

On the entangled fractional squeezing transformation

Hong-Yi Fan^{1,*}, Jun-Hua Chen^{2,†}, Peng-Fei Zhang³

¹Department of Physics, Ningbo University, Ningbo 315211, China

²Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China

³Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China

Corresponding authors. E-mail: *fhym@ustc.edu.cn, †cjh@ustc.edu.cn

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We propose an entangled fractional squeezing transformation (EFrST) generated by using two mutually conjugate entangled state representations with the following operator: $e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} e^{i\pi a_2^\dagger a_2}$; this transformation sharply contrasts the complex fractional Fourier transformation produced by using $e^{-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} e^{i\pi a_2^\dagger a_2}$ (see *Front. Phys.* DOI 10.1007/s11467-014-0445-x). The EFrST is obtained by converting the triangular functions in the integration kernel of the usual fractional Fourier transformation into hyperbolic functions, i.e., $\tan \alpha \rightarrow \tanh \alpha$ and $\sin \alpha \rightarrow \sinh \alpha$. The fractional property of the EFrST can be well described by virtue of the properties of the entangled state representations.

Keywords entangled fractional squeezing transformation, entangled state representation, squeezing operator, core operator

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1 Introduction

In Fourier optics and information optics, the fractional Fourier transformation (FrFT) is a very useful tool. The concept of the FrFT was originally introduced for signal processing in 1980 by Namias [1] as a Fourier transform of fractional order. However, the FrFT did not have a significant impact on optics until it was defined physically based on propagation in quadratic graded-index media (GRIN media) [2] by Mendlovic, Ozaktas, and Lohmann. Nowadays, the FrFT is widely employed in optical communication, image manipulation, and signal analysis.

The one-dimensional FrFT of α -th order (for a real α angle) is defined as

$$F_\alpha[f](p) = \int_{-\infty}^{\infty} V_\alpha(p, x) f(x) dx, \quad (1)$$

where

$$\begin{aligned} V(p, x) &\equiv \langle p | e^{i(\frac{\pi}{2} - \alpha) a^\dagger a} | x \rangle \\ &= \frac{e^{i(\alpha/2 - \pi/4)}}{\sqrt{2\pi \sin \alpha}} \exp \left[\frac{i(p^2 + x^2)}{2 \tan \alpha} - \frac{ipx}{\sin \alpha} \right] \end{aligned} \quad (2)$$

and $|x\rangle$ and $\langle p|$ are coordinate and momentum eigenstates, respectively. In Refs. [3–5], two mutually conju-

gate entangled state representations,

$$|\eta\rangle = \exp \left(-\frac{1}{2} |\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger \right) |00\rangle, \quad (3)$$

$$\eta = \eta_1 + i\eta_2,$$

$$|\xi\rangle = \exp \left(-\frac{1}{2} |\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger \right) |00\rangle, \quad (4)$$

$$\xi = \xi_1 + i\xi_2,$$

are introduced, where $[a_i, a_j^\dagger] = \delta_{i,j}$, $(i, j) = 1, 2$. $|00\rangle$ is the vacuum state in two-mode Fock space. $|\eta\rangle$ and $|\xi\rangle$ satisfy the completeness relations

$$\int \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1 \quad (5)$$

and

$$\int \frac{d^2\xi}{\pi} |\xi\rangle \langle \xi| = 1. \quad (6)$$

Based on this, in Ref. [6] we introduced the complex fractional Fourier transformation (CFrFT) (referred to as the entangled FrFT) with parameter α , denoted as

$$K_\alpha[f] = \int \frac{d^2\xi}{\pi} K_\alpha(\eta, \xi) f(\xi), \quad (7)$$

where

$$K_\alpha(\eta, \xi) = \frac{e^{i(\alpha - \frac{\pi}{2})}}{2 \sin \alpha} \exp \left[\frac{i(|\eta|^2 + |\xi|^2)}{2 \tan \alpha} - i \frac{\xi \eta^* + \xi^* \eta}{2 \sin \alpha} \right] \equiv \langle \eta | K_\alpha | \xi \rangle. \tag{8}$$

K_α is called the CFrFT operator, and we have shown in [6] that

$$K_\alpha = e^{-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} e^{i\pi a_2^\dagger a_2}. \tag{9}$$

$e^{i\pi a_2^\dagger a_2}$ is called the core operator, which we emphasize as being essential to the additive rule $K_\alpha \circ K_\beta[f] = K_{\alpha+\beta}[f]$ of the CFrFT.

An interesting question thus arises: if we change the integration kernel $K_\alpha(\eta, \xi)$ in Eq. (8) to

$$\mathfrak{S}_\alpha(\eta, \xi) \equiv \frac{1}{2i \sinh \alpha} \exp \left[\frac{i(|\eta|^2 + |\xi|^2)}{2 \tanh \alpha} - i \frac{\xi \eta^* + \xi^* \eta}{2 \sinh \alpha} \right], \tag{10}$$

i.e., change the triangular functions to hyperbolic functions,

$$\tan \alpha \rightarrow \tanh \alpha, \quad \sin \alpha \rightarrow \sinh \alpha, \tag{11}$$

then to what kind of transformation will $\mathfrak{S}_\alpha(\eta, \xi)$ belong? In the following we shall demonstrate that $\mathfrak{S}_\alpha(\eta, \xi)$ is just the integration kernel of a new entangled fractional squeezing transformation (EFrST). As is well-known, the squeezing transformation has been a major topic in quantum optics because it produces a squeezed state that exhibits less quantum fluctuation than a coherent state in one quadrature at the expense of more fluctuation in another quadrature [7, 8]. In this work we shall first propose an EFrST and then prove that it also satisfies additivity. The work is arranged as follows: In Section 2 we briefly review the meaning of “fractional” involved in (7) and (8); in so doing we expose that there exists a core operator for the additivity $K_\alpha \circ K_\beta[f] = K_{\alpha+\beta}[f]$. In Section 3 we demonstrate that the unitary operator $e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} e^{i\pi a_2^\dagger a_2}$ is responsible for the EFrST, where $e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)}$ is a two-mode squeezing operator. We are able to do this because we have devised an integration technique within the ordered product of operators with which we can directly perform integration over Dirac’s ket-bra operators [9, 10]. In Section 4 we prove that the EFrST also satisfies additivity, and that the operator $e^{i\pi a_2^\dagger a_2}$ is indispensable to realizing the additivity. In Section 5 we briefly discuss possible physical application of the EFrST. We expect that the EFrST may be implemented in combinations of quadratic nonlinear crystals with different phase mismatches.

2 Brief review of the meaning of “fractional” in (9)

By “fractional” we mean the additive rule (or the compositional rule) $K_\alpha \circ K_\beta[f] = K_{\alpha+\beta}[f]$; i.e., two successive CFrFTs with parameters α and β are equivalent to

$$\begin{aligned} K_\alpha \circ K_\beta[f] &= \int \frac{d^2 \xi}{\pi} \int \frac{d^2 \xi'}{\pi} K_\alpha(\eta, \xi') \cdot [K_\beta(\eta', \xi) |_{\eta'=\xi'}] f(\xi) \\ &= \int \frac{d^2 \xi}{\pi} K_{\alpha+\beta}(\eta, \xi) f(\xi) \\ &= \frac{e^{i(\alpha+\beta - \frac{\pi}{2})}}{2 \sin(\alpha + \beta)} \int \frac{d^2 \xi}{\pi} \exp \left[\frac{i(|\eta|^2 + |\xi|^2)}{2 \tan(\alpha + \beta)} - i \frac{\xi \eta^* + \xi^* \eta}{2 \sin(\alpha + \beta)} \right] f(\xi) \\ &= K_{\alpha+\beta}[f]. \end{aligned} \tag{12}$$

Note that $K_\alpha(\eta, \xi) \equiv \langle \eta | K_\alpha | \xi \rangle$ is a matrix element transition from $|\xi\rangle$ to $\langle \eta |$. In terms of Dirac’s symbol, Eq. (12) can be expressed as

$$\begin{aligned} &\int \frac{d^2 \xi}{\pi} \int \frac{d^2 \xi'}{\pi} \langle \eta | K_\alpha | \xi' \rangle_{\eta'=\xi'} \langle \eta' | K_\beta | \xi \rangle \langle \xi | f \rangle \\ &= \int \frac{d^2 \xi}{\pi} \langle \eta | K_\alpha W K_\beta | \xi \rangle \langle \xi | f \rangle \\ &= \int \frac{d^2 \xi}{\pi} \langle \eta | K_{\alpha+\beta} | \xi \rangle \langle \xi | f \rangle, \end{aligned} \tag{13}$$

where the core operator

$$W \equiv \int \frac{d^2 \xi'}{\pi} |\xi'\rangle_{\eta'=\xi'} \langle \eta' | = \exp(-i\pi a_2^\dagger a_2) \tag{14}$$

plays the role of transforming $|\eta\rangle$ to $|\xi\rangle_{\xi=\eta}$,

$$\begin{aligned} W|\eta\rangle &= \exp \left(-\frac{1}{2} |\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger e^{-i\pi} + e^{-i\pi} a_1^\dagger a_2^\dagger \right) \cdot |00\rangle = |\xi\rangle_{\xi=\eta}, \end{aligned} \tag{15}$$

and from (12) and (15) we know that

$$\begin{aligned} K_{\alpha+\beta} &= K_\alpha W K_\beta = \exp[-i(\alpha + \beta) \cdot (a_1^\dagger a_1 + a_2^\dagger a_2)] \exp(i\pi a_2^\dagger a_2). \end{aligned} \tag{16}$$

In the following we will show that W is still needed for composing the EFrST.

3 Operator responsible for the EFrST

Now we turn to examine Eq. (10). Multiplying $\mathfrak{S}_\alpha(\eta, \xi)$ by $|\eta\rangle$ from the left and $\langle \xi |$ from the right, then using the normal ordering form of the two-mode vacuum state

projector

$$|00\rangle\langle 00| =: \exp(-a_1^\dagger a_1 - a_2^\dagger a_2):, \tag{17}$$

as well as Eqs. (3) and (4), we employ our integration within ordered product (IWOP) technique to perform the following integration:

$$\begin{aligned} & \int \frac{d^2\xi}{\pi} \int \frac{d^2\eta}{\pi} |\eta\rangle \mathfrak{S}_\alpha(\eta, \xi) \langle \xi| \\ &= \int \frac{d^2\xi}{\pi} \int \frac{d^2\eta}{\pi} \exp\left(-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right) \\ & \quad \times |00\rangle \mathfrak{S}_\alpha(\eta, \xi) \langle 00| \\ & \quad \times \exp\left(-\frac{1}{2}|\xi|^2 + \xi^* a_1 + \xi a_2 - a_1 a_2\right) \\ &= \frac{1}{2i \sinh \alpha} \int \frac{d^2\xi}{\pi} \int \frac{d^2\eta}{\pi} : \exp\left[\frac{|\xi|^2}{2}(i \coth \alpha - 1) \right. \\ & \quad \left. - i \frac{\xi \eta^* + \xi^* \eta}{2 \sinh \alpha} + \xi^* a_1 + \xi a_2 + \frac{|\eta|^2}{2}(i \coth \alpha - 1) \right. \\ & \quad \left. + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger - a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right]: \\ &= \frac{1}{i \sinh \alpha + \cosh \alpha} \int \frac{d^2\eta}{\pi} : \exp\left[\frac{i|\eta|^2}{\tanh \alpha - i} \right. \\ & \quad \left. + \eta \left(a_1^\dagger - \frac{ia_2}{\sinh \alpha - i \cosh \alpha}\right) \right. \\ & \quad \left. - \eta^* \left(a_2^\dagger + \frac{ia_1}{\sinh \alpha - i \cosh \alpha}\right) + a_1^\dagger a_2^\dagger - a_1^\dagger a_1 \right. \\ & \quad \left. - a_2^\dagger a_2 + \frac{1 + i \coth \alpha}{1 - i \coth \alpha} a_1 a_2\right]: \\ &= \text{sech} \alpha : \exp\{-ia_1^\dagger a_2^\dagger \tanh \alpha + (\text{sech} \alpha - 1)a_1^\dagger a_1 \\ & \quad + (-\text{sech} \alpha - 1)a_2^\dagger a_2 + ia_1 a_2 \tanh \alpha\}:. \tag{18} \end{aligned}$$

If we let

$$\mathfrak{S}_\alpha(\eta, \xi) = \langle \eta | \mathfrak{S}_\alpha | \xi \rangle \tag{19}$$

and use the operator identity

$$e^{\lambda a_1^\dagger a_1} =: \exp[(e^\lambda - 1)a_1^\dagger a_1]: \tag{20}$$

and

$$e^{-i\pi a_2^\dagger a_2} e^{ia_1 a_2 \tanh \alpha} e^{i\pi a_2^\dagger a_2} = e^{-ia_1 a_2 \tanh \alpha} \tag{21}$$

and

$$\begin{aligned} & \text{sech} \alpha e^{-ia_1^\dagger a_2^\dagger \tanh \alpha} e^{(a_1^\dagger a_1 + a_2^\dagger a_2) \ln \text{sech} \alpha} e^{-ia_1 a_2 \tanh \alpha} \\ &= e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)}, \tag{22} \end{aligned}$$

as well as Eqs. (5) and (6), we see that

$$\begin{aligned} \mathfrak{S}_\alpha &= \text{sech} \alpha e^{-ia_1^\dagger a_2^\dagger \tanh \alpha} e^{a_1^\dagger a_1 \ln \text{sech} \alpha} e^{a_2^\dagger a_2 \ln(-\text{sech} \alpha)} \\ & \quad \times e^{ia_1 a_2 \tanh \alpha} \\ &= \text{sech} \alpha e^{-ia_1^\dagger a_2^\dagger \tanh \alpha} e^{a_1^\dagger a_1 \ln \text{sech} \alpha} e^{a_2^\dagger a_2 \ln(-\text{sech} \alpha)} \end{aligned}$$

$$\begin{aligned} & \times e^{a_2^\dagger a_2 \ln e^{i\pi}} e^{-a_2^\dagger a_2 \ln e^{i\pi}} e^{ia_1 a_2 \tanh \alpha} \\ &= \text{sech} \alpha e^{-ia_1^\dagger a_2^\dagger \tanh \alpha} e^{(a_1^\dagger a_1 + a_2^\dagger a_2) \ln \text{sech} \alpha} \\ & \quad \times e^{-ia_1 a_2 \tanh \alpha} e^{-i\pi a_2^\dagger a_2} \\ &= e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} e^{-i\pi a_2^\dagger a_2} \\ &= e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)}, \tag{23} \end{aligned}$$

where $e^{-i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)}$ is a two-mode squeezing operator. Because a two-mode squeezed state is itself an entangled state, we call $e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)}$ the entangled fractional squeezing operator, which is quite different from the operator $\exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)] \exp[i\pi a_2^\dagger a_2]$ for the entangled FrFT in Eq. (16).

4 Additivity property of the EFrST

Now we show that $e^{-i\pi a_2^\dagger a_2}$ is indispensable to realizing the additivity of two EFrSTs. Since

$$\begin{aligned} & \exp[i\pi a_2^\dagger a_2] |\eta\rangle \\ &= \exp\left(-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger e^{i\pi} + e^{i\pi} a_1^\dagger a_2^\dagger\right) |00\rangle \\ &= |\xi\rangle_{\xi=\eta} \tag{24} \end{aligned}$$

we have

$$\langle \eta | \exp[-i\pi a_2^\dagger a_2] = \langle \xi |_{\xi=\eta} \tag{25}$$

and therefore

$$\begin{aligned} \langle \eta | \mathfrak{S}_\alpha | \xi \rangle &= \langle \eta | e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi \rangle \\ &= \xi' = \eta \langle \xi' | e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi \rangle. \tag{26} \end{aligned}$$

The additivity can be discussed in terms of the Dirac symbol; i.e., by using Eqs. (25) and (5) and (6), we have

$$\begin{aligned} & (\mathfrak{S}_\alpha \circ \mathfrak{S}_\beta)[f](\eta) \\ &= \int \frac{d^2\xi'}{\pi} \mathfrak{S}_\alpha(\eta, \xi') \int \frac{d^2\xi}{\pi} [\mathfrak{S}_\beta(\eta', \xi) |_{\eta'=\xi'}] f(\xi) \\ &= \int \frac{d^2\xi}{\pi} \int \frac{d^2\xi'}{\pi} \langle \eta | e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi' \rangle_{\eta'=\xi'} \\ & \quad \times \langle \eta' | e^{-i\pi a_2^\dagger a_2} e^{i\beta(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi \rangle \langle \xi | f \rangle \\ &= \int \frac{d^2\xi}{\pi} \int \frac{d^2\xi'}{\pi} \langle \eta | e^{-i\pi a_2^\dagger a_2} e^{i\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi' \rangle \\ & \quad \times \langle \xi' | e^{i\beta(a_1^\dagger a_2^\dagger + a_1 a_2)} | \xi \rangle \langle \xi | f \rangle \\ &= \langle \eta | e^{-i\pi a_2^\dagger a_2} \exp[i(\beta + \alpha)(a_1^\dagger a_2^\dagger + a_1 a_2)] | f \rangle \\ &= \mathfrak{S}_{\alpha+\beta}[f](\eta). \tag{27} \end{aligned}$$

Alternately, we can use the properties of the hyperbolic functions

$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta}, \quad (28)$$

$$\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \sinh \beta \cosh \alpha \quad (29)$$

to directly calculate

$$\begin{aligned} & (\mathfrak{S}_\alpha \circ \mathfrak{S}_\beta)[f](\eta) \\ &= \int \frac{d^2 \eta'}{\pi} \mathfrak{S}_\alpha(\eta, \eta') \int \frac{d^2 \xi}{\pi} \mathfrak{S}_\beta(\eta', \xi) f(\xi) \\ &= \frac{1}{2i \sinh \alpha} \frac{1}{2i \sinh \beta} \\ & \times \int \frac{d^2 \eta'}{\pi} \exp \left[\frac{i(|\eta|^2 + |\eta'|^2)}{2 \tanh \alpha} - i \frac{\eta' \eta^* + \eta'^* \eta}{2 \sinh \alpha} \right] \\ & \times \int \frac{d^2 \xi}{\pi} \exp \left[\frac{i(|\eta'|^2 + |\xi|^2)}{2 \tanh \beta} - i \frac{\xi \eta'^* + \eta' \xi^*}{2 \sinh \beta} \right] f(\xi) \\ &= \frac{1}{2i \sinh(\alpha + \beta)} \int \frac{d^2 \xi}{\pi} \\ & \times \exp \left[\frac{i}{2} \left(\frac{|\eta|^2 + |\xi|^2}{\tanh(\alpha + \beta)} - \frac{\eta^* \xi + \eta \xi^*}{\sinh(\alpha + \beta)} \right) \right] f(\xi) \\ &= \mathfrak{S}_{\alpha + \beta}[f](\eta), \end{aligned} \quad (30)$$

so the two successive integration transformations are equivalent to one EFrST, which exhibits additivity.

5 Application

The above EFrST provides us with a new approach for developing information optics; i.e., we now have a new analyzing method for optical signals, rather than the usual FrFT. For example, when the state vector in $f(\xi) = \langle \xi | f \rangle$ is the two-mode Fock state

$$|f\rangle = \frac{a_1^{\dagger n} a_2^{\dagger m}}{\sqrt{n! m!}} |00\rangle = |n, m\rangle, \quad (31)$$

then

$$f(\xi) = \langle \xi | n, m \rangle = \frac{1}{\sqrt{n! m!}} H_{m, n}(\xi, \xi^*) e^{-|\xi|^2/2}, \quad (32)$$

where $H_{m, n}(\xi, \xi^*)$ is a two-variable Hermite polynomial defined by

$$H_{m, n}(\xi, \xi^*) = \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} e^{t\xi + \tau\xi^* - t\tau} \Big|_{t=0, \tau=0}. \quad (33)$$

The EFrST of $\langle \xi | n, m \rangle$, according to Eq. (10), yields

$$\begin{aligned} \mathfrak{S}_\alpha[f](\eta) &= \langle \eta | \mathfrak{S}_\alpha | n, m \rangle = \int \frac{d^2 \xi}{\pi} \langle \eta | \mathfrak{S}_\alpha | \xi \rangle \langle \xi | n, m \rangle \\ &= \frac{1}{2i \sinh \alpha \sqrt{m! n!}} \exp \left(\frac{i|\eta|^2}{2 \tanh \alpha} \right) \\ & \times \int \frac{d^2 \xi}{\pi} H_{m, n}(\xi, \xi^*) \end{aligned}$$

$$\begin{aligned} & \times \exp \left[\frac{-|\xi|^2}{2} \left(1 - \frac{i}{\tanh \alpha} \right) - i \frac{\xi \eta^* + \xi^* \eta}{2 \sinh \alpha} \right] \\ &= \frac{1}{2i \sinh \alpha \sqrt{m! n!}} \left(\frac{i + \tanh \alpha}{i - \tanh \alpha} \right)^{(m+n)/2} \\ & \times \exp \left[-\frac{i + \tanh \alpha}{i - \tanh \alpha} \frac{|\eta|^2}{2} \right] \\ & \times H_{m, n} \left(\frac{\eta^*}{\sqrt{\cosh 2\alpha}}, \frac{\eta}{\sqrt{\cosh 2\alpha}} \right). \end{aligned} \quad (34)$$

One can see that this involves a squeezing transformation from $H_{m, n}(\xi, \xi^*) \rightarrow H_{m, n} \left(\frac{\eta^*}{\sqrt{\cosh 2\alpha}}, \frac{\eta}{\sqrt{\cosh 2\alpha}} \right)$; for spin squeezing in Bose–Einstein condensates we refer to Ref. [11]. Further, if we continue to perform another EFrST \mathfrak{S}_β , then the final result entails just replacing α on the right of Eq. (34) by $\alpha + \beta$. This property may be employed to perform characteristic recognition for optical images or to design optical correlators. In deriving (34), we have used the integration formula

$$\begin{aligned} & \int \frac{d^2 \xi}{\pi} H_{n, m}(\xi, \xi^*) e^{\lambda|\xi|^2 + \mu\xi + \nu\xi^*} \\ &= \frac{\partial^{n+m}}{\partial t^n \partial \tau^m} \int \frac{d^2 \xi}{\pi} e^{\lambda|\xi|^2 + (\mu+t)\xi + (\nu+\tau)\xi^* - t\tau} \Big|_{t=0, \tau=0} \\ &= \sqrt{\left(\frac{1+\lambda}{\lambda} \right)^{m+n}} e^{-\mu\nu/\lambda} H_{n, m} \\ & \times \left(\frac{-\nu}{\sqrt{\lambda(1+\lambda)}}, \frac{-\mu}{\sqrt{\lambda(1+\lambda)}} \right). \end{aligned} \quad (35)$$

6 Conclusions

In summary, by using two mutually conjugate entangled state representations, we have successfully proposed an EFrST that obeys additivity and is an important generalization of the single-mode Fresnel transformation (a kind of single-mode fractional squeezing transformation) in Ref. [6]. The fractional property of the EFrST has been well described by virtue of the properties of the entangled state representations. We expect that the EFrST may be implemented in combinations of quadratic nonlinear crystals with different phase mismatches.

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