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RESEARCH ARTICLE

Correlated effects of noise on symmetry of an asymmetric bistable system

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The effects of correlation between additive and multiplicative noises on the symmetry of an asymmetric bistable system are investigated. The steady-state probability distribution function of the system was calculated by using analytical and numerical methods. Results indicate that i) for the case of positive correlation between noises, as the correlation strength between additive and multiplicative noises, λ, increases, the symmetry of the system is restored; ii) for the case of negative correlation between noises, as the absolute value of λ increases, the symmetry of the system is destroyed; and iii) the analytic prediction agrees well with the stochastic simulation result.

Keywords asymmetry bistable system, correlated noises, symmetrical characteristic

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1 Introduction

It is well known that noise is ubiquitous in realistic systems. Through studies conducted in the past few decades, the constructive roles of noise have become better understood. Some counterintuitive behaviors induced by noise, e.g., stochastic resonance [1, 2], noise-enhanced stability [3–5], resonant activation [6–9], anomalous diffusion [10–12], have also become the focus of stochastic research. However, in most of the previous works, noises present in the stochastic systems were usually treated as random variables uncorrelated with each other. Indeed, noises in some stochastic processes may have a common origin and thus can be correlated [13, 14]. Also, Madureira *et al.* considered that fluctuations are not independent because the noise contributions originate from both additive and multiplicative characters [15].

The effects of correlations between additive and multiplicative noise have been widely studied in bistable systems [13–20]. As a prototype model of stochastic dynamics, the dynamics properties of a bistable system, including stationary properties [16, 17], transient properties [18, 19], relaxation time and correlation function [20], are changed by the correlation between additive and multiplicative noise. The correlated noises are of great significance in investigating dynamics in bistable systems. However, most of the previous studies focused on symmetric bistable systems. Indeed, real-world manifestations of these systems are often asymmetric [21–24], with the dynamics containing even and odd functions of the state variable. Bulsara *et al.* studied noise-controlled resonance behavior in nonlinear dynamical systems with broken symmetry [25]. Gradually, the breaking and restoration of symmetry has a major impact on the scientific community, even becoming the central concept of modern sciences such as crystallography, molecular science, chemistry, and physics. The symmetry breaking phenomenon induced by time delay in a stochastic bistable system has already been discussed [26], but the symmetry breaking and revival induced by correlation, especially in an asymmetric bistable system, has not been fully explored.

In this paper, we study an asymmetric bistable system driven by cross-correlated noises. Such an investigation is significant because noise correlation is ubiquitous in nature and often changes the dynamics of a system fundamentally. Moreover, we will demonstrate how the noise correlation and a system's asymmetry collaborate in the process of symmetry restoration.

In Section 2, the analytical expressions for the stationary probability distribution function (SPDF) of an asymmetric bistable system with correlated noises are derived, and we make a comparison between analytical and simulation results. In Section 3, the discussion of the results concludes the paper.

2 The SPDF of the asymmetric system

Consider an asymmetric bistable system driven by cross-correlated noises. We break the symmetry of the system by introducing the asymmetric parameter r directly into the bistable model. The dimensionless form of the corresponding Langevin equation reads

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r + x - x^3 + x\xi(t) + \eta(t) \tag{1}$$

where $\xi(t)$ is multiplicative Gaussian white noise and $\eta(t)$ is additive Gaussian white noise with zero mean and the following statistical properties:

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

$$\langle \eta(t)\eta(t')\rangle = 2Q\delta(t-t'), \tag{3}$$

$$\langle \xi(t)\eta(t')\rangle = \langle \eta(t)\xi(t')\rangle = 2\lambda\sqrt{QD}\delta(t-t'), \tag{4}$$

where D and Q are the multiplicative and additive noise intensities respectively. λ denotes the cross-correlated intensity between $\xi(t)$ and $\eta(t)$.

The Fokker–Planck equation corresponding to the Langevin equation can be written as [27]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x)P(x,t) + \frac{\partial^2}{\partial x^2} G(x)P(x,t), \tag{5}$$

where $P(x,t)$ is the probability distribution function and

$$f(x) = r + x - x^3 + Dx + \lambda\sqrt{QD}, \tag{6}$$

$$G(x) = Dx^2 + 2\lambda\sqrt{QD}x + Q. \tag{7}$$

The SPDF of Eq. (5) with (6) and (7) is given by [28]

$$P_{st}(x) = NG(x)^{-1/2} \exp\left[-\frac{U(x)}{Q}\right] \quad \text{for } |\lambda| < 1, \tag{8}$$

where N is the normalization constant. By using the explicit forms of $f(x)$ and $G(x)$, the following generalized potential is obtained:

$$\begin{aligned} U(x) &= -\int^x \frac{r + y - y^3}{Ry^2 + 2\lambda\sqrt{R}y + 1}\mathrm{d}y \\ &= \frac{\beta_1 - r}{\sqrt{(1-\lambda^2)R}} \arctan\frac{\sqrt{R}x + \lambda}{\sqrt{1-\lambda^2}} \\ &\quad -\frac{2\lambda}{R\sqrt{R}}x + \frac{x^2}{2R} + \beta_2 \ln G(x), \end{aligned} \tag{9}$$

where

$$\beta_1 = \frac{\lambda}{\sqrt{R}}\left(1 - \frac{4\lambda^2 - 3}{R}\right), \quad \beta_2 = \frac{1}{2R}\left(\frac{4\lambda^2 - 1}{R} - 1\right). \tag{10}$$

Here $R = D/Q$ is the ratio of noise intensities.

By calculating the derivative $\frac{\mathrm{d}P_{st}(x)}{\mathrm{d}x}$ and letting $\frac{\mathrm{d}P_{st}(x)}{\mathrm{d}x} = 0$, the equation for determining the extreme point of $P_{st}(x)$ can be obtained as

$$\begin{aligned} &\frac{\beta_1 - r}{1 + R + 2\lambda R^{1/2}x} - \frac{2\lambda}{R^{3/2}} + \frac{x}{R} \\ &\quad + \frac{4\beta_2 R^{1/2}(R^{1/2}x + 2\lambda) + 1}{2(1 + 2\lambda R^{1/2}x + Rx^2)}. \end{aligned} \tag{11}$$

It is very difficult to obtain an analytic solution of Eq. (11), but from Eq. (11) we can see that the extreme point of $P_{st}(x)$ is affected by the parameters r, R and λ.

According to the expression for the SPDF (Eq. (8)) of the system, the effects of correlation λ on $P_{st}(x)$ can be studied theoretically.

Figure 1 shows $P_{st}(x)$ as a function of x for different values of λ when $r = 0.05$. There is a double-peaked structure in the SPDF of the system. For the case of weak correlation between noises, i.e., $\lambda = 0.05$, the height of the right peak is markedly higher than that of the left one. As the correlation between noises increases ($\lambda = 0.1$), the height of the right peak decreases while that of the left one increases, and the height difference between two peaks is not so remarkable and the two heights begin to approach one another. This means that the extent of the asymmetry is decreasing; in other words, the system is tending toward restoring the symmetry. This trend is more obvious when $\lambda = 0.15$. For the case of $\lambda = 0.2$, the height of the left peak is almost equal to that of the right one, indicating that the symmetry restoration phenomenon emerges as λ increases.

In Fig. 2, we show the results of negative correlation. For the case of $\lambda = -0.1$, the structure of the SPDF

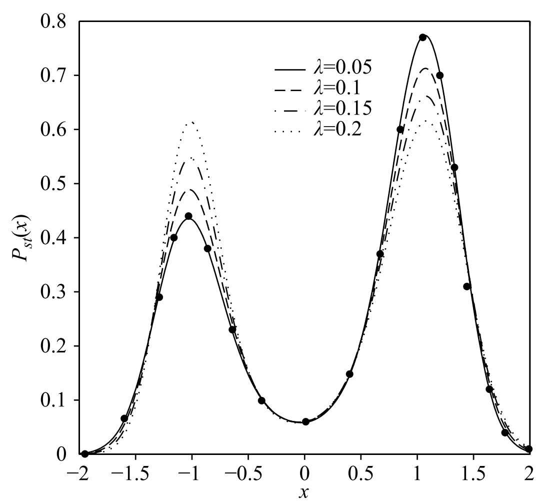

Fig. 1 $P_{st}(x)$ versus x calculated from the analytical expression (8) for $r = 0.05$, $Q = 0.1$, and $D = 0.1$, with λ taking the values 0.05, 0.1, 0.15, and 0.2. The dots indicate the results of $P_{st}(x)$ versus x determined by stochastic simulation directly from Eq. (1) for $r = 0.05$, $Q = 0.1$, $D = 0.1$, and $\lambda = 0.05$.

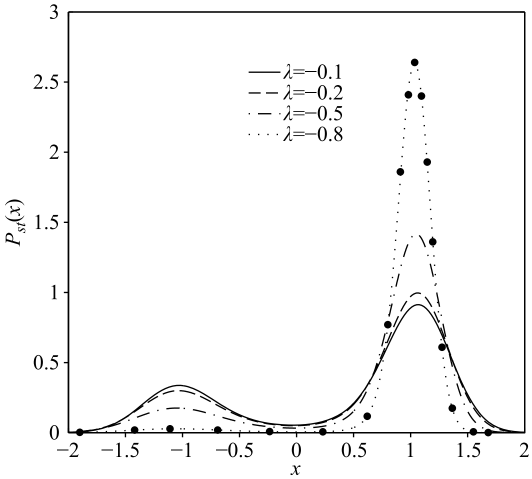

Fig. 2 $P_{st}(x)$ versus x calculated from the analytical expression (8) for $r = 0.05$, $Q = 0.1$, and $D = 0.1$, with λ taking the values -0.1, -0.2, -0.5, and -0.8. The dots indicate the results of $P_{st}(x)$ versus x determined by stochastic simulation directly from Eq. (1) for $r = 0.05$, $Q = 0.1$, $D = 0.1$, and $\lambda = -0.8$.

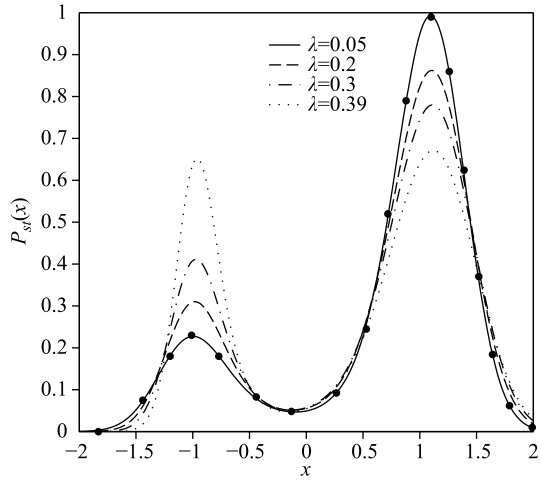

Fig. 3 $P_{st}(x)$ versus x calculated from the analytical expression (8) for $r = 0.1$, $Q = 0.1$, and $D = 0.1$, with λ taking the values 0.05, 0.2, 0.3, and 0.39. The dots indicate the results of $P_{st}(x)$ versus x determined by stochastic simulation directly from Eq. (1) for $r = 0.1$, $Q = 0.1$, $D = 0.1$, and $\lambda = 0.05$.

exhibits a typical bimodal peak. However, as one increases the extent of the negative correlation, the left peak gradually decreases and the right peak increases when $\lambda = -0.2$ and $\lambda = -0.5$. When $\lambda = -0.8$, the structure of the SPDF becomes unimodal, indicating that the symmetry of the system is destroyed as the absolute value of λ increases.

To better understand how the asymmetry of the system plays a role with the correlation between noises in the process of symmetry restoration, we should consider the situation of different values of r. Figure 3 shows $P_{st}(x)$ as a function of x for different values of λ when $r = 0.1$ for the case of positive correlation. The restoration phenomenon emerges when $\lambda = 0.39$, whereas in Fig. 1 this phenomenon occurs when $\lambda = 0.2$. This difference reveals that λ and r play opposite roles in the process. The more asymmetric the system is, the stronger the correlation is needed in this process. In the negative-correlation case, as shown in Fig. 4, the left peak disappears approximately when $\lambda = -0.6$, which is different from the case of $r = 0.05$.

We show the results of numerical simulations [11] of the Langevin equation (1) for the SPDF. As shown in Figs. 1–4, for both positive correlation (Figs. 1 and 3) and negative correlation (Figs. 2 and 4) between noises, the analytic prediction agrees well with the numerical simulation results.

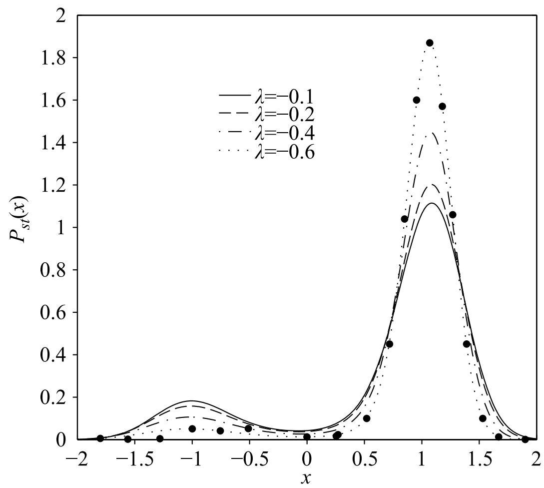

Fig. 4 $P_{st}(x)$ versus x calculated from the analytical expression (8) for $r = 0.1$, $Q = 0.1$, and $D = 0.1$, with λ taking the values -0.1, -0.2, -0.4, and -0.6. The dots indicate the results of $P_{st}(x)$ versus x determined by stochastic simulation directly from Eq. (1) for $r = 0.1$, $Q = 0.1$, $D = 0.1$, and $\lambda = -0.6$.

3 Conclusion

In this study, the effects of correlation between additive and multiplicative noises on the symmetry of an asymmetric bistable system have been investigated. The steady-state probability distribution function of the system was calculated by using analytical and numerical methods. We have presented two significant phenomena induced by noise correlation. For the case of positive correlation, the symmetry restoration behavior occurs when the correlation strength between additive and multiplicative noises, λ, increases. For the case of negative correlation, as the absolute value of λ increases, the structure of the SPDF changes from bimodal to unimodal; i.e., the symmetry of the system is destroyed. The analytic prediction agrees well with the stochastic simulation results.

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