Front. Phys., 2014, 9(4): 446–450
DOI 10.1007/s11467-014-0425-1

RESEARCH ARTICLE

Quantum phase for an electric quadrupole moment in noncommutative quantum mechanics

Halqem Nizamidin[1], **Abduwali Anwar**[1], **Sayipjamal Dulat**[2,†], **Kang Li**[3]

[1] *School of Mathematical Science, Capital Normal University, Beijing 100048, China*
[2] *School of Physics Science and Technology, Xinjiang University, Urumqi 830046, China*
[3] *Department of Physics, Hangzhou Normal University, Hangzhou 310036, China*
Corresponding author. E-mail: †*dulat98@yahoo.com*

Received November 18, 2013; accepted April 10, 2014

We study the noncommutative nonrelativistic quantum dynamics of a neutral particle, which possesses an electric qaudrupole moment, in the presence of an external magnetic field. First, by introducing a shift for the magnetic field, we give the Schrödinger equations in the presence of an external magnetic field both on a noncommutative space and a noncommutative phase space, respectively. Then by solving the Schrödinger equations both on a noncommutative space and a noncommutative phase space, we obtain quantum phases of the electric quadrupole moment, respectively. We demonstrate that these phases are geometric and dispersive.

Keywords noncommutative quantum mechanics, electric quadrupole moment, quantum phase, noncommutative phase space

PACS numbers 03.65.Vf, 02.40.Gh, 13.40.Em

1 Introduction

The approach to noncommutative quantum field theory based on star products and Seiberg–Witten maps allowing for the generalization of the standard model of particle physics to the case of noncommutative space-time. Since noncommutative quantum field theory may solve the puzzles of the standard model, there are many papers concerning the quantum field theory on a noncommutative space-time [1–4]. Apart from these studies, much researches has been devoted to the study of various aspects of quantum mechanics (QM) on a noncommutative space (NCS) and a noncommutative phase space(NCPS), because the main goal of noncommutative quantum mechanics (NCQM) is to find measurable spatial noncommutativity effects. For example, the authors of Refs. [5–8] have studied the Aharonov–Bohm phase on an NCS and an NCPS. The Aharonov–Casher phase for a spin-1/2 and a spin-1 particle on a NCS and a NCPS has been studied in Refs. [9–12]. The He–McKellar–Wilkens effect for spin one particles in noncommutative quantum mechanics has been studied in Ref. [13]. The noncommutative quantum Hall effect has been studied in Refs. [14–21]. Refs. [22, 23] have studied the spin hall effect (SHE) in the framework of noncommutative quantum mechanics. The noncommutative quantum dynamics of a neutral particle, which possesses permanent magnetic and electric dipole moments, in the presence of external electric and magnetic fields has been studied in Ref. [24], but there is no discussion about the noncommutative quantum phase of a neutral particle, which possesses an electric quadrupole moment in the literature, thus it is necessary to study these problems.

This paper is organized as follows: In Section 2, we discuss the phase of an electric quadrupole moment on a usual commutative space. In Section 3, we obtain the quantum phase of an electric quadrupole moment on an NCS. The quantum phase of an electric quadrupole on an NCPS is investigated in Section 4. Conclusions are given in the last section.

2 Quantum phase of an electric quadrupole moment on a commutative space

To discuss possible quantum phase of an electric quadrupole and multipole moment, we begin with the Lagrangian. Consider a neutral particle of mass m with an electric quadruple moment $\boldsymbol{Q}$ moving at velocity $\boldsymbol{v}$ in an electromagnetic field, the nonrelativistic Lagrangian [25] is

$$L=\frac{1}{2}m\boldsymbol{v}^2+\boldsymbol{Q}\cdot(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}) \tag{1}$$

Here we used the notation $\boldsymbol{Q}=\mathcal{Q}_i\hat{\boldsymbol{e}}_i$, with $\mathcal{Q}_i=\mathcal{Q}_{ij}\partial_j$. Then the canonical momentum is given by

$$\boldsymbol{P}=m\boldsymbol{v}-\frac{1}{c}\boldsymbol{Q}\times\boldsymbol{B} \tag{2}$$

and the Hamiltonian is

$$H=\frac{1}{2m}(\boldsymbol{P}+\frac{1}{c}\boldsymbol{Q}\times\boldsymbol{B})^2-\boldsymbol{Q}\cdot\boldsymbol{E} \tag{3}$$

When $\boldsymbol{E}=0$, the Schrödinger equation for this problem is (we choose the unit of $\hbar=c=1$)

$$\frac{1}{2m}(\boldsymbol{P}+\boldsymbol{Q}\times\boldsymbol{B})^2\Psi=E\Psi \tag{4}$$

In an analogous way as in usual quantum mechanics, the solution for Eq. (4) reads

$$\Psi=\Psi_0\exp[-\mathrm{i}\int_{x_0}^{x}(\boldsymbol{Q}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{x}] \tag{5}$$

where Ψ_0 is the solution of Eq. (4) when $\boldsymbol{B}=0$. The phase term of Eq. (5) is the quantum phase of an electric quadrupole moment on a commutative space. If we consider a particle to pass a double slits, then the integral runs from the source x_0 through one of the two slits to the screen x, the coherent pattern will depend on the phase difference of two paths. Thus we can write this nondispersive quantum phase of a neutral particle with an electric quadrupole moment moving in a magnetic field as

$$\Phi_Q=-\mathrm{i}\oint(\boldsymbol{Q}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{x} \tag{6}$$

3 Quantum phase of an electric quadrupole moment on a noncommutative space

To begin with, let us first briefly review some properties of NCS. At string scale, not only coordinate and momentum operators do not commute, but also coordinate-coordinate operators may not commute any more. Therefore, the NCS is a space where space coordinate $\hat{x}_i$ and momentum coordinate $\hat{p}_i$ operators satisfy the following commutation relations:

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\Theta_{ij},\ [\hat{p}_i,\hat{p}_j]=0,\ [\hat{x}_i,\hat{p}_j]=\mathrm{i}\delta_{ij},\ i,j=1,2,3 \tag{7}$$

Here Θ_{ij} is totally antisymmetric real tensor which represent the noncommutative property of the coordinate on an NCS. By replacing the normal product with a star product, the Schrödinger equation on a commuting space will change into the Schrödinger equation on an NCS, i.e., on an NCS, the Schrödinger equation can be written as

$$H(x,p)*\Psi=E\Psi \tag{8}$$

Here $H(x,p)$ is the Hamiltonian operator of the usual quantum system. The Moyal–Weyl (or star) product between two functions is defined by

$$\begin{aligned}\hat{f}(\hat{\vec{x}})\hat{g}(\hat{\vec{x}})&\equiv f(\vec{x})*g(\vec{x})\\&=\exp\left[\frac{\mathrm{i}\Theta_{ij}}{2}\partial_{x_i}\partial_{y_j}\right]f(\vec{x})g(\vec{y})|_{\vec{x}=\vec{y}}\end{aligned} \tag{9}$$

where $f(\vec{x})$ and $g(\vec{x})$ are two arbitrary, infinitely differentiable functions on a commutative space, and $\hat{f}(\hat{\vec{x}})$ and $\hat{g}(\hat{\vec{x}})$ are the corresponding functions on an NCS.

On a NCS the star product can be replaced with a Bopp's shift [26], i.e., the star product can be changed into the ordinary product by replacing $H(x,p)$ with $H(\hat{x},\hat{p})$. Thus the Schrödinger equation can be written as

$$H(\hat{x},\hat{p})\Psi=H(x_i-\frac{1}{2}\Theta_{ij}p_j,p_i)\Psi=E\Psi \tag{10}$$

Here x_i and p_i are coordinate and momentum operators in usual quantum mechanics. Thus Eq. (10) is actually defined on a commutative space, and the noncommutative effects can be evaluated through the Θ-related terms.

Now we are in the position to discuss how to calculate quantum phase of an electric quadrupole moment on an NCS. When magnetic field is involved, the Schrödinger equation (8) becomes

$$H(x_i,p_i,B_i)*\Psi=E\Psi \tag{11}$$

To replace the star product in Eq. (11) with a usual product, we need to replace x_i, p_i and B_i with shifts as follows:

$$\begin{aligned}&x_i\to x_i-\frac{1}{2}\Theta_{ij}p_j\\&p_i\to p_i,\quad i,j=1,2\cdots,n\\&B_i\to B_i+\frac{1}{2}\Theta_{lj}p_l\partial_jB_i\end{aligned} \tag{12}$$

Thus the Schrödinger equation (11) in the presence of magnetic field becomes

$$H(x_i-\frac{1}{2}\Theta_{ij}p_j,p_i,B_i+\frac{1}{2}\Theta_{lj}p_l\partial_jB_i)\Psi=E\Psi \tag{13}$$

Now let us consider a particle of mass m, with an electric quadruple moment $\boldsymbol{Q}$, moving at velocity $\boldsymbol{v}$ in a magnetic field, Schrödinger equation is

$$\frac{1}{2m}(p_k + \epsilon_{ijk}\mathcal{Q}_i B_j + \frac{1}{2}\epsilon_{ijk}\mathcal{Q}_i \Theta_{ln} p_l \partial_n B_j)^2 \Psi = E\Psi \tag{14}$$

The solution of Eq. (14) reads

$$\Psi = \Psi_0 \exp[-\mathrm{i}\oint(\epsilon_{ijk}\mathcal{Q}_i B_j + \frac{1}{2}\epsilon_{ijk}\mathcal{Q}_i\Theta_{ln}p_l\partial_n B_j)\mathrm{d}x_k] \tag{15}$$

where Ψ_0 is the solution of Eq. (14) when $B_j = 0$. The phase term of Eq. (15) is the quantum phase of an electric quadrupole moment on an NCS. Thus we can write this phase as

$$\Phi_Q^{\mathrm{NCS}} = \Phi_Q + \Phi_Q^{\theta} \tag{16}$$

where

$$\Phi_Q = -\mathrm{i}\oint \epsilon_{ijk}\mathcal{Q}_i B_j \mathrm{d}x_k \tag{17}$$

and

$$\Phi_Q^{\theta} = -\frac{\mathrm{i}}{2}\oint \epsilon_{ijk}\mathcal{Q}_i\Theta_{ln}[mv_l - (\boldsymbol{Q}\times\boldsymbol{B})_l]\partial_n B_j \mathrm{d}x_k \tag{18}$$

where we used the relation $p_l = mv_l - (\boldsymbol{Q}\times\boldsymbol{B})_l + \mathcal{O}(\Theta)$, and we omitted the second order terms of the Θ; the first term Φ_Q is the quantum phase of an electric quadrupole moment in usual quantum mechanics; the second term Φ_Q^{θ} is the correction to the usual quantum phase of an electric quadrupole moment due to space–space noncommutativity.

In three dimensional NCS, $i, j = 1, 2, 3$, we can define a vector $\theta = (\theta_1, \theta_2, \theta_3)$, with θ_i satisfies $\Theta_{ij} = \epsilon_{ijk}\theta_k$ or $\theta_i = \frac{1}{2}\epsilon_{ijk}\Theta_{jk}$. Then Eq. (16) has the form

$$\begin{aligned}\Phi_Q^{\mathrm{NCS}} = & -\mathrm{i}\oint(\boldsymbol{Q}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{x} \\ & -\frac{\mathrm{i}}{2}m\oint\theta\cdot[\boldsymbol{v}\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x} \\ & +\frac{\mathrm{i}}{2}\oint\theta\cdot[(\boldsymbol{Q}\times\boldsymbol{B})\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x}\end{aligned} \tag{19}$$

This equation is the noncommutative version of the quantum phase of an electric quadrupole moment. The first term in Eq. (19) is the usual quantum phase of an electric quadrupole on a commutative space. The other NCS correction terms depend on the velocity of the particle.

4 Quantum phase of an electric quadrupole on a noncommutative phase space

In this section we discuss the NCPS correction to the quantum phase of a neutral particle which possesses electric quadrupole moment. Because the case of both space-space and momentum–momentum noncommutativity is different from the case of only space–space noncommutativity. Thus on a NCPS the momentum operator in Eq. (7) satisfies the following commutation relations:

$$[\hat{p}_i, \hat{p}_j] = \mathrm{i}\bar{\Theta}_{ij}, \qquad i, j = 1, 2, \cdots, n \tag{20}$$

Here $\bar{\Theta}_{ij}$ is totally antisymmetric real tensor which represent the noncommutative property of the momentum. On an NCPS the star product in Eq. (9) becomes

$$\begin{aligned}(f * g)(x, p) &= \mathrm{e}^{\frac{\mathrm{i}}{2\alpha^2}\Theta_{ij}\partial_i^x\partial_j^x + \frac{\mathrm{i}}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p\partial_j^p} f(x, p)g(x, p) \\ &= f(x, p)g(x, p) + \frac{\mathrm{i}}{2\alpha^2}\Theta_{ij}\partial_i^x f\partial_j^x g|_{x_i = x_j} \\ &\quad + \frac{\mathrm{i}}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p f\partial_j^p g|_{p_i = p_j}\end{aligned} \tag{21}$$

To replace the star product in Schrödinger equation (11) with a usual product, first we need to replace x_i and p_i with a generalized Bopp's shift as follows:

$$\begin{aligned} x_i &\to x_i - \frac{1}{2\alpha^2}\Theta_{ij}p_j \\ p_i &\to p_i + \frac{1}{2\alpha^2}\bar{\Theta}_{ij}x_j \end{aligned} \tag{22}$$

and then we need to replace B_i with the generalized shift as

$$B_i \to \alpha B_i + \frac{1}{2\alpha}\Theta_{lj}p_l\partial_j B_i \tag{23}$$

Here α is a scaling constant related to the noncommutativity of phase space. Thus on an NCPS the Schrödinger equation (11) becomes

$$\begin{aligned} & H(x_i - \frac{1}{2\alpha^2}\Theta_{ij}p_j, p_i + \frac{1}{2\alpha^2}\bar{\Theta}_{ij}x_j, \\ & B_i + \frac{1}{2\alpha^2}\Theta_{lj}p_l\partial_j B_i)\Psi = E\Psi \end{aligned} \tag{24}$$

Now the Schrödinger equation for the neutral particle with quadrupole moment on an NCPS has the form

$$\begin{aligned} & \frac{1}{2m'}(p_k + \frac{1}{2\alpha^2}\bar{\Theta}_{kj}x_j + \epsilon_{ijk}\mathcal{Q}_i B_j \\ & \quad + \frac{1}{2\alpha^2}\epsilon_{ijk}\mathcal{Q}_i\Theta_{ln}p_l\partial_n B_j)^2\Psi = E\Psi \end{aligned} \tag{25}$$

with $m' = m/\alpha^2$. Thus the total phase shift including the contributions due to both space-space and momentum-momentum non-commutativity on 3-dimensional NCPS is

$$\begin{aligned}\Phi_Q^{\mathrm{NCPS}} = & -\mathrm{i}\oint(\boldsymbol{Q}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{x} \\ & -\frac{\mathrm{i}}{2}m\oint\theta\cdot[\boldsymbol{v}\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x} \\ & +\frac{\mathrm{i}}{2}\oint\theta\cdot[(\boldsymbol{Q}\times\boldsymbol{B})\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x} + \Phi_Q^{\bar{\theta}} \\ = & \ \Phi_Q + \Phi_Q^{\theta} + \Phi_Q^{\bar{\theta}}\end{aligned} \tag{26}$$

Where the $\varPhi_Q^{\bar{\theta}}$ is the first order modification term due to momentum–momentum non-commutativity, and it has the form

$$\begin{aligned}\varPhi_Q^{\bar{\theta}} &= \frac{-\mathrm{i}}{2\alpha^4}(1-\alpha^4)\oint \theta\cdot[\boldsymbol{v}\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x}\\ &\quad -\frac{\mathrm{i}}{2\alpha^2}(1-\alpha^2)\oint \theta\cdot[(\boldsymbol{Q}\times\boldsymbol{B})\times\vec{\nabla}(\boldsymbol{Q}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{x}\\ &\quad -\frac{\mathrm{i}}{2\alpha^2}\oint \bar{\varTheta}_{ij}x_j\mathrm{d}x_i \end{aligned}\tag{27}$$

The term $\varPhi_Q$ is the usual phase for an electric quadrupole moment on a commutative space; the term $\varPhi_Q^{\theta}$ is the correction due to noncommutativity of space; the other term $\varPhi_Q^{\bar{\theta}}$ is the additional phase due to the momentum-momentum noncommutativity. Note that this is a dispersive geometric phase that depends on the velocity of the particle and also on the magnetic field. It is obvious from Eq. (27) that, when $\alpha = 1$, we have $\bar{\varTheta}_{ij} = 0$ as well as $\varPhi_Q^{\bar{\theta}} = 0$, so this phase returns to its expression in Eq. (19) on an NCS.

5 Conclusions

There are two methods, namely, star product and shift method, to study physical effects on an NCS and an NCPS. In this paper, using the shift method, we studied the nonrelativistic quantum dynamics of a neutral particle which possesses electric quadrupole moment, in the presence of magnetic field. The consideration of the NCS and NCPS accumulated additional phase differences. In order to obtain additional phase differences, in Section 3, first, we gave the Schrödinger equation in the presence of magnetic field, by replacing the star product with the Bopp's shift, we derived the quantum phases of an electric quadrupole moment with NCS corrections. The new term $\varPhi_Q^{\theta}$ in Eq. (18) represents the additional phase difference on a NCS. Furtherly, we also demonstrated that these are geometric dispersive phases, since they depend on the velocity of the particle. Usually, a geometric phase is a local effect, while a topological phase is nonlocal. In Section 4, by solving the Schrödinger equation in the presence of magnetic field, we obtained the NCPS correction to the usual quantum phase of an electric quadrupole moment. The new term $\varPhi_Q^{\bar{\theta}}$ in Eq. (27) represents the additional phase difference on an NCPS and demonstrates that these are geometric dispersive phase too, as they depend on the velocity of the particle. If the present experimental observations on the quantum phase for quadrupole are available, an estimation for the bound on the parameter of non-commutativity θ could be made explicitly, and thus high-precision measurements in quantum mechanical systems might be able to reveal the noncommutativity of space.

Acknowledgements The work was supported by the National Natural Science Foundation of China (Grant Nos. 11165014 and 11175053).

References

1. S. Godfrey and M. A. Doncheski, Signals for noncommutative QED in $e\gamma$ and $\gamma\gamma$ collisions, *Phys. Rev. D*, 2001, 65(1): 015005
2. M. Haghighat and M. M. Ettefaghi, Parton model in Lorentz invariant noncommutative space, *Phys. Rev. D*, 2004, 70(3): 034017
3. A. Devoto, S. Chiara, and W. W. Repko, Noncommutative QED corrections to $\mathrm{e}^+\mathrm{e}^- \to \gamma\gamma\gamma$ at linear collider energies, *Phys. Rev. D*, 2005, 72(5): 056006
4. X. Calmet, Quantum electrodynamics on noncommutative spacetime, *Eur. Phys. J. C*, 2007, 50(1): 113
5. M. Chaichian, A. Demichev, P. Prešnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Aharonov–Bohm effect in noncommutative spaces, *Phys. Lett. B*, 2002, 527(1–2): 149
6. M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Quantum theories on noncommutative spaces with nontrivial topology: Aharonov–Bohm and Casimir effects, *Nucl. Phys. B*, 2001, 611(1–3): 383
7. H. Falomir, J. Gamboa, M. Loewe, F. Méndez, and J. Rojas, Testing spatial noncommutativity via the Aharonov–Bohm effect, *Phys. Rev. D*, 2002, 66(4): 045018
8. K. Li and S. Dulat, The Aharonov–Bohm effect in noncommutative quantum mechanics, *Eur. Phys. J. C*, 2006, 46(3): 825
9. B. Mirza and M. Zarei, Non-commutative quantum mechanics and the Aharonov–Casher effect, *Eur. Phys. J. C*, 2004, 32(4): 583
10. K. Li and J. H. Wang, The topological AC effect on noncommutative phase space, *Eur. Phys. J. C*, 2007, 50(4): 1007
11. B. Mirza, R. Narimani, and M. Zarei, Aharonov–Casher effect for spin-1 particles in a non-commutative space, *Eur. Phys. J. C*, 2006, 48(2): 641
12. S. Dulat and K. Li, The Aharonov–Casher effect for spin-1 particles in non-commutative quantum mechanics, *Eur. Phys. J. C*, 2008, 54(2): 333
13. S. Dulat, K. Li, and J. Wang, The He–McKellar–Wilkens effect for spin one particles in non-commutative quantum mechanics, *J. Phys. A: Math. Theor.*, 2008, 41(6): 065303
14. B. Harms and O. Micu, Noncommutative quantum Hall effect and Aharonov–Bohm effect, *J. Phys. A*, 2007, 40(33): 10337
15. O. F. Dayi and A. Jellal, Hall effect in noncommutative coordinates, *J. Math. Phys.*, 2002, 43(10): 4592

16. O. F. Dayi and A. Jellal, Erratum: “Hall effect in noncommutative coordinates” [J. Math. Phys. 43, 4592 (2002)], *J. Math. Phys.*, 2004, 45(2): 827 (E)
17. A. Kokado, T. Okamura, and T. Saito, Noncommutative phase space and the Hall effect, *Prog. Theor. Phys.*, 2003, 110(5): 975
18. S. Dulat and K. Li, Quantum Hall effect in noncommutative quantum mechanics, *Eur. Phys. J. C*, 2009, 60(1): 163
19. B. Chakraborty, S. Gangopadhyay, and A. Saha, Seiberg–Witten map and Galilean symmetry violation in a noncommutative planar system, *Phys. Rev. D*, 2004, 70(10): 107707, arXiv: hep-th/0312292
20. F. G. Scholtz, B. Chakraborty, S. Gangopadhyay, and A. G. Hazra, Dual families of noncommutative quantum systems, *Phys. Rev. D*, 2005, 71(8): 085005
21. F. G. Scholtz, B. Chakraborty, S. Gangopadhyay, and J. Govaerts, Interactions and non-commutativity in quantum Hall systems, *J. Phys. A*, 2005, 38(45): 9849
22. Ö. F. Dayi and M. Elbistan, Spin Hall effect in noncommutative coordinates, *Phys. Lett. A*, 2009, 373(15): 1314
23. K. Ma and S. Dulat, Spin Hall effect on a noncommutative space, *Phys. Rev. A*, 2011, 84(1): 012104
24. E. Passos, L. R. Ribeiro, C. Furtado, and J. R. Nascimento, Noncommutative Anandan quantum phase, *Phys. Rev. A*, 2007, 76(1): 012113
25. C. C. Chen, Topological quantum phase and multipole moment of neutral particles, *Phys. Rev. A*, 1995, 51(3): 2611
26. T. Curtright, D. Fairlie, and C. Zachos, Features of time-independent Wigner functions, *Phys. Rev. D*, 1998, 58(2): 025002