment. The quantum states in the subspaces are projected states ruled by a subdynamic kinetic equation. These projected states can be used to perform general, large-scale decoherence-free quantum computing.

Keywords quantum information, subdynamics, decoherence-free

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1 Introduction

Although many publications in the last several years have formulated a remarkable theory for decoherence-free (DF) subspaces and subsystems, in which quantum computing is performed in a DF subspace despite the total space being subject to decoherence [1–18], the DF subspaces cannot be exactly established without introducing an approximation. Many proposals [19–26] have used the Born–Markov approximation or restrictions on the type of decoherence (e.g., symmetric and collective decoherence), even though some papers also considered DF subspaces that do not invoke the Born–Markov approximation and seems to be general regarding the type of decoherence [19]. Some well-known methods have been developed for protecting quantum systems from noise: dynamical decoupling and quantum error correction. In dynamical decoupling, designed pulses are applied to the system for protection, allowing the damaging effects of noise to nearly average away, while in quantum error correction, protected logical qubits are encoded as collective states of many physical qubits to permit the detection and overturning of damage due to noise. Some authors considered combining these two methods of error control to explore quantum computing in DF conditions [20, 23, 27]. In addition, optimal control [28] and topological protection [29] have also been developed. However, thus far, it is still difficult to apply the most general framework of DF subspaces and subsystems in practical quantum computing. In this respect, the theory in the literature is insufficient.

In this work, a different approach is used. We present quantum computation in DF triangular subspaces based on the Schrödinger-type subdynamic kinetic equation (SSKE) [30, 31] by using operator projected method. The basic characteristic of this type of quantum computing is that the eigen projectors in the subspace are remarkably invariant. This property can be used to realize general, large-scale DF quantum logic operations. As a demonstration of the application of the proposed scheme, the quantum computation of 2000 qubits or more in triangular subspaces is presented. The remainder of this article is mainly divided into two parts: the first part introduces the basic principle, and second part describes the design. For clarity, we start from the principle of our formalism.

2 Principle of formalism

Consider a general quantum open system S interacting with an environment B, the Hamiltonian of which is expressed as $H = H_S + H_B + \lambda H_{int}$ with coupling number λ. If one chooses the eigen projectors and its orthogonal projectors of a free Hamiltonian $H_0 = H_S + H_B$ as P_n and Q_n, respectively (this is possible because the free Hamiltonian is assumed to be easily diagonalized), then by means of the subdynamics theory [32], one can introduce a creation or destruction correlation operator (as a type of resolvent):

$$C_n = \lambda \frac{1}{E_n - Q_n H Q_n} Q_n H_{int} P_n = D_n^{\dagger}, \tag{1}$$

where E_n is an eigenvalue of H_S. This allows one to construct an SSKE for a projected state as

$$\mathrm{i}\frac{\partial}{\partial t}\phi_{proj} = \Theta\phi_{proj}, \tag{2}$$

where the projected wave function ϕ_{proj} is defined as

$$\phi_{proj}(t) = \sum_n P_n \Pi_n \varphi(t). \tag{3}$$

Π_n is expressed as

$$\Pi_n = (P_n + C_n)(P_n + D_n C_n)^{-1}(P_n + D_n). \tag{4}$$

$\varphi(t)$ satisfies the original Schrödinger equation, and the intermediate operator Θ is defined as

$$\Theta = H_0 + \lambda \sum_n P_n H_{int} C_n. \tag{5}$$

The creation operator C_n is independent of the representation with respect to the projectors P_n and Q_n, and it is not necessarily self-adjoint in an extended functional space. This reveals that the eigenvalues of the intermediate operator may be complex, and the evolution operator may correspond to two non-unitary semigroup evolutions. By using the physical boundary conditions, such as the ϵ-rule [30], we can determine which semigroup is correct. Therefore, the evolution of the projected density operator for the open system can be time-asymmetric and may exist in a generalized functional space beyond the Hilbert space, such as the rigged Hilbert space [33, 34]. This time-asymmetric evolution is consistent with the second law of thermodynamics.

Moreover, by replacing the Hamiltonian H with the Liouvillian L and the wave function φ with the density operator ρ, and by using the same approach as above, the SSKE can be transformed to the Liouvillian-type subdynamic equation (LSKE) for a quantum open system:

$$\mathrm{i}\frac{\partial}{\partial t}\rho_{proj} = [\Theta, \rho_{proj}]. \tag{6}$$

The creation (destruction) correlation operator can be used by means of the subdynamics theory. In fact, by introducing the eigen projectors of the total Hamiltonian, Π_n, using the Heisenberg equation and by considering the eigenvalue problem of Π_n and the definition of the creation (destruction) operators, one can obtain the basic operator equation of the creation operator. Its solutions can be expressed as retarded or advanced integrals corresponding to the two kinds of time-evolution semigroups, $t \in [0, +\infty)$ or $t \in (-\infty, 0]$, and continuations of the up and down half complex planes. In the same manner, the basic equation for the destruction operator and its relevant solution can be obtained. The second-order approximation for the LSKE also corresponds to the Master, Boltzmann, Pauli, and Fokker–Plank equations of kinetic theory and Brownian motion. For example, from the LSKE for a time-independent open system S, one can easily deduce the general Master equation by using Born–Markovian approximation. Indeed, assuming the projector

$$P = \frac{\exp(-\beta H_B)}{\mathrm{tr}_B \exp(-\beta H_B)} \mathrm{tr}_B, \tag{7}$$

then it gives the reduced density operator for the system S as $P\rho = \rho_S$ on the basis of a Markovian equation that can be reduced to a type of Lindblad equation or Boltzmann equation [36].

This construction of SSKE or LSKE in subspace can be related to the original Schrödinger or Liouville equation. For instance, by using the intertwining relations from subdynamics theory [35, 37] $\Omega\Theta\Omega^{-1} = H$, one can arrive at the original Schrödinger equation,

$$\mathrm{i}\frac{\partial}{\partial t}\phi = \Omega\Theta\Omega^{-1}\phi = H\phi, \tag{8}$$

where the similarity operator is defined as

$$\Omega = \sum_n (P_n + C_n), \tag{9}$$

which is also not necessarily unitary. The eigenvectors of the time-independent total Hamiltonian H can be given by the eigenvectors of H_0, ϕ_n, as

$$|\varphi_n\rangle = (P_n + C_n)|\phi_n\rangle \tag{10}$$

with the same structure of eigenvalues as Θ.

An interesting advantage of using the above formalism is that it allows the construction of a precise DF subspace. It is remarkable that the projected space in which subdynamics works is naturally a kind of DF subspace on choosing a suitable basis. In fact, if a system is subject to decoherence by the interaction from the environment, the spectral decomposition of Θ can be expressed by the subdynamical formalism

$$\Theta = H_0 = \sum_n E_n P_n, \tag{11}$$

or

$$\Theta = \sum_n (E_n + \Delta E_n) P_n = \Omega^{-1}(H + \Delta H)\Omega, \tag{12}$$

where it is clear the eigen projector (included eigenvector) P_n for Θ is invariant. One only needs to consider a phase shift ΔE_n for the evolution state because it leads to a type of decoherence in the subspaces. For example, if the evolution of the entangled states in the subspace is $\sum_n \exp(-\mathrm{i}E_n t)\phi_n(0)$ before the interaction with the environment, then after the interaction, the evolution of the states in the subspace becomes $\sum_n \exp\left(-\mathrm{i}\left(E_n+\Delta E_n\right)t\right)\phi_n(0)$. Therefore, the problem reduces to one of finding a procedure to cancel the change of the eigenvalues. The key idea to realizing this procedure is to use partial triangulation.

3 Quantum computing of 2000 qubits

To demonstrate the above general formalism, we demonstrate an example of a practical application by considering a typical quantum computing system S of 2000 qubits consisting of 1000 two-spins $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, as 1000 two electrons of two ^{31}P atoms are confined in 1000 different single-electron spin-resonance transistors with germanium/silicon heterostructures [38, 39] or in 1000 two-quantum dots [40–44]. The number 2000 is arbitrarily chosen, and a large number is chosen to guarantee that the quantum computing system has a sufficiently large number of qubits suitable for practical applications; the capacity of 2000 qubits is far beyond that of classical computing. Then, by ignoring the influence of the environment, the Hamiltonian can be written in the Heisenberg model as

$$H_S(\tau)=\sum_m^{1000} J_m\left(\tau\right)\boldsymbol{S}_1^m\cdot\boldsymbol{S}_2^m, \tag{13}$$

where $J_m(\tau)$ is the time-dependent exchange-coupling parameter determined by the specific model considerations and the index m denotes the mth two-qubit system; each two-qubit system may have the same structure with different physical parameters. In the case of spins of the two electrons (e.g., confined in two vertically (laterally) coupled quantum dots [41]), J_m is the mth difference in the energies of a two-electron ground state, a spin singlet at zero magnetic field, and the lowest spin-triplet state; J_m is also a function of the electric and magnetic fields and the interdot distance. Using the relationship between $\boldsymbol{S}_1^m\cdot\boldsymbol{S}_2^m$ and the square of the sum of $\boldsymbol{S}_1^m$ and $\boldsymbol{S}_2^m$, the eigenvalues and eigenvectors of $\boldsymbol{S}_1^m\cdot\boldsymbol{S}_2^m$ can be determined from

$$\boldsymbol{S}_1^m\cdot\boldsymbol{S}_2^m=\frac{1}{2}\left(\boldsymbol{S}^{m2}-\frac{3}{2}\right) \tag{14}$$

using $E_1=E_2=E_3=\frac{1}{2}$ corresponding to

$$\left\{|\phi_1\rangle=|11\rangle,|\phi_2\rangle=|00\rangle,|\phi_3\rangle=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)\right\} \tag{15}$$

and $E_4=-3/4$ corresponding to

$$|\phi_4\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle). \tag{16}$$

Assuming that the working qubits are a combination of the direct tensor product of the four states, such that

$$\begin{aligned}&|\varphi_1\rangle\equiv|00\rangle\otimes|00\rangle\otimes\cdots\otimes|00\rangle,\\&\quad\cdots\\&|\varphi_{4^{1000}}\rangle\equiv|10\rangle\otimes|11\rangle\otimes\cdots\otimes|11\rangle,\end{aligned} \tag{17}$$

where

$$\begin{aligned}&|00\rangle=|\uparrow\uparrow\rangle=|\phi_2\rangle,\\&|01\rangle=|\uparrow\downarrow\rangle=\frac{\sqrt{2}}{2}\left(|\phi_3\rangle+|\phi_4\rangle\right),\\&|10\rangle=|\downarrow\uparrow\rangle=\frac{\sqrt{2}}{2}\left(|\phi_3\rangle-|\phi_4\rangle\right),\\&|11\rangle=|\downarrow\downarrow\rangle=|\phi_1\rangle.\end{aligned} \tag{18}$$

An advantage of constructing the DF subspace for this 2000-qubit model is that in any two-qubit system, the eigenstates $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, and $|\phi_4\rangle$ are invariant in the decoherence. Therefore, $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ remain invariant, enabling all working states $|\varphi_1\rangle$, $\cdots$, $|\varphi_{4^{1000}}\rangle$ to be stable without decoherence.

The quantum logic gate C_N can be extended on the basis of a combination of the sequence of operations to two qubits,

$$U_{CN}=\sum_{m=1}^{1000}U_{CNm}, \tag{19}$$

where

$$U_{CNm}=\left(\mathrm{e}^{\mathrm{i}(\pi/2)S_1^z}\mathrm{e}^{-\mathrm{i}(\pi/2)S_2^z}U_{sw}^{1/2}\mathrm{e}^{\mathrm{i}\pi S_1^z}U_{sw}^{1/2}\right)_m \tag{20}$$

is the mth Control-Not operator for the mth two qubits ($|10\rangle_m,|11\rangle_m,|00\rangle_m,|01\rangle_m$). A quantum Control-Not (CN) gate of any two qubits can be given by the sequence of operations,

$$\mathrm{e}^{\mathrm{i}(\pi/2)S_1^z}\mathrm{e}^{-\mathrm{i}(\pi/2)S_2^z}U_{sw}^{1/2}\mathrm{e}^{\mathrm{i}\pi S_1^z}U_{sw}^{1/2}, \tag{21}$$

where U_{sw} is an ideal swap operator that can exchange the quantum states of qubits 1 and 2 and is generally determined by an evolution operator $U_s\left(J\left(\tau\right)\right)$ by adjusting the coupling time τ between the two spins in the evolution of the system. For a particular spin–spin coupling duration, τ_s, where

$$\begin{aligned}&\int_0^{\tau_s}J(\tau)\mathrm{d}\tau=\pi\left(\mathrm{mod}\ 2\pi\right),\\&U_{sw}=U_s\left(\pi\right),\end{aligned} \tag{22}$$

the swap operator that can exchange the quantum states of qubits 1 and 2 is given by [40]

$$\begin{aligned}U_{sw}&=\mathrm{e}^{-\mathrm{i}\int_0^{\tau_s}\ \Theta(\tau)\mathrm{d}\tau}\\&=\sum_{n=1}^{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}|\phi_n\rangle\langle\phi_n|+\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}|\phi_4\rangle\langle\phi_4|.\end{aligned} \tag{23}$$

Generally, the interaction part of the Hamiltonian between the system and environment is assumed as H_{int}, which is unknown, while the Hamiltonian H_B of the environment is assumed to be diagonal with known eigenvectors. Then, the complete set of eigenvectors for the free part of $H_0 = H_S(\tau_s) + H_B$ is denoted as

$$\left\{|\varphi_j \otimes \{k\}\rangle, \langle\{k\} \otimes \varphi_j|, j = 1, \cdots, 4^{1000}\right\}. \tag{24}$$

Owing to the presence of the environment, a non-ideal action of the swap operation must be considered because the environment may introduce decoherence in the ideal swap operation. Therefore, one can generally assume that the eigen projectors P_{nk} of H_0 is formally given by

$$P_{nk} = |\varphi_j \otimes \{k\}\rangle\langle\{k\} \otimes \varphi_j|. \tag{25}$$

In order to control the induced decoherence, by using the definition of the eigen projectors P_{nk}, the spectral decomposition of the intermediate operator $\mathit{\Theta}$ is given by two cases:

$$\mathit{\Theta} = \sum_{m=1}^{1000}\sum_{n=1}^{4}\left(\langle\varphi_{nk}|H_0|\varphi_{nk}\rangle P_{nk},\right)_m, \text{ without the interaction,} \tag{26}$$

and

$$\begin{aligned}\mathit{\Theta} = {} & \sum_{m=1}^{1000}\sum_{n=1}^{4}\left(\langle\varphi_{nk}|H_0|\varphi_{nk}\rangle P_{nk}\right)_m + \sum_{m=1}^{1000}\left(\langle\varphi_{1k}|\lambda H_{int} Q_{1k}\frac{1}{E_{1k}(t) - Q_{1k}H(t)Q_{1k}}Q_{1k}\lambda H_{int}|\varphi_{1k}\rangle P_{1k}\right)_m \\ & + \sum_{m=1}^{1000}\left(\langle\varphi_{2k}|\lambda H_{int} Q_{2k}\frac{1}{E_{2k}(t) - Q_{2k}H(t)Q_{2k}}Q_{2k}\lambda H_{int}|\varphi_{2k}\rangle P_{2k}\right)_m, \text{ with the interaction,}\end{aligned} \tag{27}$$

where E_{1k} and E_{2k} are the eigenvalues of H_0 corresponding to the eigen projectors of $\mathit{\Theta}$, P_{1k}, and P_{2k}, respectively. This shows that the eigen projectors P_{nk} of $\mathit{\Theta}$ are invariant and independent of the interaction H_{int} in the constructed subspace $\mathcal{H}_S \otimes \mathit{\Phi}_{sub}$ for $\varphi_j \in \mathcal{H}_S$, $\{k\} \in \mathit{\Phi}_{sub}$, although the eigenvalues for $n = 1, 2$ are changed. Here, $\mathit{\Phi}_{sub}$ is a subdynamic space, which can be triangulated as $\mathit{\Phi}_{utri}$ by introducing an upper-triangular inner product: for any operator A in $\mathit{\Phi}_{sub}$, $\langle k|A|k'\rangle \neq 0$ for $k' \geqslant k$ and $\langle k|A|k'\rangle = 0$ for $k' < k$ are introduced. Then, it can be easily seen that in the interaction part of $\mathit{\Theta}$, if the left matrix $\langle\varphi_{nk}|H_{int}|\varphi_{nk'}\rangle \neq 0$, then the triangulation always gives $k' > k$, while the middle matrix $\langle\varphi_{nk'}|\frac{1}{E_{nk}(t) - Q_{nk}H(t)Q_{nk}}|\varphi_{nk''}\rangle \neq 0$ always gives $k'' \geqslant k'$; both make the index of k always increase to k'' so that the right matrix is $\langle\varphi_{nk''}|H_{int}|\varphi_{nk}\rangle = 0$ because $k'' > k$. Therefore, the triangulation enables the interaction part of $\mathit{\Theta}$ to be zero.

In the subspace $\mathcal{H}_S \otimes \mathit{\Phi}_{utri}$, the quantum Control-Not logical operation U_{CN} can be executed by using a sequence of operations, and its relevant swap operator U_{sw} remains invariant before and after interaction with the environment. This can be described by the following formula:

$$\begin{aligned}& \mathrm{tr}_B \mathrm{e}^{-\mathrm{i}\int_0^{\tau_s}\mathit{\Theta}(\tau)\mathrm{d}\tau} \\ & \quad = \sum_{m=1}^{1000}\sum_{n=1}^{3}\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}|\phi_n\rangle\langle\phi_n| + \mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}|\phi_4\rangle\langle\phi_4|\right)_m c,\end{aligned} \tag{28}$$

where $c = \mathrm{tr}_B(H_B)$ is a constant in the upper-triangular subspace $\mathit{\Phi}_{utri}$, implying that it does not influence the original swap operation. Moreover, the operators $\mathrm{e}^{\mathrm{i}\pi S_1^z}$, $\mathrm{e}^{-\mathrm{i}(\pi/2)S_2^z}$, and $\mathrm{e}^{\mathrm{i}(\pi/2)S_1^z}$ directly interact with spin 2 (qubit 2) and spin 1 (qubit 1), respectively, in the original Hilbert space $\mathcal{H}_S$. Their interactions are obviously independent of the subspace $\mathit{\Phi}_{utri}$. By combining these two points, it is clear that the upper triangulation has no effect on the original CN logic operation, U_{CN}. This implies that in this upper-triangular subspace, U_{CN} remains invariant with the DF property, and triangulating the inner product for the construction of a suitable subspace cannot influence or change the original operation U_{CN}.

4 General discussion

From the preceding discussion, the procedure to construct the upper-triangular subspaces is planned as follows: if one performs the relevant quantum logic operations using the state φ, then one should adopt the normal inner product with respect to the index n for the quantum computing system and the upper-triangular inner product with respect to the index k for the environment.

Remember that the evolution of the states are described by $\mathrm{tr}_B \exp(-\mathrm{i}\mathit{\Theta}t)\varphi(0)$ in the upper-triangular subspace. The properties or rules in the upper-triangular subspaces are as follows:

i) Working states $|\varphi_1\rangle, \cdots, |\varphi_{4^{1000}}\rangle$, which are a combination of the eigenstates $(|\phi_1\rangle, \cdots, |\phi_4\rangle)$.

ii) Evolution $\varphi_j(t) = \mathrm{tr}_B \exp(-\mathrm{i}\Theta t)\varphi_j$, which relates to Θ operator in the triangular subspace.

iii) For any operator A, $\langle k|A|k'\rangle \neq 0$ for $k' \geqslant k$ and $\langle k|A|k'\rangle = 0$ for $k' < k$;

vi) For any state ψ satisfying the Schrödinger equation, the corresponding SSKE is $\mathrm{i}\frac{\partial\phi}{\partial t} = \Theta\phi$, where the transformation of the states between the original Hilbert space and the projected triangular subspace is given by $|\phi\rangle = (P + D)|\psi\rangle$. However, if $\psi = \varphi$, then $\phi = (P + D)\varphi = \varphi$, where $P\varphi = \varphi$ and $D\varphi = 0$.

v) The quantum CN logic operation remains unchanged in the subspace because the invariance of $(U_{sw})_m$ determines the invariance of $(U_{CN})_m$ so that U_{CN} is stable.

Finally, the physical realization of the subdynamic space is as follows: if one measures the working states $\varphi_1, \cdots, \varphi_4$ in the original system, then on the basis of Eq. (10), they are related to the working states in the projected subdynamic system by

$$|\varphi_1\rangle = |\phi_1\rangle, \quad |\varphi_2\rangle = |\phi_2\rangle + \frac{h_{12}}{\frac{1}{2} - h_{11}|\phi_1\rangle\langle\phi_1|}|\phi_1\rangle, \quad |\varphi_3\rangle = |\phi_3\rangle + \frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11}|\phi_1\rangle\langle\phi_1| - h_{22}|\phi_2\rangle\langle\phi_2|}\left(|\phi_2\rangle + |\phi_1\rangle\right),$$

$$|\varphi_4\rangle = |\phi_4\rangle + \frac{h_{13} + h_{23}}{-\frac{3}{4} - h_{11}|\phi_1\rangle\langle\phi_1| - h_{22}|\phi_2\rangle\langle\phi_2| - h_{33}|\phi_3\rangle\langle\phi_3|}\left(|\phi_3\rangle + |\phi_2\rangle + |\phi_1\rangle\right). \tag{29}$$

Hence, the inversive states $\phi_1, \cdots, \phi_4$ in the subdynamic space can be transformed by

$$|\phi_1\rangle = |\varphi_1\rangle, \qquad |\phi_2\rangle = -\frac{h_{12}}{\frac{1}{2} - h_{11}}|\varphi_1\rangle + |\varphi_2\rangle,$$

$$|\phi_3\rangle = \left(\frac{h_{12}}{\frac{1}{2} - h_{11}} - 1\right)\frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11} - h_{22}}|\varphi_1\rangle - \frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11} - h_{22}}|\varphi_2\rangle + |\varphi_3\rangle,$$

$$|\phi_4\rangle = \left(\frac{h_{12}}{\frac{1}{2} - h_{11}} - 1 + \frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11} - h_{22}} - \frac{h_{12}}{\frac{1}{2} - h_{11}}\frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11} - h_{22}}\right)\frac{h_{13} + h_{23}}{-\frac{3}{4} - h_{11} - h_{22} - h_{33}}|\varphi_{,1}\rangle$$

$$+ \left(\frac{h_{13} + h_{23}}{\frac{1}{2} - h_{11} - h_{22}} - 1\right)\frac{h_{13} + h_{23}}{-\frac{3}{4} - h_{11} - h_{22} - h_{33}}|\varphi_2\rangle - \frac{h_{13} + h_{23}}{-\frac{3}{4} - h_{11} - h_{22} - h_{33}}|\varphi_3\rangle + |\varphi_4\rangle, \tag{30}$$

where $h_{ij} = \langle\phi_i|H_{int}|\phi_j\rangle$ for $i < j$ and $i, j = 1, 2, 3$. This formula can be used to transform the states in the original system in the Hilbert space to the subsystem in the subspace in practice.

For extending the 2000-qubit system, the interaction of the environment with the system may be quite different in different models, and a general N-qubit system for constructing the DF subspace by triangulation is proposed here.

Suppose a general quantum computing system uses working states that are the eigenvectors of the free Hamiltonian H_0. Thus, the spectral decomposition for the intermediate operator based on the above subdynamic formalism is given by

$$\Theta = \sum_n \mathrm{tr}_k E_{nk}(t)P_{nk} = H_S, \tag{31}$$

where P_{nk} is an eigen projector of Θ and E_{nk} is an eigenvalue of Θ. Now, suppose the system is subject to decoherence induced by a general unknown H_{int} and the matrix of the Hamiltonian becomes off-diagonal with respect to the index k related to the environment. Again, the upper-triangular subspace can be constructed using the rule of the upper-triangular inner product, such that

$$\langle\varphi_{nk}|H_{int}|\varphi_{n'k'}\rangle \neq 0, \text{ for } k' \geqslant k;$$
$$\langle\varphi_{nk}|H_{int}|\varphi_{n'k'}\rangle = 0, \text{ for } k' < k. \tag{32}$$

This leads to an upper-triangular matrix H_{int}. By utilizing this upper-triangular property, the interaction part in the operator becomes zero. Thus, the spectral decomposition maintains invariance, and $\Theta = H_S$, where P_{nk} is now chosen as an upper-triangular projector with respect to k. This allows a constructed triangular subspace to be DF.

It may be necessary to emphasize that the states required to perform quantum computing in the DF (triangular) subspace are the projected states and are ruled by SSKE, where the triangular inner product only applies to the index applicable to the environment; here, it stands for k, while for other indexes related to the original quantum computing system, such as an N-qubit (where N is arbitrary) quantum computing system, the normal inner product remains. In general, if the change of the environment induces an unknown H_B and H_{int}, then the CN operation for N qubits in the triangular subspace is still described by $\mathrm{tr}(H_B)U_{CN}$ for any integer N. This shows that $\mathrm{tr}(H_B)$ is transformed to a constant; consequently, U_{CN} is invariant in the DF condition.

Therefore, with the basic concept of the above triangular subspace, for the possibility of implementing quantum computation using our model with a very large number of qubits, we suggest that it is necessary to build an extra correction system (or combine an extra correction system with the original error-correction system) in the original quantum measuring system. The task of this correction system is to transform the states in the original space to the subdynamic triangular space so that for any two-qubit state in the two coupling quantum dots, if one measures the working states $\varphi_1, \cdots, \varphi_4$ in the original system, then the correction system directly transforms them to the states $\phi_1, \cdots, \phi_4$ in the subdynamic space; further, if the evolution of the working states is $\mathrm{e}^{-\mathrm{i}(E_1+\Delta E_1)t}\varphi_1, \cdots, \mathrm{e}^{-\mathrm{i}(E_4+\Delta E_4)t}\varphi_4$, then the correction system directly transforms them to $\mathrm{e}^{-\mathrm{i}E_1 t}\phi_1, \cdots, \mathrm{e}^{-\mathrm{i}E_4 t}\phi_4$ without considering the Hamiltonian and relevant calculations, since from the properties of the subdynamic space and triangular subspace, it is known that if $\mathrm{e}^{-\mathrm{i}(E_1+\Delta E_1)t}\varphi_1, \cdots, \mathrm{e}^{-\mathrm{i}(E_4+\Delta E_4)t}\varphi_4$ are in the original space, then $\mathrm{e}^{-\mathrm{i}(E_1+\Delta E_1)t}\phi_1, \cdots, \mathrm{e}^{-\mathrm{i}(E_4+\Delta E_4)t}\phi_4$ are in the subdynamic space, while $\mathrm{e}^{-\mathrm{i}E_1 t}\phi_1, \cdots, \mathrm{e}^{-\mathrm{i}E_4 t}\phi_4$ are in the triangular subspace, where notice again that $E_1 = E_2 = E_3 = \frac{1}{2}$, $E_4 = -\frac{3}{4}$ and $\phi_1, \cdots, \phi_4$ are the eigenvalues and eigenvectors of the free Hamiltonian, respectively. Hence, by using this correction system, one can project the working states in the original system to the DF states in the triangular subspace for any coupling quantum dots. Furthermore, the 2000-qubit (or more) system can be extended by protecting every state in two coupling quantum dots in this manner.

5 Conclusion

In conclusion, a model for quantum computing with 2000 (or more) qubits in a DF subspace has been presented, which highlights that the rule of the upper-triangular inner product in the subspace provides a suitable DF condition. The original quantum CN logic operation cannot restrict or change if this CN operator is suitably designed within the upper-triangular subspace. The entanglement between the system and environment is canceled within the triangular subspace, although decoherence exists in the original total space. The role of the upper-triangular inner product with respect to the index for the environment is to cancel the decoherence (phase shift) due to the environment. This indicates that in a quantum computing process, if one chooses the rule of the upper-triangular inner product related to the index of the environment and keeps the rule of the ordinary inner product related to the index of the quantum computing system during relevant calculations of matrix elements of the Hamiltonian, the decoherence in the subsystems can be completely canceled, although the decoherence in the original system still exists. This provides an efficient physical method to realize DF quantum computation in triangular subdynamic spaces. Moreover, this approach is exact and general for quantum computation with an arbitrary number of qubits in a complicated environment.

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