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RESEARCH ARTICLE

On the core of the fractional Fourier transform and its role in composing complex fractional Fourier transformations and Fresnel transformations

Hong-Yi Fan[1,2,*], **Jun-Hua Chen**[2,†]

[1] *Department of Physics, Ningbo University, Ningbo 315211, China*
[2] *Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China*
Corresponding authors. E-mail: *fhym@ustc.edu.cn, †cjh@ustc.edu.cn

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By a quantum mechanical analysis of the additive rule $F_\alpha[F_\beta[f]] = F_{\alpha+\beta}[f]$, which the fractional Fourier transformation (FrFT) $F_\alpha[f]$ should satisfy, we reveal that the position–momentum mutual-transformation operator is the core element for constructing the integration kernel of FrFT. Based on this observation and the two mutually conjugate entangled-state representations, we then derive a core operator for enabling a complex fractional Fourier transformation (CFrFT), which also obeys the additive rule. In a similar manner, we also reveal the fractional transformation property for a type of Fresnel operator.

Keywords fractional Fourier transform, core operator, IWOP technique, entangled state of continuum variables, Fresnel operator

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1 Introduction

In a recent review paper [1], by performing integration within an ordered product (IWOP) of operators, we explored a quantum optical version of classical optical transformations. In this way, we developed some topics in classical optics theory using research results in quantum optics and vice-versa, i.e., not only have we found new quantum mechanical unitary operators that correspond to known optical transformations, but we also proposed some new classical optical transformations. For example, we found a Fresnel operator (GFO) corresponding to the famous Fresnel transform (GFT) in classical optics. In this paper, using the same technique, we shall reveal that a core operator is involved in the quantum-mechanical treatment of fractional Fourier transforms (FrFT). Then, we shall demonstrate the role of this core operator in composing complex fractional Fourier transformations (CFrFTs). In this manner, we shall also reveal the fractional property of Fresnel transformation caused by a type of Fresnel operators.

The FrFT is a very useful tool in Fourier optics and information optics, especially in optical communication, image manipulations, and signal analysis [1–10]. The concept of the FrFT was originally used for signal processing in 1980 by Namias [2] as a Fourier transform of fractional order. However, FrFT did not have a significant impact on optics until the FrFT was physically defined based on propagation in quadratic graded-index media (GRIN media) [6] by Mendlovic, Ozaktas, and Lohmann.

Other than its Fourier-transform nature, the essence of FrFT, denoted as $F_\alpha[f]$, is its additive rule (or the compositional rule), i.e.,

$$F_\alpha\left[F_\beta\left[f\right]\right] = F_{\alpha+\beta}[f], \tag{1}$$

which gives rise to the name “fractional” transformation. In this work by virtue of quantum mechanical representation theory and the IWOP technique we shall reveal the essence of the “fractional” transformation by finding its core operator, which is shown to be the position–momentum mutual-transformation operator. Based on this observation, and using the two mutually conjugate entangled-state representations, we shall derive a core operator that allows for complex, fractional Fourier transformations (CFrFT) [11], and which also obeys the additive rule. Then, we shall also reveal the fractional nature of the Fresnel operator [12].

The work is arranged as follows: In Section 2, using the known integration kernel of FrFT, we use Dirac's representation to derive the FrFT operator. Then, in Section 3 we emphasize that the position-momentum mutual-transformation operator is the core of FrFT, and then derive its concrete form using the IWOP technique [13–15]. In Sections 4 and 5, based on the two mutually conjugate entangled-state representations, we also derive the core operator for constructing CFrFT, which manifestly exhibits the additive rule. In Section 6, we demonstrate the fractional nature of the Fresnel operator.

2 Obtaining the FrFT operator by virtue of the IWOP method

The one-dimensional FrFT of the α-th (the real α-angle) is defined in Refs. [5, 6]

$$F_\alpha[f](p) = \int_{-\infty}^{\infty} K_\alpha(p,x) f(x)\mathrm{d}x \tag{2}$$

where

$$K_\alpha(p,x) = \frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{p^2+x^2}{\tan\alpha} - \frac{2px}{\sin\alpha}\right)\right], \tag{3}$$

noting that $\frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{\sin\alpha}} = \sqrt{1-\mathrm{i}\cot\alpha}$. The conventional Fourier transform is simply $F_{\pi/2}$.

We hope to derive the FrFT operator K_α such that

$$\langle p|K_\alpha|x\rangle = K_\alpha(p,x). \tag{4}$$

In the context of quantum mechanics, a function f becomes the quantum state $|f\rangle$, and the value $f(x)$ becomes the matrix element $\langle x|f\rangle$. The typical Fourier transform is simply the change of basis between the coordinate representation $|x\rangle$ and the momentum representation $|p\rangle$

$$\langle p|f\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|f\rangle \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} f(x)\mathrm{e}^{-\mathrm{i}px}. \tag{5}$$

In Fock space, $|x\rangle$ and $|p\rangle$ are respectively expressed as

$$|x\rangle = \pi^{-1/4}\exp\left(-\frac{x^2}{2} + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{6}$$

$$|p\rangle = \pi^{-1/4}\exp\left(-\frac{p^2}{2} + \mathrm{i}\sqrt{2}pa^\dagger + \frac{a^{\dagger 2}}{2}\right)|0\rangle. \tag{7}$$

where $\left[a, a^\dagger\right] = 1$, and $|0\rangle$ is the vacuum state annihilated by a, $a|0\rangle = 0$. Multiplying $K_\alpha(p,x)$ by $|p\rangle$ from the left and by $\langle x|$ from the right, and then integrating over $\mathrm{d}p$ and $\mathrm{d}x$, using the IWOP method and $|0\rangle\langle 0| =: \exp(-a^\dagger a)$:, while also using the identity $\exp\left\{fa^\dagger a\right\} =: \exp\left[\left(\mathrm{e}^f - 1\right)a^\dagger a\right]:$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle\int_{-\infty}^{\infty}\mathrm{d}x\langle x|K_\alpha(p,x)\\
&= \frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}}\iint_{-\infty}^{\infty}\frac{\mathrm{d}p\mathrm{d}x}{\sqrt{\pi}}\exp\left[\frac{\mathrm{i}}{2}\left(\frac{p^2+x^2}{\tan\alpha}-\frac{2px}{\sin\alpha}\right)\right.\\
&\quad\left.-\frac{p^2}{2}+\mathrm{i}\sqrt{2}pa^\dagger+\frac{a^{\dagger 2}}{2}\right]\\
&\quad\times|0\rangle\langle 0|\exp\left(-\frac{x^2}{2}+\sqrt{2}xa-\frac{a^2}{2}\right)\\
&=: \exp\left[\left(\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}-1\right)a^\dagger a\right] := \exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a\right].
\end{aligned} \tag{8}$$

On the other hand, from the completeness relation

$$\int_{-\infty}^{\infty}\mathrm{d}p\,|p\rangle\langle p| = 1, \qquad \int_{-\infty}^{\infty}\mathrm{d}x|x\rangle\langle x| = 1, \tag{9}$$

we know

$$\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle\int_{-\infty}^{\infty}\mathrm{d}x\langle x|K_\alpha(p,x) = K_\alpha, \tag{10}$$

so

$$K_\alpha = \exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a\right], \tag{11}$$

which is the FrFT operator. It then follows that

$$\begin{aligned}
&\frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}}\exp\left[\frac{\mathrm{i}}{2}\left(\frac{p^2+x^2}{\tan\alpha}-\frac{2px}{\sin\alpha}\right)\right]\\
&= \langle p|\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a}|x\rangle.
\end{aligned} \tag{12}$$

Now, we analyze the appearance of $\mathrm{e}^{\mathrm{i}\frac{\pi}{2}a^\dagger a}$ in more detail. As will be soon demonstrated, it is the core for FrFT.

3 The position–momentum mutual-transformation operator as the core of FrFT

The key feature of FrFT is composite ability and additivity. Let $K_\beta(p,x)$ be another β-th FrFT

$$K_\beta(p,x) = \langle p|\,K_\beta|x\rangle = \langle p|\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^\dagger a}\,|x\rangle. \tag{13}$$

When K_α and K_β are to be combined as a single FrFT $K_{\alpha+\beta}$, i.e., to perform a fractional Fourier transform repeatedly on $f(x)$, they must be connected smoothly as the following two-fold integral

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x' K_\alpha\left(p,x'\right)\left[K_\beta\left(p',x\right)|_{p'=x'}\right]f(x). \tag{14}$$

In terms of the Dirac symbolism, this is expressed as

$$\begin{aligned}
&\int\mathrm{d}x\int\mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a}\,|x'\rangle_{p'=x'}\\
&\times\langle p'|\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^\dagger a}|x\rangle\langle x|f\rangle,
\end{aligned} \tag{15}$$

in which we examine

$$\int \mathrm{d}x' \langle p|\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^{\dagger}a}\,|x'\rangle_{p'=x'}\,\langle p'|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^{\dagger}a}\,|x\rangle \equiv I_{\alpha,\beta}. \tag{16}$$

Using $\langle x'\,|x\rangle = \delta(x'-x)$, we have

$$\begin{aligned} I_{\alpha,\beta} &= \int \mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^{\dagger}a}\,|x'\rangle \\ &\quad\times \int \mathrm{d}p'\delta\left(x'-p'\right)\langle p'|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^{\dagger}a}|x\rangle \\ &= \int \mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^{\dagger}a}\,|x'\rangle\,\langle x'| \\ &\quad\times \int \mathrm{d}p'\,|x\rangle_{x=p'}\,\langle p'|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^{\dagger}a}\,|x\rangle. \end{aligned} \tag{17}$$

Then, using $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}a^{\dagger}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}a^{\dagger}a} = a^{\dagger}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}a^{\dagger}$ and Eqs. (6) and (7), we see that

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}\,|p'\rangle &= \pi^{-1/4}\exp\left(-\frac{p'^{2}}{2}+\sqrt{2}p'a^{\dagger}-\frac{a^{\dagger 2}}{2}\right) \\ &= |x\rangle_{x=p'}. \end{aligned} \tag{18}$$

Thus, $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}$ is the position–momentum mutual-transformation operator. Using Eq. (18), we can express Eq. (17) as

$$\begin{aligned} I_{\alpha,\beta} &= \int \mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^{\dagger}a}\,|x'\rangle\,\langle x'|\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}a^{\dagger}a} \\ &\quad\times \int \mathrm{d}p'\,|p'\rangle\,\langle p'|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\beta\right)a^{\dagger}a}|x\rangle \\ &= \langle p|K_{\alpha}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}K_{\beta}\,|x\rangle. \end{aligned} \tag{19}$$

Then, according to the definition of K_{α} in Eq. (11), we see that

$$K_{\alpha}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}K_{\beta} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha-\beta\right)a^{\dagger}a} = K_{\alpha+\beta}, \tag{20}$$

noting that

$$K_{\alpha}K_{\beta} = \mathrm{e}^{\mathrm{i}(\pi-\alpha-\beta)a^{\dagger}a} \neq K_{\alpha+\beta}. \tag{21}$$

Therefore, the appearance of $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}$ in the center of $K_{\alpha}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^{\dagger}a}K_{\beta}$ is inevitable for the additivity of FrFT. We now have a better understanding of why the kernel $K_{\alpha}(p,x)$ should be defined in the seemingly unnatural terms $\langle p|\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^{\dagger}a}|x\rangle$ instead of naively using $\langle p|\,\mathrm{e}^{-\mathrm{i}\alpha a^{\dagger}a}\,|x\rangle$.

The position–momentum mutual-transformation operator can be expressed as $\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\,\langle p|_{p=x}$. In fact, using IWOP, we perform the integration

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\,\langle p|_{p=x} &= \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}} \\ &\quad\times :\exp\left(-x^{2}+\sqrt{2}xa^{\dagger}-\frac{a^{\dagger 2}}{2}-a^{\dagger}a-\mathrm{i}\sqrt{2}xa+\frac{a^{2}}{2}\right): \\ &=:\exp\left[-\left(1+\mathrm{i}\right)a^{\dagger}a\right]:=\exp\left(-\frac{\mathrm{i}\pi}{2}a^{\dagger}a\right), \end{aligned} \tag{22}$$

which confirms that

$$\begin{aligned} \exp\left(-\frac{\mathrm{i}\pi}{2}a^{\dagger}a\right)|p'\rangle &= \int_{-\infty}^{\infty}\mathrm{d}x|x\rangle\langle p|_{p=x}\,|p'\rangle \\ &= \int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\,\delta\left(x-p'\right) = |x\rangle_{x=p'}. \end{aligned} \tag{23}$$

4 Two mutually conjugate entangled-state representations for finding the core of a complex fractional Fourier transformation

In Refs. [15–18] we introduced two mutually conjugate entangled-state representations

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^{2}+\eta a_{1}^{\dagger}-\eta^{*}a_{2}^{\dagger}+a_{1}^{\dagger}a_{2}^{\dagger}\right)|00\rangle \tag{24}$$

and

$$|\xi\rangle = \exp\left(-\frac{1}{2}|\xi|^{2}+\xi a_{1}^{\dagger}+\xi^{*}a_{2}^{\dagger}-a_{1}^{\dagger}a_{2}^{\dagger}\right)|00\rangle\,. \tag{25}$$

Here, $|\eta=\eta_{1}+\mathrm{i}\eta_{2}\rangle$ is the common eigenvector of two-particles' relative coordinates $X_{1}-X_{2}$ and total momentum $P_{1}+P_{2}$,

$$\begin{aligned} \left(X_{1}-X_{2}\right)|\eta\rangle &= \sqrt{2}\eta_{1}\,|\eta\rangle\,, \\ \left(P_{1}+P_{2}\right)|\eta\rangle &= \sqrt{2}\eta_{2}\,|\eta\rangle\,. \end{aligned} \tag{26}$$

It was Einstein *et al.* [19] who first noted that the commutativity of $\left[\left(X_{1}-X_{2}\right),\left(P_{1}+P_{2}\right)\right]=0$ would lead to the concept of quantum entanglement. Here, $|\xi=\xi_{1}+\mathrm{i}\xi_{2}\rangle$ is the common eigenvector of $\left(X_{1}+X_{2}\right)$ and $\left(P_{1}+P_{2}\right)$,

$$\begin{aligned} \left(X_{1}+X_{2}\right)|\xi\rangle &= \sqrt{2}\xi_{1}\,|\xi\rangle\,, \\ \left(P_{1}-P_{2}\right)|\xi\rangle &= \sqrt{2}\xi_{2}\,|\xi\rangle\,, \end{aligned} \tag{27}$$

where $|\eta\rangle$ and $|\xi\rangle$ both possess completeness and the orthogonality:

$$\begin{aligned} &\int\frac{\mathrm{d}^{2}\eta}{\pi}\,|\eta\rangle\,\langle\eta| = 1, \\ &\langle\eta|\,\eta'\rangle = \pi\delta\left(\eta-\eta'\right)\delta\left(\eta^{*}-\eta'^{*}\right) \equiv \pi\delta^{(2)}\left(\eta-\eta'\right), \\ &\int\frac{\mathrm{d}^{2}\xi}{\pi}\,|\xi\rangle\,\langle\xi| = 1, \\ &\langle\xi|\,\xi'\rangle = \pi\delta\left(\xi-\xi'\right)\delta\left(\xi^{*}-\xi'^{*}\right) \equiv \pi\delta^{(2)}\left(\xi-\xi'\right). \end{aligned} \tag{28}$$

The overlap

$$\langle\eta\ \ |\xi\rangle = \frac{1}{2}\exp[\left(\eta^{*}\xi-\eta\xi^{*}\right)/2] \tag{29}$$

is a Fourier transformation kernel in complex form. The core for constructing the complex fractional Fourier transformation (CFrFT) transiting from $|\xi\rangle$ to $\langle\eta|$ can be found by examining the additivity, i.e., when performing complex fractional Fourier transforms K_α and K_β repeatedly on $f(\xi)$, the following two-fold integral

$$\int \frac{\mathrm{d}^2\xi}{\pi} \int \frac{\mathrm{d}^2\xi'}{\pi} K_\alpha(\eta,\xi') \left[K_\beta(\eta',\xi)|_{\eta'=\xi'}\right] f(\xi) \tag{30}$$

must be smoothly connected as a single CFrFT $K_{\alpha+\beta}$ with $(\alpha+\beta)$-order. Using Dirac notation, Eq. (30) is expressed as

$$\int \frac{\mathrm{d}^2\xi}{\pi} \int \frac{\mathrm{d}^2\xi'}{\pi} \langle\eta| K_\alpha |\xi'\rangle_{\eta'=\xi'} \langle\eta'| K_\beta |\xi\rangle \langle\xi| f\rangle, \tag{31}$$

in which we examine

$$\int \frac{\mathrm{d}^2\xi'}{\pi} \langle\eta| K_\alpha |\xi'\rangle_{\eta'=\xi'} \langle\eta'| K_\beta |\xi\rangle \equiv J_{\alpha,\beta}. \tag{32}$$

Using Eq. (28), we have

$$\begin{aligned} J_{\alpha,\beta} &= \int\frac{\mathrm{d}^2\xi'}{\pi} \langle\eta| K_\alpha |\xi'\rangle \int \mathrm{d}^2\eta' \delta^{(2)}(\xi'-\eta') \langle\eta'| K_\beta |\xi\rangle \\ &= \int \frac{\mathrm{d}^2\xi'}{\pi} \langle\eta| K_\alpha |\xi'\rangle \langle\xi'| \int \frac{\mathrm{d}^2\eta'}{\pi} |\xi\rangle_{\xi=\eta'} \langle\eta'| K_\beta |\xi\rangle. \end{aligned} \tag{33}$$

If we can find an operator W that can play the role of transforming

$$W|\eta\rangle = |\xi\rangle_{\xi=\eta} \tag{34}$$

then

$$\begin{aligned} J_{\alpha,\beta} &= \int \frac{\mathrm{d}^2\xi'}{\pi} \langle\eta| K_\alpha |\xi'\rangle \langle\xi'| W \int \frac{\mathrm{d}^2\eta'}{\pi} |\eta'\rangle \langle\eta'| K_\beta |\xi\rangle \\ &= \langle\eta| K_\alpha W K_\beta |\xi\rangle = \langle\eta| K_{\alpha+\beta} |\xi\rangle. \end{aligned} \tag{35}$$

Comparing (24) with (25), we see that

$$W = \exp\left(-\mathrm{i}\pi a_2^\dagger a_2\right). \tag{36}$$

Thus, W is the core operator for constructing the CFrFT:

$$\begin{aligned} W|\eta\rangle &= \exp\left(-\frac{|\eta|^2}{2} + \eta a_1^\dagger - \eta^* a_2^\dagger \mathrm{e}^{-\mathrm{i}\pi} + \mathrm{e}^{-\mathrm{i}\pi} a_1^\dagger a_2^\dagger\right)|00\rangle \\ &= |\xi\rangle_{\xi=\eta}, \end{aligned}$$

so that W is the $|\eta\rangle - |\xi\rangle$ mutual-transformation operator.

5 The kernel of CFrFT (improved)

We now know that the operator for enabling the use of a CFrFT can be chosen as

$$K_\alpha = \exp\left[-\mathrm{i}\alpha\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] \exp\left(\mathrm{i}\pi a_2^\dagger a_2\right), \tag{37}$$

since

$$K_\alpha W K_\beta = K_{\alpha+\beta}. \tag{38}$$

We then calculate the kernel $K_\alpha(\eta,\xi)$ of the CFrFT, using the two-mode coherent-state representation $|z_1 z_2\rangle$ with the over-completeness

$$\int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} |z_1 z_2\rangle \langle z_1 z_2| = 1 \tag{39}$$

and the overlap

$$\begin{aligned} \langle\eta| z_1 z_2\rangle &= \exp\Big[-\frac{1}{2}\left(|\eta|^2 + |z_1|^2 + |z_2|^2\right) + \eta^* z_1 \\ &\qquad -\eta z_2 + z_1 z_2\Big], \\ \langle z_1' z_2' |\xi\rangle &= \exp\Big[-\frac{1}{2}\left(|\xi|^2 + |z_1'|^2 + |z_2'|\right)^2 + \xi z_1'^* \\ &\qquad +\xi^* z_2'^* - z_1'^* z_2'^*\Big]. \end{aligned} \tag{40}$$

Thus, we have

$$\begin{aligned} K_\alpha(\eta,\xi) &= \int \frac{\mathrm{d}^2 z_1}{\pi} \frac{\mathrm{d}^2 z_2}{\pi} \frac{\mathrm{d}^2 z_1'}{\pi} \frac{\mathrm{d}^2 z_2'}{\pi} \langle\eta| z_1 z_2\rangle \langle z_1 z_2| \\ &\quad \times :\exp\left[\left(\mathrm{e}^{-\mathrm{i}\alpha} - 1\right) a_1^\dagger a_1 - \left(\mathrm{e}^{-\mathrm{i}\alpha} + 1\right) a_2^\dagger a_2\right]: \\ &\quad \times |z_1' z_2'\rangle \langle z_1' z_2' \ |\xi\rangle \\ &= \int \frac{\mathrm{d}^2 z_1}{\pi} \frac{\mathrm{d}^2 z_2}{\pi} \frac{\mathrm{d}^2 z_1'}{\pi} \frac{\mathrm{d}^2 z_2'}{\pi} \exp[-\frac{1}{2}|\eta|^2 - |z_2|^2 \\ &\quad -\eta z_2 - |z_1'|^2 - \frac{1}{2}|\xi|^2 + \xi z_1'^* - |z_1|^2 + \eta^* z_1 + z_1 z_2 \\ &\quad +\mathrm{e}^{-\mathrm{i}\alpha} z_1^* z_1' - |z_2'|^2 + \xi^* z_2'^* - z_1'^* z_2'^* - \mathrm{e}^{-\mathrm{i}\alpha} z_2^* z_2'] \\ &= \frac{\mathrm{e}^{\mathrm{i}\left(\alpha - \frac{\pi}{2}\right)}}{2\sin\alpha} \exp\left[\frac{\mathrm{i}\left(|\eta|^2 + |\xi|^2\right)}{2\tan\alpha} - \mathrm{i}\frac{\xi\eta^* + \xi^*\eta}{2\sin\alpha}\right]. \end{aligned} \tag{41}$$

At this point, we should mention that in Ref. [11], Fan *et al.* calculated $\langle\eta| \exp\left[\mathrm{i}\left(\frac{\pi}{2} + \alpha\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] |\xi\rangle$, but the additive rule of CFrFT was not examined. The present paper offers a supplement.

In particular, when $\alpha = \pi/2$, using

$$\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a^\dagger a} a^\dagger \mathrm{e}^{\mathrm{i}\frac{\pi}{2}a^\dagger a} = a^\dagger \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}a^\dagger \tag{42}$$

and Eq. (29), we can verify that

$$\begin{aligned} K_{\alpha=\pi/2}(\eta,\xi) &= \langle\eta| \exp\left(-\mathrm{i}\frac{\pi}{2}a_1^\dagger a_1 + \mathrm{i}\frac{\pi}{2}a_2^\dagger a_2\right) |\xi\rangle \\ &= \langle\eta| - \mathrm{i}\xi\rangle = \frac{1}{2}\exp\left(\frac{-\mathrm{i}\xi\eta^* - \mathrm{i}\xi^*\eta}{2}\right) \\ &= \frac{1}{2}\exp\left(-\mathrm{i}\frac{\xi\eta^* + \xi^*\eta}{2}\right), \end{aligned} \tag{43}$$

which is also a Fourier transformation.

6 The fractional transformation property for a type of Fresnel operator

In this section, drawing from the core-operator concept, we demonstrate that the Fresnel operator also involves the fractional transformation property (but not the fractional Fourier transformation).

As we demonstrated in Ref. [1], the Fresnel operator in the coherent state representation is

$$F\left(s,r\right)=\sqrt{s}\int\frac{\mathrm{d}^2z}{\pi}\left|sz-rz^*\right\rangle\left\langle z\right|, \tag{44}$$

where s and r are complex numbers, obeying $|s|^2-|r|^2=1$, and where $|z\rangle$ is the coherent state

$$|z\rangle=\exp\left(za^\dagger-z^*a\right)|0\rangle. \tag{45}$$

After performing the integration in (46), using the IWOP technique, we see that

$$\begin{aligned}F\left(s,r\right)=&\exp\left(-\frac{r}{2s^*}a^{\dagger 2}\right)\exp\left[\left(a^\dagger a+\frac{1}{2}\right)\ln\frac{1}{s^*}\right]\\&\times\exp\left(\frac{r^*}{2s^*}a^2\right).\end{aligned} \tag{46}$$

By introducing a real A,B,C and D obeying $AD-BC=1$, through the following relation:

$$\begin{aligned}s&=\frac{1}{2}\left[A+D-\mathrm{i}\left(B-C\right)\right],\\r&=-\frac{1}{2}\left[A-D+\mathrm{i}\left(B+C\right)\right],\end{aligned} \tag{47}$$

or conversely

$$\begin{aligned}A&=\frac{1}{2}\left(s-r+s^*-r^*\right),\quad B=\frac{1}{2}\left(s+r-s^*-r^*\right),\\C&=-\frac{\mathrm{i}}{2}\left(s-r-s^*+r^*\right),\quad D=\frac{1}{2}\left(s+r+s^*+r^*\right),\end{aligned} \tag{48}$$

we obtain the matrix element of $F\left(s,r\right)$ in coordinate representation $\langle x|$

$$\begin{aligned}&\left\langle x'\right|F\left(s,r\right)\left|x\right\rangle\\&=\frac{1}{\sqrt{2\pi\mathrm{i}B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2-2x'x+Dx'^2\right)\right]\end{aligned} \tag{49}$$

which is simply the kernel of the optical Fresnel transform in classical optics. This is the reason we call $F\left(s,r\right)$ the Fresnel operator. Now we let $s=s^*=\cosh\alpha$, and $r=-\mathrm{i}\sinh\alpha$, so that $F\left(s,r\right)\rightarrow F(\cosh\alpha,-\mathrm{i}\sinh\alpha)\equiv F(\alpha)$, that is

$$\begin{aligned}F\left(s,r\right)\rightarrow F(\alpha)=&\exp\left(\frac{\mathrm{i}}{2}a^{\dagger 2}\tanh\alpha\right)\\&\times\exp\left\{\left(a^\dagger a+\frac{1}{2}\right)\ln\operatorname{sech}\alpha\right\}\exp\left(\frac{\mathrm{i}}{2}a^2\tanh\alpha\right)\\=&\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right],\end{aligned} \tag{50}$$

which is a squeezing operator [20], since

$$F(\alpha)aF^{-1}(\alpha)=a\cosh\alpha-\mathrm{i}a^\dagger\sinh\alpha. \tag{51}$$

In this case,

$$A=D=\cosh\alpha,\quad B=\sinh\alpha=C. \tag{52}$$

Since

$$|x'|_{x'=p}\rangle=\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}|p\rangle,\quad\langle p|\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}=\langle x'|_{x'=p}|, \tag{53}$$

for $F(\alpha)$, using Eqs. (50) and (52), we can re-express Eq. (49) as

$$\begin{aligned}\left\langle x'\right|F(\alpha)\left|x\right\rangle&=\left\langle p\right|_{p=x'}\left|\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\right|x\rangle\\&=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\exp\{\frac{\mathrm{i}}{2\sinh\alpha}\left[\cosh\alpha\left(x^2+p^2\right)-2px\right]\}\\&=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\mathrm{e}^{\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha}-\frac{\mathrm{i}xp}{\sinh\alpha}}.\end{aligned} \tag{54}$$

Comparing Eq. (12) with Eq. (54) we see that they have a similar form, with the exception that $\tan\alpha\rightarrow\tanh\alpha$, $\sin\alpha\rightarrow\sinh\alpha$. Thus, we see that the core operator $\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$ also plays a role in the Fresnel transformation. It remains to be determined if $\langle x'|F(\alpha)|x\rangle$ obeys the additive rule (fractional transformation property).

7 The additivity of fractional squeezing transformations

By recalling that the Fresnel operator defined in Eq. (46) possesses the group multiplication property [1],

$$\begin{aligned}F\left(s,r\right)F\left(s',r'\right)=&\sqrt{ss'}\int\frac{\mathrm{d}^2z\mathrm{d}^2z'}{\pi^2}\\&\times\ \left|sz-rz^*\right\rangle\left\langle z\left|s'z'-r'z'^*\right\rangle\right\langle z'\right|\\=&\frac{1}{\sqrt{s''^*}}\exp\left(-\frac{r''}{2s''^*}a^{\dagger 2}\right):\exp\left[\left(\frac{1}{s''^*}-1\right)a^\dagger a\right]:\\&\times\exp\left(\frac{r''^*}{2s''^*}a^2\right)\\=&\sqrt{s''}\int\frac{\mathrm{d}^2z}{\pi}\left|s''^*z-r''z^*\right\rangle\left\langle z\right|=F\left(s'',r''\right),\end{aligned} \tag{55}$$

where

$$s''=ss'+rr'^*,\ r''=r's+rs'^* \tag{56}$$

or

$$\begin{aligned}&\begin{pmatrix}s''&-r''\\-r^{*\prime\prime}&s^{*\prime\prime}\end{pmatrix}=\begin{pmatrix}s&-r\\-r^*&s^*\end{pmatrix}\begin{pmatrix}s'&-r'\\-r'^*&s'^*\end{pmatrix}\\&|s''|^2-|r''|^2=1,\end{aligned} \tag{57}$$

We let $s = s^* = \cosh\alpha$, $r = -\mathrm{i}\sinh\alpha$, $s' = s'^* = \cosh\beta$, and $r' = -\mathrm{i}\sinh\beta$. Then, from Eq. (56) we see that

$$\begin{aligned} s'' &= \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta = \cosh(\alpha+\beta), \\ r'' &= -\mathrm{i}\sinh\beta\cosh\alpha - \mathrm{i}\sinh\alpha\cosh\beta \\ &= -\mathrm{i}\sinh(\alpha+\beta). \end{aligned} \tag{58}$$

Correspondingly, using Eq. (48) we have

$$A'' = \cosh(\alpha+\beta), \quad B'' = \sinh(\alpha+\beta). \tag{59}$$

It then follows from Eq. (54) and Eq. (59) that the fractional transformation property of the Fresnel operator can be demonstrated by

$$\begin{aligned} &\langle x'|\, F(\alpha)F(\beta)|x\rangle = \langle x'|\, F(\alpha+\beta)|x\rangle \\ &\quad = \frac{1}{\sqrt{2\pi\mathrm{i}\sinh(\alpha+\beta)}} \\ &\quad\times\ \exp\left[\frac{\mathrm{i}}{2}\left(\frac{x^2+p^2}{\tanh(\alpha+\beta)} - \frac{2xp}{\sinh(\alpha+\beta)}\right)\right]. \end{aligned} \tag{60}$$

By combining Eq. (54) and Eq. (60), we have exposed the additive property of the Fresnel transformations caused by a type of Fresnel operator $F(\alpha)$.

In summary, by quantum–mechanical analysis of the additive rule which the fractional Fourier transformation should satisfy, we have revealed that the position–momentum mutual-transformation operator is the core for constructing the integration kernel of a FrFT. Based on this observation, we have found the core operator for CFrFT, which is the mutual transform operator between the two conjugate entangled states $|\xi\rangle$ and $|\eta\rangle$. We also identified $K_\alpha = \exp[-\mathrm{i}\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)]\exp(\mathrm{i}\pi a_2^\dagger a_2)$ as the operator for enabling a CFrFT. This observation also helps to reveal the fractional-transformation property for a type of Fresnel operator.

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