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RESEARCH ARTICLE

Studying bi-partite entangled state representations via the integration over ket–bra operators in $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering

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For two particles' relative position and total momentum we have introduced the entangled state representation $|\eta\rangle$, and its conjugate state $|\xi\rangle$. In this work, for the first time, we study them via the integration over ket–bra operators in $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering, where $\mathfrak{Q}$-ordering means all Qs are to the left of all Ps and $\mathfrak{P}$-ordering means all Ps are to the left of all Qs. In this way we newly derive $\mathfrak{P}$-ordered (or $\mathfrak{Q}$-ordered) expansion formulas of the two-mode squeezing operator which can show the squeezing effect on both the two-mode coordinate and momentum eigenstates. This tells that not only the integration over ket–bra operators within normally ordered, but also within $\mathfrak{P}$-ordered (or $\mathfrak{Q}$-ordered) are feasible and useful in developing quantum mechanical representation and transformation theory.

Keywords integration over ket–bra operators, $\mathfrak{Q}$-ordering, $\mathfrak{P}$-ordering, entangled state representation

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Early in 1935, Einstein, Podolsky and Rosen (EPR) [1] in their famous paper arguing the incompleteness of quantum mechanics introduced the common eigenfunction of two particles' relative position $Q_1 - Q_2$ (with center of mass coordinate q_0) and their total momentum $P_1 + P_2$ (with eigenvalue $p_0 = 0$)

$$\psi(q_1, p_1; q_2, p_2) = \delta(q_1 - q_2 + q_0)\delta(p_1 + p_2) \tag{1}$$

which describes a sharply correlated two-particle system. In an entangled quantum state, measurement performed on one part of the system provides information on the remaining part. For example, if one measures the momentum of particle 1 and finds $p_1 = k$, then the outcome of a subsequent measurement of momentum on particle 2 is $p_2 = -k$ with certainty. Thus there is a mysterious nonlocal entanglement between separated quantum objects. The EPR's argument has stimulated many discussions on the nonlocality and entanglememt inherent in quantum mechanics. Remarkably, in Refs. [2, 3] the simultaneous eigenstate $|\eta\rangle$ of the commutative operators $(Q_1 - Q_2, P_1 + P_2)$ (where Q_i, P_i are the coordinate and momentum operator respectively) in two-mode Fock space is explicitly constructed

$$|\eta\rangle = \exp[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_2^\dagger a_1^\dagger]|00\rangle \tag{2}$$

$\eta = \eta_1 + i\eta_2$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $(a_i, a_i^\dagger)$, $i = 1, 2$, are two-mode Bose annihilation and creation operators in Fock space. The conjugate state of $|\eta\rangle$ is

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger + a_2^\dagger a_1^\dagger\right]|00\rangle$$
$$\xi = \xi_1 + i\xi_2 \tag{3}$$

since it is the common eigenvector of $(Q_1 + Q_2, P_1 - P_2)$. The entangled states representation have many applications in discussing quantum computation, quantum teleportation, quantum cryptography, and quantum superdense coding.

On the other hand, in quantum mechanics theory the basic operator ordering is coordinate-momentum operator ordering [4], the physical essence of quantum mechanics is characteristic of $[Q, P] = i\hbar$, (we set $\hbar = 1$ for later's convenience). This fundamental commutative relation is the basis of other complicated operators' ordering. In Refs. [5–7] the integration theory within $\mathfrak{Q}$-ordering and $\mathfrak{P}$-ordering is introduced, where $\mathfrak{Q}$-ordering means all Qs are to the left of all Ps and $\mathfrak{P}$-ordering means all Ps are to the left of all Qs, because the order of operators Q and P within the symbol $\mathfrak{Q}(\mathfrak{P})$ can be permuted, for instance, $\mathfrak{Q}(QP) = \mathfrak{Q}(PQ) = QP$, $\mathfrak{Q}(\mathfrak{P})$-ordered products can be integrated or differentiated with respect to c-number provided the integration is convergent, and we can derive some new fundamental operator identities about their mutual reordering. The $\mathfrak{Q}$- and $\mathfrak{P}$-ordered formulas of the Wigner operator are also deduced in Refs. [5–7], which together with the Weyl quantization rule gives a convenient method for arranging operators into either $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering via the Weyl–Wigner [8, 9] correspondence rule.

In this work, for the first time, we study bi-partite entangled state representations via the integration method within $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering of operators, as an remarkable example, we derive $\mathfrak{Q}$-ordered and $\mathfrak{P}$-ordered expansion formulas of the two-mode squeezing operator. This will demonstrate that not only the integration over ket–bra within normally ordered, but also within $\mathfrak{P}$-ordered (or $\mathfrak{Q}$-ordered) operators are feasible and useful in developing representation and transformation theory. The work is arranged as follows: in Section 2 we briefly review the properties of $|\eta\rangle$ and $|\xi\rangle$. In Section 3 we derive the δ-function form of $|\eta\rangle\langle\eta|$, $|\xi\rangle\langle\xi|$, and the $\mathfrak{Q}$-ordered and $\mathfrak{P}$-ordered form of $|\eta\rangle\langle\xi|$. In Section 4 we develop the integration over ket-bra operators in $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering for the entangled state representation, remarkablely, we newly derive $\mathfrak{P}$-ordered (or $\mathfrak{Q}$-ordered) expansion formulas of the two-mode squeezing operator.

1 Brief review of $|\eta\rangle$ and $|\xi\rangle$

From Eq. (1) we see that $|\eta\rangle$ state obeys the eigenvector equations

$$(a_1 - a_2^\dagger)|\eta\rangle = \eta|\eta\rangle, \quad (a_2 - a_1^\dagger)|\eta\rangle = -\eta^*|\eta\rangle \tag{4}$$

where $[a_i, a_j^\dagger] = \delta_{ij}$, $i,j = 1,2$, a_i, $a_i^\dagger$ are related to (Q_i, P_i) by

$$Q_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \quad P_i = \frac{1}{\sqrt{2}i}(a_i - a_i^\dagger) \tag{5}$$

It then follows from Eqs. (5) and (4) that

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \quad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle \tag{6}$$

$[(Q_1 - Q_2), (P_1 + P_2)] = 0$, $|\eta\rangle$ is qualified to make up a new quantum mechanical representation, we name it the entangled state representation with continuous variable, not only because that $|\eta\rangle$ satisfies the completeness relation

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1, \quad \mathrm{d}^2\eta \equiv \mathrm{d}\eta_1\mathrm{d}\eta_2 \tag{7}$$

but also possesses the orthonormal property

$$\langle\eta'|\eta\rangle = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*) \tag{8}$$

On the other hand, $|\xi\rangle$ obeys the eigenvector equations

$$(a_1 + a_2^\dagger)|\xi\rangle = \xi|\xi\rangle, \quad (a_1^\dagger + a_2)|\xi\rangle = \xi^*|\xi\rangle \tag{9}$$

$|\xi\rangle$ is the common eigenstate of $Q_1 + Q_2$ and $P_1 - P_2$, $[Q_1 + Q_2, P_1 - P_2] = 0$, it follows

$$(Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_1|\eta\rangle, \quad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle$$
$$\xi = \xi_1 + i\xi_2 \tag{10}$$

Its completeness relation and orthogonal property are

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1 \tag{11}$$

and

$$\langle\xi'|\xi\rangle = \pi\delta(\xi' - \xi\delta(\xi'^* - \xi^*) \tag{12}$$

The overlap is

$$\langle\eta|\xi\rangle = \frac{1}{2}\mathrm{e}^{(\xi\eta^* - \xi^*\eta)/2} \tag{13}$$

2 The $\mathfrak{P}$-ordering of $\delta\left(\xi_1 - \frac{Q_1+Q_2}{\sqrt{2}}\right) \cdot \delta\left(\eta_2 - \frac{P_1+P_2}{\sqrt{2}}\right)$

Let us prove

$$|\eta\rangle\langle\eta|\xi\rangle\langle\xi| = \pi^2\delta\left(\eta_1 - \frac{Q_1 - Q_2}{\sqrt{2}}\right)\delta\left(\eta_2 - \frac{P_1 + P_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\xi_1 - \frac{Q_1 + Q_2}{\sqrt{2}}\right)\delta\left(\xi_2 - \frac{P_1 - P_2}{\sqrt{2}}\right) \tag{14}$$

In fact, using the eigenvector Eqs. (6) and (10),

$$\begin{aligned}
&|\eta\rangle\langle\eta|\xi\rangle\langle\xi| \\
&= \int \mathrm{d}^2\eta'\delta(\eta_1 - \eta_1')\delta(\eta_2 - \eta_2')|\eta'\rangle\langle\eta'| \\
&\quad\times\int \mathrm{d}^2\xi'|\xi'\rangle\langle\xi'|\delta(\xi_1 - \xi_1')\delta(\xi_2 - \xi_2') \\
&= \pi^2\delta\left(\eta_1 - \frac{Q_1 - Q_2}{\sqrt{2}}\right)\delta\left(\eta_2 - \frac{P_1 + P_2}{\sqrt{2}}\right) \\
&\quad\times\int \frac{\mathrm{d}^2\eta'}{\pi}|\eta'\rangle\langle\eta'|\int \frac{\mathrm{d}^2\xi'}{\pi}|\xi'\rangle\langle\xi'|
\end{aligned}$$

$$\times\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right)$$
$$=\pi^2\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right) \quad (15)$$

It follows

$$|\eta\rangle\langle\eta|=\pi\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)\delta\left(\eta_1-\frac{P_1+P_2}{\sqrt{2}}\right) \quad (16)$$

and

$$|\xi\rangle\langle\xi|=\pi\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right) \quad (17)$$

so from Eq. (13) we see

$$\frac{1}{2}|\eta\rangle\langle\xi|\mathrm{e}^{(\xi\eta^*-\xi^*\eta)/2}=|\eta\rangle\langle\eta|\xi\rangle\langle\xi|$$
$$=\pi^2\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right) \quad (18)$$

or

$$\frac{1}{2}|\xi\rangle\langle\eta|\mathrm{e}^{-(\xi\eta^*-\xi^*\eta)/2}$$
$$=\pi^2\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right)\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right) \quad (19)$$

Like the fact that Bose operator a is commutable with $a^\dagger$ within the normal ordering symbol, Q is commutable with P within the $\mathfrak{Q}$-ordering symbol, i.e., though $[Q,P]=i$, we have $\mathfrak{Q}[PQ]=\mathfrak{Q}[QP]=QP$, so we can perform integration over c-number within $\mathfrak{Q}(\cdots)$, this is called the technique of interation within $\mathfrak{Q}$-ordering. For example, we convert the following operator into its $\mathfrak{Q}$-ordering form, as the first step, we have

$$\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)$$
$$=\iint_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v}{4\pi^2}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v}\mathrm{e}^{i\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)u} \quad (20)$$

Using the Baker–Hausdorff formula we put $\mathrm{e}^{i\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)u}$ on the left of $\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v}$, which yields the $\mathfrak{Q}$-ordering form, and within the $\mathfrak{Q}$-ordering symbol Q is commutable with P, so we can perform the following integration

$$\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)$$
$$=\mathfrak{Q}\left[\iint_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v}{4\pi^2}\mathrm{e}^{i\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)u}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v+iuv}\right]$$
$$=\mathfrak{Q}\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{2\pi}\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}+v\right)\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)}$$
$$=\frac{1}{2\pi}\mathfrak{Q}\left[\mathrm{e}^{-i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)}\right] \quad (21)$$

This is the $\mathfrak{Q}$-ordering of $\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)$. Similarly, The $\mathfrak{P}$-ordering form of $\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\cdot\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)$ is

$$\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)$$
$$=\iint_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v}{4\pi^2}\mathrm{e}^{i\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)u}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v}$$
$$=\mathfrak{P}\left[\iint_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v}{4\pi^2}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v}\mathrm{e}^{i\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)u-iuv}\right]$$
$$=\mathfrak{P}\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{2\pi}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)v}\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}-v\right)\right]$$
$$=\frac{1}{2\pi}\mathfrak{P}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)} \quad (22)$$

3 $\mathfrak{P}$-ordered expansion formulas of the two-mode squeezing operator

The two-mode squeezed state [10, 11], composed of an idler-mode and signal-mode generated from a parametric down-conversion, is itself an entangled state of these two modes in a frequncy domain. This enlightens us to study the following ket–bra integration in the entangled state representation, which means the squeezing transformation from $|\eta\rangle$ to $|\mu\eta\rangle$,

$$S\equiv\mu\int\frac{\mathrm{d}^2\eta}{\pi}|\mu\eta\rangle\langle\eta|=\mu\int\frac{\mathrm{d}^2\eta}{\pi}\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|\mu\eta\rangle\langle\eta|$$
$$=\mu\int\frac{\mathrm{d}^2\eta}{\pi}\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\eta|\mathrm{e}^{-(\xi\eta^*-\xi^*\eta)/2}$$
$$\times\frac{1}{2}\mathrm{e}^{(1-\mu)(\xi\eta^*-\xi^*\eta)/2} \quad (23)$$

where we used Eqs. (11) and (13). Then from Eqs. (19) and (22) we have

$$S=\mu\pi^2\int\frac{\mathrm{d}^2\eta}{\pi}\int\frac{\mathrm{d}^2\xi}{\pi}\delta\left(\xi_2-\frac{P_1-P_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)\delta\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)$$
$$\times\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)\mathrm{e}^{(1-\mu)(\xi\eta^*-\xi^*\eta)/2}$$

$$= \frac{\pi\mu}{2}\mathfrak{P}[\int \frac{\mathrm{d}^2\eta}{\pi}\int \frac{\mathrm{d}^2\xi}{\pi}\delta\left(\xi_2 - \frac{P_1-P_2}{\sqrt{2}}\right) \times \mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)} \times\delta\left(\eta_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)]\mathrm{e}^{(1-\mu)i(\xi_2\eta_1-\xi_1\eta_2)} \tag{24}$$

which is now within $\mathfrak{P}$-ordering, and all Qs are commutable with all Ps, so we can perform this integration

$$\begin{aligned} S &= \frac{\pi\mu}{2}\mathfrak{P}\left[\int\frac{\mathrm{d}\eta_2}{\pi}\int\frac{\mathrm{d}\xi_1}{\pi}\mathrm{e}^{i\left(\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\left(\xi_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)} \times\mathrm{e}^{(1-\mu)i\left(\frac{(P_1-P_2)(Q_1-Q_2)}{2}-\xi_1\eta_2\right)}\right] \\ &= \mu\mathfrak{P}\left[\int\mathrm{d}\eta_2\delta\left(\mu\eta_2-\frac{P_1+P_2}{\sqrt{2}}\right)\mathrm{e}^{-i\eta_2\frac{Q_1+Q_2}{\sqrt{2}}} \times\mathrm{e}^{i\frac{(P_1+P_2)(Q_1+Q_2)}{2}}\mathrm{e}^{(1-\mu)i\frac{(P_1-P_2)(Q_1-Q_2)}{2}}\right] \\ &= \mathfrak{P}\left(\mathrm{e}^{-i\frac{(P_1+P_2)(Q_1+Q_2)}{2\mu}}\mathrm{e}^{i\frac{(P_1+P_2)(Q_1+Q_2)}{2}} \times\mathrm{e}^{(1-\mu)i\frac{(P_1-P_2)(Q_1-Q_2)}{2}}\right) \\ &= \mathfrak{P}\mathrm{e}^{-i(1-\mu)\frac{(P_1+P_2)(Q_1+Q_2)}{2\mu}}\mathrm{e}^{(1-\mu)i\frac{(P_1-P_2)(Q_1-Q_2)}{2}} \\ &= \mathfrak{P}\{\mathrm{e}^{-i(P_1Q_1+P_2Q_2)(\cosh\lambda-1)+i(P_2Q_1+P_1Q_2)\sinh\lambda}\} \end{aligned} \tag{25}$$

where we have put $\mu=\mathrm{e}^\lambda$; or

$$S=\mathfrak{P}\left[\mathrm{e}^{-i\begin{pmatrix}P_1 & P_2\end{pmatrix}\begin{pmatrix}\cosh\lambda-1 & -\sinh\lambda\\ -\sinh\lambda & \cosh\lambda-1\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}}\right] \tag{26}$$

this is the $\mathfrak{P}$-ordered expansion formulas of the two-mode squeezing operator. Then we use the operator identity [7]

$$\mathfrak{P}\mathrm{e}^{-i\boldsymbol{P}(\mathrm{e}^{\Lambda}-1)\boldsymbol{Q}} = \mathrm{e}^{-i\boldsymbol{P}\Lambda\boldsymbol{Q}} \tag{27}$$

and

$$\ln\begin{pmatrix}\cosh\lambda & -\sinh\lambda\\ -\sinh\lambda & \cosh\lambda\end{pmatrix} = \begin{pmatrix}0 & -\lambda\\ -\lambda & 0\end{pmatrix} \tag{28}$$

to obtain the final integration result

$$\begin{aligned} S &= \mu\int\frac{\mathrm{d}^2\eta}{\pi}|\mu\eta\rangle\langle\eta| \\ &= \mathrm{e}^{-i\begin{pmatrix}P_1 & P_2\end{pmatrix}\begin{pmatrix}0 & -\lambda\\ -\lambda & 0\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}} \\ &= \exp[i\lambda(P_1Q_2+P_2Q_1)] \\ &= \exp[\lambda(a_1a_2-a_1^\dagger a_2^\dagger)] \end{aligned} \tag{29}$$

Thus we have performed the integration over the ket–bra operators within $\mathfrak{P}$-ordering in the entangled state representation and obtain for the first time the $\mathfrak{P}$-ordered squeezing operator.

4 Applications of $\mathfrak{P}$-ordered squeezing operator

The form of the $\mathfrak{P}$-ordered squeezing operator has some advantages. First, from Eq. (25) we immediately write down the result of squeezing on the two-mode coordinate eigenstate $|q_1,q_2\rangle$

$$S|q_1,q_2\rangle = \mathrm{e}^{-i(P_1q_1+P_2q_2)(\cosh\lambda-1)+i(P_2q_1+P_1q_2)\sinh\lambda}|q_1,q_2\rangle \tag{30}$$

then using

$$\mathrm{e}^{-iyP_i}|q_i\rangle = \mathrm{e}^{y\frac{\mathrm{d}}{\mathrm{d}q_i}}|q_i\rangle = |q_i+y\rangle \tag{31}$$

we have

$$\begin{aligned} S|q_1,q_2\rangle &= \mathrm{e}^{-iP_1[q_1(\cosh\lambda-1)-q_2\sinh\lambda]-iP_2[q_2(\cosh\lambda-1)-q_1\sinh\lambda]}|q_1,q_2\rangle \\ &= |q_1\cosh\lambda-q_2\sinh\lambda, q_2\cosh\lambda-q_1\sinh\lambda\rangle \end{aligned} \tag{32}$$

from which we immediately obtain the wave function of the two-mode squeezed number state

$$\begin{aligned} &\langle n_1,n_2|S|q_1,q_2\rangle \\ &= \langle n_1,n_2|q_1\cosh\lambda-q_2\sinh\lambda, q_2\cosh\lambda-q_1\sinh\lambda\rangle \\ &= \frac{1}{\sqrt{\pi 2^{n_1+n_2}n_1!n_2!}}H_{n_1}(q_1\cosh\lambda-q_2\sinh\lambda)H_{n_2}(q_2\cosh\lambda-q_1\sinh\lambda) \\ &\quad\times\exp\left[-\frac{(q_1\cosh\lambda-q_2\sinh\lambda)^2+(q_2\cosh\lambda-q_1\sinh\lambda)^2}{2}\right] \end{aligned} \tag{33}$$

where $\langle n_1, n_2|$ is the two-mode number state, so operator S entangles the two-mode coordinate eigenstates, and S itself is an entangling operator. Second, Eq. (32) directly leads to the coordinate representation of S

$$S = \iint_{-\infty}^{\infty} \mathrm{d}q_1\mathrm{d}q_2 |q_1 \cosh\lambda - q_2 \sinh\lambda, \\ q_2\cosh\lambda - q_1\sinh\lambda\rangle\langle q_1, q_2| \tag{34}$$

On the other hand, from Eq. (25) and

$$\langle p_i|\mathrm{e}^{-iyQ_i} = \mathrm{e}^{y\frac{\mathrm{d}}{\mathrm{d}p_i}}\langle p_i| = \langle p_i + y| \tag{35}$$

where $\langle p_i|$ is the momentum eigenvector, we immediately know the squeezing effect on $\langle p_1, p_2|$

$$\begin{aligned}&\langle p_1, p_2|\, S \\ &= \langle p_1, p_2|\, \mathrm{e}^{-i(p_1Q_1+p_2Q_2)(\cosh\lambda-1)+i(p_2Q_1+p_1Q_2)\sinh\lambda} \\ &= \langle p_1, p_2| \\ &\quad\times \mathrm{e}^{-i[p_1(\cosh\lambda-1)-p_2\sinh\lambda]Q_1-i[p_2(\cosh\lambda-1)-p_1\sinh\lambda]Q_2} \\ &= \langle p_1\cosh\lambda - p_2\sinh\lambda, p_2\cosh\lambda - p_1\sinh\lambda|\end{aligned} \tag{36}$$

and we obtain the momentum representation of S

$$S = \iint \mathrm{d}p_1\mathrm{d}p_2 |p_1, p_2\rangle\langle p_1\cosh\lambda - p_2\sinh\lambda, \\ p_2\cosh\lambda - p_1\sinh\lambda| \tag{37}$$

Thirdly, enlightened by Eq. (32) we can introduce the generalized two-mode squeezing operator by introducing a new parameter r

$$S(r) = \iint \mathrm{d}q_1\mathrm{d}q_2 |q_1\cosh\lambda - q_2\mathrm{e}^{-r}\sinh\lambda, \\ q_2\cosh\lambda - \mathrm{e}^{r}q_1\sinh\lambda\rangle\langle q_1, q_2| \tag{38}$$

For the two-mode squeezed vacuum state $S(r)|00\rangle$ and for the two two-mode quadratures

$$x_1 = \frac{Q_1+Q_2}{2}, \quad x_2 = \frac{P_1+P_2}{2} \tag{39}$$

we can employ Eq. (38) to conveniently derive the variance

$$\langle(\Delta x_1)^2\rangle \\ = \frac{1}{4}(\cosh^2\lambda + \sinh^2\lambda\cosh 2r - \sinh 2\lambda\cosh r) \tag{40}$$

$$\langle(\Delta x_2)^2\rangle \\ = \frac{1}{4}(\cosh^2\lambda + \sinh^2\lambda\cosh 2r + \sinh 2\lambda\cosh r) \tag{41}$$

so the squeezing can be adjusted with the variance of r, and the uncertainty relation for the two quadratures

$$\Delta x_1 \Delta x_2 = \frac{1}{4}\sqrt{1+\sinh^4\lambda\sinh^2 2r} \tag{42}$$

Thus, the $\mathfrak{P}$-ordered squeezing operator cannot only show us the behaviour of squeezing effect on both $|q_1, q_2\rangle$ and $|p_1, p_2\rangle$ manifestly, but also can directly deduce the uncertainty relation for the squeezed vacuum state.

In summary, not only the integration over ket–bra operators within normally ordered [12–14], but also within $\mathfrak{P}$-ordered (or $\mathfrak{Q}$-ordered) are feasible and useful in developing representation and transformation theory significantly.

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