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RESEARCH ARTICLE

Temporal inequalities for sequential multi-time actions in quantum information processing

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A new kind of temporal inequalities are discussed, which apply to algorithmic processes, involving a finite memory processing unit. They are an alternative to the Leggett–Grag ones, as well as to the modified ones by Brukner *et al.* If one considers comparison of quantum and classical processes involving systems of finite memory (of the same capacity in both cases), the inequalities give a clear message why we can expect quantum speed-up. In a classical process one always has clearly defined values of possible measurements, or in terms of the information processing language, if we have a sequential computations of some function depending on data arriving at each step on an algorithm, the function always has a clearly defined value. In the quantum case only the final value, after the end of the algorithm, is defined. All intermediate values, in agreement with Bohr's complementarity, cannot be ascribed a definite value.

Keywords temporal inequalities, quantum information

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1 Introduction

There is an on-going discussion on what is the main reason of quantum speedup in computation. Recently it was suggested [1], that the reason might be violation of principles of the so-called macro-realism, defined in Refs. [2, 3]. Unfortunately macro-realism has an assumption which is clearly violated in case of micro-systems. This is the so-called non-invasiveness, which assumes that a measurement performed upon a system does not change possible results of later measurements. This is clearly violated by quantum systems [4], in the case of which measurement is performed via an initial interaction with the system, which due to the fragility of micro-objects disturbs the system, and clearly cannot be treated as non-invasive. Still macro-realism uses another principal assumption – realism. This allows one to preform algebraic calculations involving values of measurements that would have been obtained had we chosen different measurement settings than these which actually were chosen (in the original version: had the measurements been done at different moments of time). This clearly is against the spirit of Copenhagen Interpretation of quantum mechanics, and uses notions outside of the quantum theory (there are no theoretical tools in quantum mechanics, which allow one to obtain such values). That is, we make an assumption which is allowable only if one assumes hidden variables. As temporal sequences of actions involve time-separated events, in the discussed case Bell's Theorem is inapplicable (as it concerns spatially separated events). Thus, hidden variables are not ruled out in modeling of such processes. Also Kochen–Specker Theorem does not rule out hidden variables for measurements which as sequential in time.

In the following it will be shown that if one assumes the standard properties of an algorithm, that is at a given moment one can precisely define the state of the processing unit, and that at later moment of time, t_1, the state of the processing unit is defined by its state at an earlier time, say t_0, and the input data and operations performed between t_0 and t_1, then for a processing unit of a finite memory (or equivalently of finite number of fully distinguishable states, or in more recent wording, of a finite dimensionality), one can derive temporal-inequalities, which are never violated by a classical algorithm, but can be violated by a quantum one [4]. Specific forms of such inequalities are defined by the task function of the algorithm.

The reasoning will be presented for the simplest processing unit, which has just two distinguishable states (a transistor for the classical case, a qubit, say photon's polarization, for the quantum case), and therefore memory capacity of one bit. Note that the dependence of state of the processing unit on its initial state, and intermediate operations performed upon it, is common in physics, from Newton's mechanics, to quantum mechanics, namely that the initial state of the system defines full initial conditions for its subsequent evolution (under possible external influences). The evolution is determined by the operations on the system (e.g. in case of a classical particle the applied forces). Also, a system cannot have a bigger memory capacity than the maximal number of its distinguishable states.

2 New inequalities, an example

A very simple form of inequalities will presented here [5]. However, hopefully it would be obvious for the reader that various generalizations are possible. The computational problem chosen below was selected due to its extreme simplicity, and the fact that its quantum equivalent, involving photon's polarization, has been realized in a laboratory some years ago. The specific task function has been considered earlier in studies of reduction of communication complexity in quantum realizations of some computational processes. A very important feature of the simple problem is that is does not have a deterministic classical solution (under the constraint of a finite memory – here one bit), and with the size of the problem the probability to get a correct solution via classical methods in an exponential way approaches the one for a random guess.

Consider the following problem. We want to use a computing device to give the value of the following task function of bit sequences y'_l and x_l, with $l = 1, 2, \cdots, n$:

$$T_n = (-1)^{\sum_{l=1}^{n} y'_l} \cos\left(\frac{\pi}{2}\sum_{k=1}^{n} x_k\right) \tag{1}$$

with a promise that

$$p(x_1, x_2, \cdots) = 2^{-N+1}\left|\cos\left(\frac{\pi}{2}\sum_{l=1}^{n} x_l\right)\right|$$

and that y_l's are completely random, and independent of each other. Under such a promise one has

$$\left(\sum_{l=1}^{n} x_l\right) \bmod 2 = 0$$

(their sum is always even), and the values of the function are binary, $T_n = \pm 1$. The bits y'_l and x_l arrive in pairs, and in a sequential order time order. The value of n is undefined before the start of the process. Still, when it is declared that the final $l = n$ has been reached, our computing unit must give a definite correct answer provided the promise for x_l's is met.

Let us introduce the basic ideas behind the new inequalities, the ideas which govern classical processing units. As it was said above, we shall take the simplest possible situation. Imagine a single transistor (our processing unit) which can be in two states, denoted as $A = \pm 1$, current, no current. This our *sole* processing unit, there is no pre-processing of data. The length of the sequence n is undefined, the system may be asked at any moment to produce the result, provided the promise is met (if it is not met the task function in such a case is 0, and therefore is not binary, and thus in such a case bigger set of distinguishable states is required). At each instant of time, t_k, an operation is performed on the system which may change the value of A. The operation is governed by two external input bits x_k, y'_k. The pair will be denoted $X_k = 2^1 y'_k + 2^0 x_k = 0, 1, 2, 3$. It was assumed that y'_k is completely random, whereas the distribution of x_k's is given by $p(x_1, x_2, \cdots)$. This assumption is introduced here because the randomness of y'_k implies that our two-distinguishable-state system is incapable to store information about both bits. After say l operations the state of the system, right after t_l, is denoted as $A_l = A(X_1, X_2, \cdots, X_l)$. We notice an obvious thing, in the case of a transistor used in Turing-like operations: as the unit has one bit memory it forgets the reason why it is in the current state (as this would require a bigger memory, to store e.g. the previous y'_l's and x_l's,):

$$A_l = A_l(X_l, A_{l-1}) \tag{2}$$

The current state is defined by the state of the system before the last operation, A_{l-1} , and by the last operation X_l.

For technical reasons let us replace y'_l by $y_l = (-1)^{y'_l}$. Thus, $y_l = \pm 1$. With such a notation operations performed are to give at the end of the process A_n: an answer to the question about the value of a binary task function

$$T_n = \prod_{l=1}^{n} y_l f(x_1, \cdots, x_n) \tag{3}$$

under a promise that the distribution of x_k's obeys $p(x_1, x_2, \cdots)$.

What is the probability to get a correct result for systems obeying the above assumptions?

To answer this question let us introduce the following function of merit, which gives the average product of the correct value, T_l of the task function for a certain n, such that the promise is met, and the answer of the computing system A_n. Obviously if $T_nA_n = 1$ one has $A_n = T_n$ and if $T_nA_n = -1$ the answer is wrong. The function reads

$$F = \langle A_nT_n\rangle_{\rm avg} = \sum_{x_1,x_2,\cdots,x_n=0,1}\ \sum_{y_1,y_2,\cdots,y_n=\pm1} \frac{1}{2^n}$$
$$\cdot p(x_1,\cdots,x_n)A(X_1,X_2,\cdots,X_n)\prod_{l=1}^{n} y_l f(x_1,\cdots,x_n)$$

To find the upper bound of F, note the following:

- According to our assumptions A_n depends on the current values x_n, y_n, and the state of the system before t_n, that is A_{n-1}. Thus $A_n = A_n(x_n, y_n, A_{n-1})$. It is a binary function of its three arguments.
- There are just *four* binary functions of a binary variable. To use this fact one can treat x_n and y_n as fixed, which we will denoted by putting them into subscripts. Thus $A_n = A_{x_n,y_n}(A_{n-1})$. The functional dependence on A_{n-1} can only have only the following form:

$$A_{x_n,y_n}(A_{n-1}) = D_{x_n,y_n} + C_{x_n,y_n}A_{n-1}$$

  where C and D can have values ±1 or 0, under the constraint that $|C_{x_n,y_n}| + |D_{x_n,y_n}| = 1$.
- On the same grounds $C_{x_ny_n}$ must be of the form $C_{x_n,y_n} = e_n(x_n) + c_n(x_n)y_n$, with e and c_n having the same properties as D and C. Similarly one must have $D_{x_ny_n} = f(x_n) + d(x_n)y_n$.
- Thus

$$A_{x_n,y_n}(A_{n-1}) = f(x_n) + d(x_n)y_n + [e_n(x_n) + c_n(x_n)y_n]A_{n-1}$$

  However, one has

$$\sum_{y_n,y_{n-1}=\pm1} y_ny_{n-1}\{f(x_n) + d(x_n)y_n + [e_n(x_n) + c_n(x_n)y_n]A_{n-1}\} = \sum_{y_{n-1}=\pm1} y_{n-1}c_n(x_n)A_{n-1}$$

  After the summation the only last term may not vanish, because in principle A_{n-1} can be dependent on y_{n-1}. Thus the optimal form of A_n is

$$A_n = y_nc_n(x_n)A_{n-1}$$

With a similar reasoning one shows that the optimal form of A_{n-1} is $y_{n-1}c_{n-1}(x_{n-1})A_{n-2}$, and so on. Continuing like that we arrive at the final formula which is

$$F = \langle A_nT_n\rangle_{\rm avg} = \sum_{x_1,x_2,\cdots,x_n=0,1} K(x_1,\cdots,x_n)\prod_{k=1}^{n} c_k(x_k)$$

where all $c_k(x_k)$ take values ±1 (or 0, but then they do not contribute), and the coefficients K are given by $K(x_1,\cdots,x_n) = p(x_1,\cdots,x_n)f(x_1,\cdots,x_n)$. The upper bound for F can be reached in a similar way as for any Bell inequality.

For the example for the considered example of a task function the bound is given by

$$\sum_{x_1,x_2,\cdots,x_n=0,1} K(x_1,\cdots,x_n)\prod_{k=1}^{n} c_k(x_k) \leqslant 2^{-N+1}$$

where $N = n/2$ for n even and $N = \frac{n+1}{2}$ for n odd. This can be put in an alternative form, e.g. for n odd,

$$\sum_{x_1,x_2,\cdots,x_n=0,1}\ \sum_{y_1,y_2,\cdots,y_n}\prod_{l=1}^{n} y_l \cos\left(\frac{\pi}{2}\sum_{k=1}^{n} x_k\right) \cdot A(X_1,X_2,\cdots,X_n) \leqslant 2^{\frac{n-1}{2}}2^n$$

The trivial factor in the bound, 2^n, is due to the random distribution of y_k's.

Please note that these are inequalities which are applicable to a system which undergoes a series of transformations governed by external parameters X_k. They bound performance of any classical protocol attempting to calculate our example of task function, while using a one bit device. As they involve actions (operations) at different moments of time, here denoted by the subscript, one could call them temporal inequalities. However, they are based on different assumptions than the usual ones. It will be argued that the assumptions are more natural than the ones of macro-realism.

3 Assumptions listed

To derive the inequalities following general assumptions were used. Note that, they are modified with respect to [5].

- *Realism.* Theory allows to use all variables $V(t)$, the values of which are eigenvalues of an observable $\hat{V}$, which represent the values which could have been obtained, had the given observable been measured at time t, regardless what was the actual case. $V(t)$'s may be unknown, but fixed numbers (just like stochastic variables in classical probabilistic processes). That is, any function of $V(t)$ and $W(t')$ has a definite, but perhaps unknown, value (here $W(t')$ is a realistic value linked with another observable $\hat{W}$ at time t', of whatever relation with t).

- *Temporal causality.* Future does not influence past. Note that this, if *realism* holds, implies that values $V(t)$ do not depend on what happens at *later* times.
- *Freedom.* The experimenter is free to choose the observables to be measured, and operations that direct evolution of the system.
- *Finite memory.* The system's memory is assumed to be finite (in the above example it is just *one* bit). Thus the system cannot store too much information about its past (in the example: no information about past).

The last assumption *in the case of the considered example* implies the following *dependence on initial state*: the state of the system at t_{k+1} depends on its earlier state at t_k and on the operations performed on it between t_k and t_{k+1}, as well as its possible free evolution in this time interval. Such dependence-on-initial-state assumption holds generally in physics *for transistors in microchips, in classical mechanics,* etc. [6]. *In quantum mechanics the wave function depends on $\psi(t_k)$ and the Hamiltonian (operation) from t_k to t_{k+1}.* Thus this is just a reflection of the usual thinking in physics. Note that together with the assumption of realism this implies that values $V(t_{k+1})$ depend on the state of the system at t_k, and subsequent operations and evolution in the time interval between t_k and t_{k+1}.

If one compares these assumptions with ones for macro-realism, one sees that non-invasiveness is not needed anymore (it states that the realistic values at a later time do not depend on what was actually measured at an earlier time). This is replaced by something as innocent, and clearly satisfied by information processing systems: finite memory. Please note, that the sequential operations on the system are *highly invasive*: they may entirely change the state of the processing unit.

The finite memory assumption seems to be highly constraining, if we think the presented analysis concerns particles or whichever physical objects described by very many parameters, some perhaps hidden ones (such issues were addressed in Ref. [7] in the context of … noncontextuality). However the presented reasoning is applied to an algorithm involving a central processing unit of a finite memory. Even if the actual physical system, "transistor", has many other degrees of freedom, they are irrelevant to the algorithm, or even can cause errors (a too high temperature definitely does not help performing correct operations, etc.). The system must change its state depending on the current operation in the way an algorithm allows. Otherwise it is faulty.

4 The quantum protocol

To make a fair comparison with the quantum case one must limit the possible quantum processes to operations on a *single qubit*, as its memory capacity is just one bit, just like it is for our "transistor". Only in this way the classical-quantum comparison is fair, and additionally one clearly sees that the fourth assumption holds in the quantum case for the same value of memory capacity.

Surprisingly an experiment, violating the classical constraint presented above, was realized some years ago by the group of Weinfurter [8]. The experiment was a demonstration of communication complexity reduction by quantum processes [9]. However, it can be interpreted as a realization of a quantum protocol violating the temporal inequalities.

The theory behind the experiment runs as follows. One starts with single photon in a polarization state $|\psi_i\rangle = 2^{-1/2}(|H\rangle + |V\rangle)$, where H and V represent horizontal and vertical polarization. Then one acts sequentially on the flying polarization qubit with unitary phase-shift transformations of the form $|H\rangle\langle H| + \mathrm{e}^{\mathrm{i}\pi/2 X_k}|V\rangle\langle V|$, in accordance with the local inputs x_k, y_k. This is done by suitable orientable wave plates. After all n phase shifts the state is

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(|H\rangle + \mathrm{e}^{\mathrm{i}\pi/2(\sum_{k=1}^{n} X_k)}|V\rangle) \qquad (4)$$

Due to the constraint (promise) that the sum over all X_k must be even, the phase factor $\mathrm{e}^{\mathrm{i}\pi/2(\sum_{k=1}^{n} X_k)}$ is equal to the dichotomic function T_n to be computed. A measurement of the polarization qubit in the basis given by $2^{-1/2}(|H\rangle + |V\rangle)$, giving algorithm output value 0, and $2^{-1/2}(|H\rangle - |V\rangle)$, related with 1, reveals the value of T_n, with fidelity $\langle A_n T_n\rangle_{\text{avg}} = 1$. The inequality is violated *exponentially* (in terms of the number of operations n).

The actual experiment used photon pairs of the spontaneous down conversion process (for a review of such methods see Ref. [4]). The detection an idler photon at a trigger counter was heralding the other one (signal). After a suitable preparation of the start polarization, by a half-wave plate and a polarizer in the trigger arm, the signal was sent via a sequence of birefringent crystals (five of them, each imposing phase shifts in accordance to the local input X_l). Finally the polarization of the signal photon was measured, by a system consisting of a half-wave plate, a polarizing beam-splitter, and two detectors. To make a fair comparison with the classical protocol, whenever the final detection station failed to register the signal photon, a random random ± 1 answer was assumed. Still, despite the detection inefficiency problem, the quantum protocol outperformed the classical one. The classical bound was violated.

5 Conclusions

The studied example shows that one does not have to impose the doubtful non-invasiveness assumption for classical information processing (for inequalities involving sequential measurements based on non-invasiveness see e.g., Refs. [11–15]), and still it is possible to find, for a classical system of a memory of a final capacity, temporal inequalities which are violated by quantum processes involving quantum systems of the same information capacity. The example presented here is for a two-state physical realization of processing unit, however it can be generalized to more complicated situations. This will be presented elsewhere.

The answer to the opening question is: the reason for the speedup is the non-realistic nature of the quantum realm. Only the first of the four assumptions is not satisfied by the quantum system of the experiment (if one sticks to interpretations of quantum mechanics involving only its formalism).

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