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RESEARCH ARTICLE

States and transitions in mixed networks

Ying Zhang[1], **Wen-Hui Wan**[2,†]

[1] *Department of Physics, Beijing Normal University, Beijing 100875, China*
[2] *Department of Physics, Beijing Institute of Technology, Beijing 100081, China*
Corresponding author. E-mail: [†] *7520120010@bit.edu.cn*
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A network is named as mixed network if it is composed of N nodes, the dynamics of some nodes are periodic, while the others are chaotic. The mixed network with all-to-all coupling and its corresponding networks after the nonlinearity gap-condition pruning are investigated. Several synchronization states are demonstrated in both systems, and a first-order phase transition is proposed. The mixture of dynamics implies any kind of synchronous dynamics for the whole network, and the mixed networks may be controlled by the nonlinearity gap-condition pruning.

Keywords mixed network, phase transition, synchronization state

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1 Introduction

Networks may undergo transitions between different regimes or phases [1]. Synchronization processes describe the coherent dynamics in large populations of coupled dynamical units, either physical systems or biological systems [2, 3]. Strogatz pointed out the importance of the structure of interactions between units on the emergence of synchronization [4] in 2001. Since then, transition toward synchronization has widely studied in networks with nontrivial topological features. It has been found that both the critical point and the stability of the fully synchronized state were strongly influenced by the networked interaction patterns [5–9].

The order of the synchronization transitions can be first or second according to whether the order parameter varies continuously or discontinuously at a critical value of the control parameter. In the Kuramoto model with an all-to-all regular network topology [10], a first-order phase transition has initially been found when the natural frequencies are uniformly distributed (evenly spaced frequencies) [11]. Like all symmetric frequency distributions, the natural frequencies are centered at zero, the effective frequency of the synchronization state is zero, and that means all the populations become static states in this synchronization state.

Since lots of properties about the structure and dynamics of a system have been understood by the simple graph representation [12,13], the phase transition toward synchronization has been widely investigated by considering the nontrivial network topology. Recently, the abrupt percolation transition was discovered in random [14] and scaled-free networks [15, 16]. When the Kuramoto model is modified into a scale-free network topology, the explosive synchronization will be found if the natural frequencies of the oscillators are positively correlated with their degrees [17]. The effective frequency of this synchronization state is not centered at zero, which means the synchronization state is periodic. It has been reported that a positive correlation between the node degrees and corresponding oscillator's natural frequencies leads to an explosive transition even in networked chaotic oscillators, which has been proved in an experimental work [18]. Recently, conditions for the occurrence of a first-order transition have been proposed as several rules of construction for the modified Kuramoto system [19], and the basin of attraction of this kind of explosive synchronization has been investigated [20].

On the one hand, the role of the networks structure has been studied widely, and on the other, the units investigated share similar dynamical features, i.e., either all the elements behave periodic or all the elements behave chaotic. In the real complex system, the dynamical features of the units may be mixed with these two features. It is unclear on the mixed networks that how the global network behaves: whether there is any synchronization state? It is called a synchronization state if all the units evolve in the same period, in analogy to that all the phase oscillators in the same effective frequencies in the Kuramoto systems [17]. In detail, we may further study phases of the units, in order to check whether the synchronized units evolve in the same phase, or in other words, is it a phase synchronization state? We may also inspect the order of the transitions, and check whether there is any abrupt transition happening? In this paper, we report various collective dynamics in the networks with mixed dynamics: there are fully synchronized states; there is an abrupt transition; the dynamical feature of the synchronization states can be regular or chaotic, and the network topology is important for the emergence of transitions or the synchronization states.

2 Results

The logistic map is widely encountered in the population ecology literature, as a model of population change for species with non-overlapping generations and density-dependent dynamics influenced solely by intraspecific interactions [21–23]. For example, a plant may have bumper harvest in one year, bad harvest in the next year, then bumper harvest again, and this is the way of period-2. The logistic difference equation was promoted by the ecologist Robert May as an example of a simple ecological model with very complicated dynamics, and it can be written as [24]

$$X(n+1) = f(X(n)) = rX(n)[1 - X(n)] \tag{1}$$

where r is a dimensionless population growth factor, which acts as the nonlinear parameter, and $X(n)$ is the population of the nth generation of a certain species. When $r > 1$, there is a fixed point $X^* \equiv \frac{r-1}{r}$ as the steady state of the map, and this fixed point will lose its stability at $r = 3$. As increasing the value of r, a series of period doubling bifurcations from period-2 will transition into chaos at the critical point $r_c = 3.5699$. The iterative dynamics of a single map will be chaotic with r in $(r_c, 4)$ or periodic with r in $[3, r_c)$. Both of the two kinds of maps will present in our mixed networks.

For a biocoenosis, there are many kinds of species; they are interacted with each other. We can model them into a network with nodes as logistic maps. In the network, some species are evolving in regular ways such as period-2, the others are evolving in chaotic ways. For the units adapted in our networks, their nonlinear parameters are considered as the case of evenly spaced. The dimensionless population growth factor of the ith species is

$$r_i = 3 + \frac{i-1}{N} \tag{2}$$

so that, for all isolated units in their original dynamics, there is about 57% of units behave regular and about 43% of units behave chaotic. Therefore, a mixed network can be constructed by units with various dynamics.

Let us consider an unweighted and undirected network of N coupled units. The population of the nth generation of the ith species, denoted by X_i^n $(i = 1, \cdots, N)$, evolves in steps according to the equations as

$$X_i^{n+1} = f(X_i^n) + \frac{d}{k_i}\sum_{j=1}^{N} A_{ij}[f(X_j^n) - f(X_i^n)] \tag{3}$$

where d stands for the dimensionless coupling strength between different species and k_i denotes the link degree of the ith species. The connections among the species are encoded in the adjacency matrix A. A_{ij} is 1 if species i and j are connected, and 0 otherwise. $A_{ii} \equiv 0$ for any species due to the meaning of connections. Finally, the growth factor of the ith species is denoted by r_i, and the number of nodes N is fixed at 1000 in this work.

A first-order phase transition has been found associated hysteresis [15–20]. In order to check the existence of the explosive synchronization transition, we simulated the system in two ways: forward and backward continuations. The former way is computed by gradually increasing the values of coupling strength d from δd to 1. Contrarily, the backward continuation is performed by decreasing the values of d from $1 - \delta d$ to 0. In both forward and backward continuations, $\delta d = 0.001$.

The simplest network structure is all-to-all coupling, just like the original Kuramoto model. To highlight the rich dynamics exhibited by the mixed network, Fig. 1 and Fig. 2 show the bifurcation diagrams for several typical species in the network. By comparing the forward process in Fig. 1 and the backward process in Fig. 2, it is obviously that synchronization states occur at a large coupling strength. The whole networks, either the original periodic species or the original chaotic species, oscillate in chaotic dynamics around $d = 0.35$. The idea of "chaos+chaos=periodic" has been proposed in Ref. [25, 26], and here we find both "periodic+chaos=periodic" and "periodic+chaos=chaos" in the mixed nonlinear networks. Figure 3 presents the period numbers for all the species in the network corresponding to the two processes in Fig. 1 and Fig. 2. Since the sum steps we count is 32, only lower number is meaningful, and period number is 32 for chaotic species. There are two synchronization states: period-8 and period-4 oscillations with $d > 0.4$. In order to identify the phases of the periodic species, we check their corresponding dynamics in the network at coupling strengths concerned. The collective periodic states are found in the same phase: all the species produce their maximum populations (i.e., $X_{i,\max}$) at the

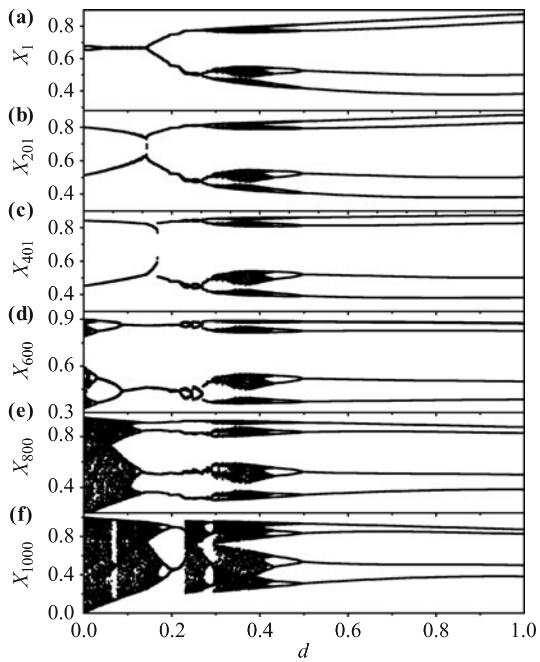

Fig. 1 Bifurcation diagram for the all-to-all networks. X_j vs. d resulting from the forward simulations, where $j = 1$ in **(a)**, 201 in **(b)**, 401 in **(c)**, 600 in **(d)**, 800 in **(e)**, and 1000 in **(f)**, respectively.

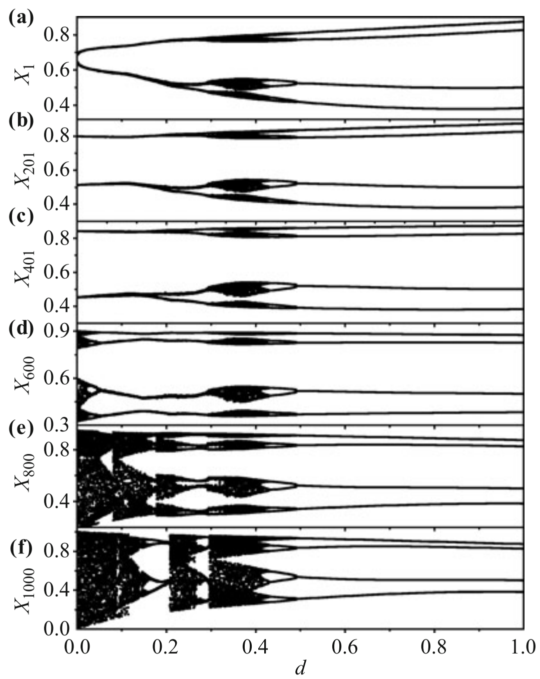

Fig. 2 Bifurcation diagram for the all-to-all networks. X_j vs. d resulting from the backward simulations, where $j = 1$ in **(a)**, 201 in **(b)**, 401 in **(c)**, 600 in **(d)**, 800 in **(e)**, and 1000 in **(f)**, respectively.

same step, and also produce their minimum populations (i.e., $X_{i,\min}$) at the other same step, but their maximum or minimum are different values. There is no hysteresis in Fig. 3, and there is no abrupt phase transition in the all-to-all coupling model.

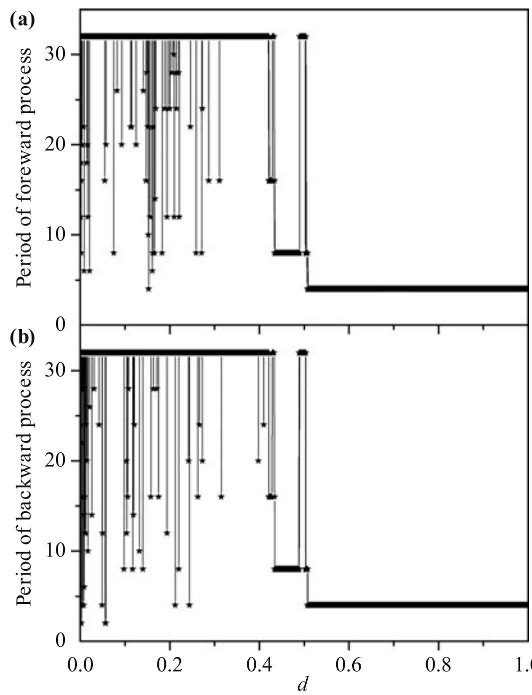

Fig. 3 Period of all the species for the all-to-all networks. **(a)** Period vs. d resulting from the forward simulations. **(b)** Period vs. d resulting from the backward simulations. The sum steps checked is 32, so only the data about low period is meaningful.

For the occurrence of abrupt phase transitions in networks of phase oscillators, Ref. [19] proposed several conditions. This frequency-degree correlation feature has been demonstrated helpfully to obtain an explosive transition. In our networks, we apply this condition to remove some of links from the all-to-all network configuration studied above. The nonlinearity gap-condition for the construction of our networks is as the following. First, a critical value of Z is fixed as the pruning gap. Then, each link will be checked. The link will be removed with probability Z if the nonlinear parameter difference between the two nodes is smaller than the fixed gap: $|r_i - r_j| < Z$, while will be remained with probability $(1-Z)$ otherwise. It is obviously that the critical value Z should be in [0, 1]. By this way, we obtain the nonlinearity gap-condition for networks studied in this work. As the critical value Z is increased, more and more links are removed, connectivity of the network becomes lower. Behavior of the system changes as mean degree $\langle k \rangle$ decreases progressively.

Let us check the network when $Z = 0.75$, a lot of links have been pruned, and the average link degree is only 62. The bifurcation diagrams of several typical species in the networks are plotted in Fig. 4 and Fig. 5. Figure 4 shows the results of the forward process, and Fig. 5 shows the results of the backward process. With the increasing of the coupling strength, the species of original periodicity are driven into chaos by the coupling strength around

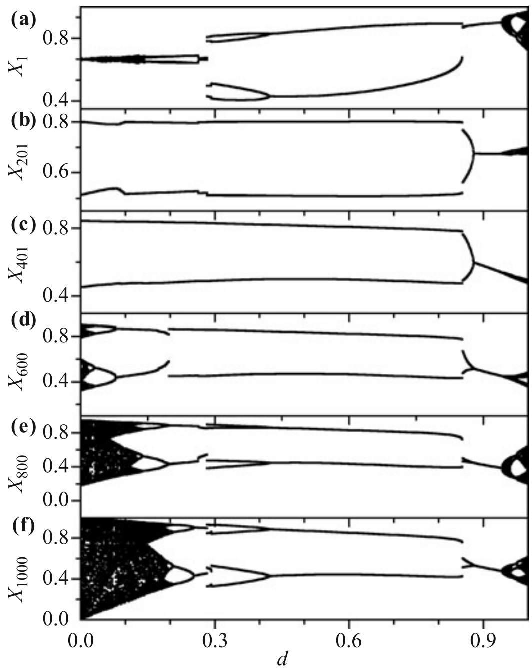

Fig. 4 Bifurcation diagram for the nonlinearity gap-condition networks. The pruning gap is $Z = 0.75$, and the average degree after pruning is $\langle k \rangle = 62$. X_j vs. d resulting from the forward simulations, where $j = 1$ in **(a)**, 201 in **(b)**, 401 in **(c)**, 600 in **(d)**, 800 in **(e)**, and 1000 in **(f)**, respectively.

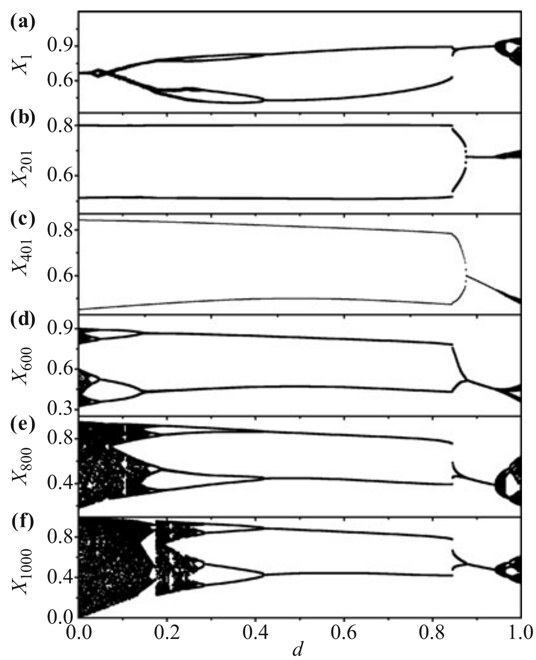

Fig. 5 Bifurcation diagram for the nonlinearity gap-condition networks. The pruning gap is $Z = 0.75$, and the average degree after pruning is $\langle k \rangle = 62$. X_j vs. d resulting from the backward simulations, where $j = 1$ in **(a)**, 201 in **(b)**, 401 in **(c)**, 600 in **(d)**, 800 in **(e)**, and 1000 in **(f)**, respectively.

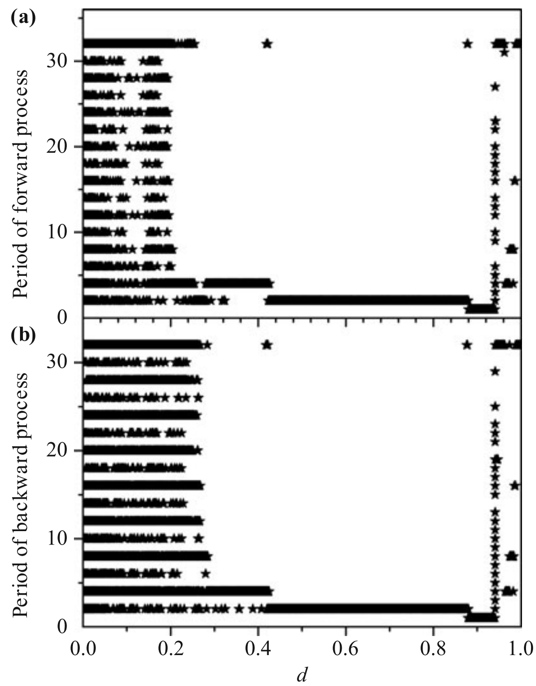

Fig. 6 Period of all the species for the nonlinearity gap-condition networks. The pruning gap is $Z = 0.75$, and the average degree after pruning is $\langle k \rangle = 62$. **(a)** Period vs. d resulting from the forward simulations. **(b)** Period vs. d resulting from the backward simulations. The sum steps checked is 32, so only the data about low period is meaningful.

$d = 1$ through fixed points. The species of original chaos are driven into a series of periodicity, then also turned into another basin of chaos through fixed points. Figure 6 presents the period numbers for all the species in the network corresponding to the two processes in Fig. 4 and Fig. 5. It is obviously that all the species can be expected in collective dynamics with $d > 0.4$. In order to identify the phases of the periodic species, we check the corresponding trajectories of each species in the network at coupling strength concerned. We confirm that there are several synchronization states: fixed points, period-2oscillation, period-4 oscillation, and maybe chaos in some sense. In those states, all the species produce their maximum populations (i.e., $X_{i,\max}$) at the same step, then their minimum populations (i.e., $X_{i,\min}$) at the other same step, although their maximum or minimum have different values. For typical examples of these collective states, Fig. 7 shows the populations of all the species at certain generation with the coupling strength d of 0.35, 0.6, 0.9, and 1.0, respectively. They are periodic states and chaos. The network with mean degree $\langle k \rangle = 62$ can be pruned further by random pruning, but the behavior of the whole network changes little. Similar to the case in Ref. [19], behavior of the system maintains due to large nonlinearity mismatches, although the size of discontinuity decreases.

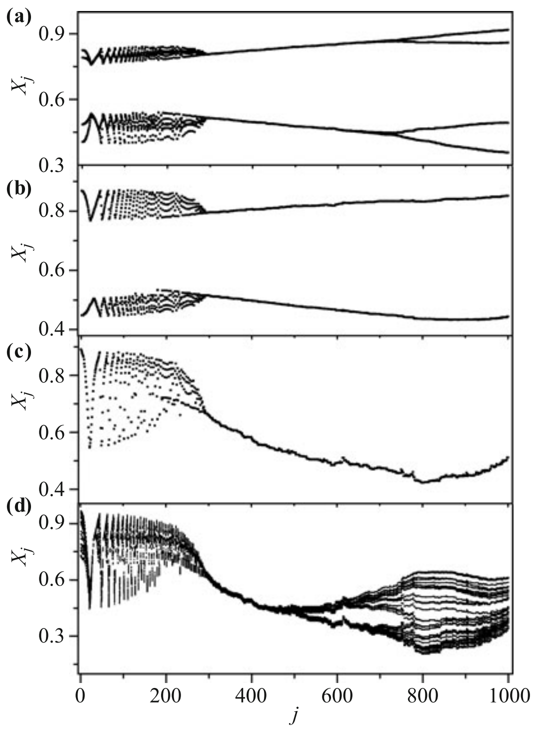

Fig. 7 Populations of all the species in the nonlinearity gap-condition networks. X_j vs. j at the 32 step, and the coupling strength d is 0.35 in panel **(a)**, 0.6 in panel **(b)**, 0.9 in panel **(c)**, and 1.0 in panel **(d)**, respectively.

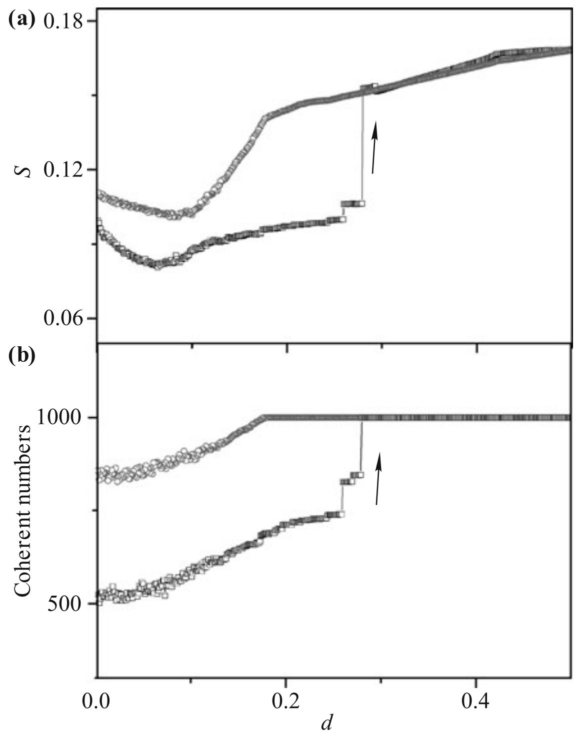

Fig. 8 Phase synchronization level and the coherent numbers in the nonlinearity gap-condition networks. **(a)** S vs. d; **(b)** The coherent numbers versus d. The line with squares denotes results from the forward simulations, and the line with circles denotes results from the backward simulations.

There seems an abrupt change happening around $d = 0.28$ in Fig. 6. In order to clarify the order of this transition, we refer to an order parameter defined in the networks of N Kuramoto oscillators [19]. The order parameter for the network of N species (Logistic maps) at the nth generation can be proposed as $s(n) = \frac{1}{N}\sum_{i=1}^{N}|X_i^n - X_i^*|$, where X_i^* is a time average of X_i with enough steps. The level of synchronization can be monitored by checking the value of $S = \langle s(n)\rangle_T$, where $\langle\cdots\rangle_T$ denotes a time average with $T \gg 1$. The results of phase synchronization level S is reported in panel (a) of Fig. 8, there is an abrupt transition around $d = 0.28$, although it is not a typical hysteresis. We propose the coherent number to describe how many species go up together at certain step, which is a rough estimate for the coherent degree of the networks. The coherent numbers in both processes of forward and backward are shown in panel (b) of Fig. 8. States in the upper curve imply to be more coherent referring to the panels either (a) or (b) in Fig. 8. It should be regarded as an first-order phase transition, because of the discontinuity in phase diagram shown in Fig. 8, although this transition is not toward a fully synchronization state.

3 Discussion

In conclusion, we demonstrate that there are multiply synchronization states in the mixed networks. Various synchronization states are reported in the all-to-all coupling networks, such as periodic two, periodic four and periodic eight. The states are not only synchronization states, but also phase synchronization states in general sense. According to the progressive dynamics of the system in forward process and backward process, the phase transitions are not abrupt ones when the all-to-all network is driven into periodic synchronization states in order.

On the other hand, the nonlinearity gap-condition is introduced into the mixed network. As the coupling strength is increased gradually, the whole network transitions into coherent states, from periodic states into fixed points, and finally into global chaotic dynamics. Except the emergence of chaos, the forward process reflects part of an inverse period-doubling bifurcation. The periodic states are not only synchronization states, but also phase synchronization states in general sense, this feature is same to that in the all-to-all coupling networks. The global chaos is in a coherent basin of chaos, which is different from the original basin of any single chaotic maps. Whether the chaotic species are synchronized in general sense will be studied in further work.

Abrupt transition happens in networks when the frequencies of the nodes are positively correlated to the node degree. In our model, the frequencies of the nodes correspond to the periodicity of the nodes, and the periodicity of the nodes depends on their nonlinear parameters. The gap-pruning sets up a positive correlation between the node degree and its nonlinear parameter, i.e., positive correlation between the node degree and its dynamics. This network transitions into the coherent branch of states as the coupling is increased, while the coherent branch keeps its stability in a backward process. A first-order transition with an abrupt jumping up is observed. Large nonlinearity mismatches and lower connectivity are important to the occurrence of the abrupt phase transition and this agrees with the results in Refs. [18–21].

In complex networks, even with regular topology, there may be mixed units of different type of dynamics, how the regular ones are influenced by the chaotic ones, or vice versa? Interaction will help them coherent and synchronous, driving them into a big whole unit. Rich synchronization states can be expected in the mixed networks. Each unit of the network may oscillate in phase or be fixed at ease, even behave in collective chaos. On the other hand, if coupling strength of the networks was fixed, we may adjust the critical value Z for pruning to remove or graft a certain number of links, thus the behavior of the network will be controlled, and the order of the phase transition may be controlled in the same way.

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