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RESEARCH ARTICLE

Photon diffusion in a relativistically expanding sphere

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A relativistically expanding sphere exists in many explosive astrophysical systems, including gamma-ray bursts, neutron star mergers, and some supernovae. In this paper we investigate the photon diffusion process in a relativistically expanding sphere, which is important for understanding the energetic and radiative characters of the above mentioned explosive systems. The following contents are discussed in the frame work of special relativity: random walks of photons by scattering with electrons, photospheres, photon diffusion, and the energy flux density emerging from the surface of the expanding sphere. Some of the results are also applicable to the Universe since the Universe is also a spherical expanding system.

Keywords special relativity, explosive processes, gamma-ray bursts, supernovae, neutron star mergers, radiative transfer, scattering, cosmology

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1 Introduction

Violent explosion events are ubiquitous in astronomy, including supernovae, gamma-ray bursts (GRBs), and mergers of compact objects (white dwarfs, neutron stars, and black holes) [1–4]. According to the modern theory of cosmology, the Universe started from an extremely powerful explosion event — the Big Bang [5–7]. But of course, the Big Bang happened only once (at least in the Universe that we are living in, to our knowledge so far).

In many of the above mentioned cases, the explosion processes produce a spherical or quasi-spherical (at least to the first order approximation) matter that expands rapidly. In some extreme cases, the expansion speed of the sphere can be trans-relativistic (e.g., mergers of neutron stars) or even ultra-relativistic (e.g., GRBs). The Big Bang is also a relativistic process, since relative to a fixed position in space the cosmic expansion speed can be close to the speed of light at a very far distance [5].

In this paper we investigate the photon diffusion process in a relativistically expanding sphere. The expanding sphere can arise from GRBs [8, 9], hypernovae [10, 11, a type of supernovae with extremely large explosion energy] or mergers of neutron stars [1]. The sphere can also be a part of an expanding Milne universe [12]. The expansion can be either trans-relativistic, super-relativistic, or sub-relativistic. In other words, we take a general treatment of the expansion speed, including the non-relativistic expansion as a specific case.

Understanding of the photon diffusion process in a relativistic expanding sphere is essential for determination of the radiation properties of the expanding sphere, including the luminosity and the spectra of emissions of the above mentioned transient astrophysical systems.

In our treatment, we assume that the expanding sphere has a kinetic energy much larger than the gravitational binding energy so that the particles in the sphere move freely. This approximation is valid for an explosion process in the stage when the initial acceleration process has finished but the later deceleration process has not begun yet. This is a difference between our model and a real universe since in a real universe the gravity of matter (and radiation in a very early universe) plays an important role.

The paper is organized as follows. In Section 2, we describe the kinematics of a uniform and relativistically expanding sphere, including its similarity to an expanding Milne universe. In Section 3, we study the random walk process for photons in the sphere under the scattering interaction with free electrons. This is the basis for understanding the physics of the photon diffusion process. In Section 4, we describe the definition of a photosphere surface, which is a key concept for understanding the blackbody radiation from the sphere and the cosmic microwave background (CMB) in a universe. In Section 5, we discuss photon diffusion and the relation between the effective temperature of the emerging blackbody radiation and the temperature of the radiation at the sphere center. In Section 6, we present the formulas for calculating the energy spectral density flux for the blackbody radiation emerging from the sphere surface.

Finally, in Section 7 we summarize the results and deliver the conclusion of this research.

2 Kinematics of an expanding sphere

In this paper, we assume that a spherical matter produced by some kind of explosion events expands with a velocity proportional to the distance to the explosion center and constant in time, with mass and radiation distributed uniformly within the sphere to the first order approximation. The electromagnetic emission from the surface of the sphere contributes a small perturbation to the adiabatic expansion of the radiation inside the sphere.

The geometry of a freely expanding, homogeneous, and spherically symmetric matter sphere is like a Milne universe (see, e.g., Ref. [12])[1], except that the matter sphere has a finite volume and a surface boundary but the Milne universe has an infinite volume and no boundary in the comoving frame. In other words, the expanding sphere investigated in this paper to the first order can be considered as a part of the Milne universe.

In fact, if the expansion speed of the outer surface of the sphere approaches the speed of light, then the spatial volume in the comoving frame becomes infinite [see Eqs. (6) and (12) below] and the sphere becomes a Milne universe.

Let us denote the time in the rest frame at the center of the sphere by t, and the spherical coordinates by $\{r, \theta, \phi\}$. The center of the expanding sphere is at $r = 0$. The mass and energy density inside the sphere are assumed to be low enough to allow us ignore the effect of general relativity. Then, the spacetime metric is just the Minkowski metric

$$g_{ab} = -c^2 \mathrm{d}t_a \mathrm{d}t_b + \mathrm{d}r_a \mathrm{d}r_b + r^2 \mathrm{d}\Omega_{ab}^2 \tag{1}$$

where

$$\mathrm{d}\Omega_{ab}^2 \equiv \mathrm{d}\theta_a \mathrm{d}\theta_b + \sin^2\theta \mathrm{d}\phi_a \mathrm{d}\phi_b \tag{2}$$

is the metric on a two-dimensional sphere of a unit radius.

Define the coordinates η and ξ by

$$t = \eta\cosh\xi, \quad r = c\eta\sinh\xi \tag{3}$$

the metric can then be rewritten as the Milne metric

$$g_{ab} = -c^2 \mathrm{d}\eta_a \mathrm{d}\eta_b + c^2\eta^2(\mathrm{d}\xi_a \mathrm{d}\xi_b + \sinh^2\xi\, \mathrm{d}\Omega_{ab}^2) \tag{4}$$

A particle comoving with the expanding sphere has a constant velocity $v = \beta c$ $(0 \leqslant \beta < 1)$. The world line of the particle is a straight line defined by

$$r = c\beta t, \quad \theta = \text{constant}, \quad \phi = \text{constant} \tag{5}$$

Eq. (5) demonstrates that, at any moment of constant t, we have $v \propto r$. This is a characteristic velocity profile for a typical spherical explosion event.

As noted in the Introduction, the model defined above applies only when the explosion event is in the free-expansion stage.

By Eq. (3), along the worldline of the particle, we have

$$\xi = \operatorname{arctanh}\beta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \tag{6}$$

and

$$\eta = \gamma^{-1} t \tag{7}$$

where

$$\gamma \equiv (1-\beta^2)^{-1/2} = \cosh\xi \tag{8}$$

is the Lorentz factor of the particle.

Therefore, η represents the proper time in the rest frame of the particle, and ξ measures the spatial velocity of the particle. The spatial hypersurface Σ_η defined by η = constant is a homogeneous and isotropic 3-dimensional space with a negative curvature, which is just the surface of simultaneity of an observer comoving with the particle. By Eq. (4), on Σ_η the spatial distance measured in the radial direction is

$$l = c\eta\int_{\xi_1}^{\xi_2} \mathrm{d}\xi = c\eta(\xi_2 - \xi_1) \tag{9}$$

In Fig. 1, we show the spacetime diagram of the expanding sphere. The sphere has an outer boundary at $r = R(t) = c\beta_R t$, where the expansion velocity is $v_R = c\beta_R$, and the Lorentz factor at the surface boundary of the sphere is

$$\varGamma \equiv \gamma_R = (1-\beta_R^2)^{-1/2} \tag{10}$$

The area of a sphere of radius ξ with the center at $\xi = r = 0$ is

$$\mathcal{A} = 4\pi r^2 = 4\pi c^2\eta^2\sinh^2\xi \tag{11}$$

The total spatial volume of the sphere, defined on the spatial hypersurface Σ_η, is

[1] A Milne universe is a cosmological model free of matter, in which the expansion of the universe is balanced by the negative spatial curvature. It is a future asymptotic state of a Friedmann universe with a negative spatial curvature and a zero cosmological constant [5, 6].

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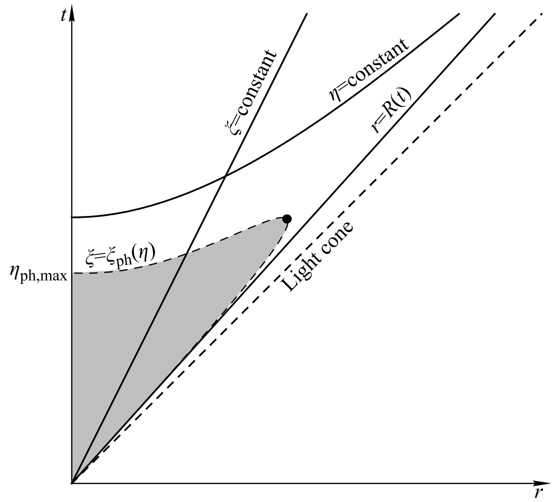

Fig. 1 The spacetime diagram of an expanding sphere is as that of a Milne universe. Each point in the diagram represents a two-sphere. The hypersurface $\xi = \xi_{\rm ph}(\eta)$ defines the photosphere 3-surface, within which (the shaded region) a photon needs to scatter $N > 1$ times with electrons before the photon reaches the surface of the sphere. The photosphere hypersurface becomes spacelike when $\eta > 0.493\eta_c$, and ends at $\eta = \eta_{\rm ph,max}$ [see Eq. (64) and the discussion following it].

$$\Sigma = 4\pi c^3\eta^3 \int_0^\xi \sinh^2 \xi {\rm d}\xi = \pi c^3\eta^3(\sinh 2\xi - 2\xi) \quad (12)$$

When $\xi \ll 1$ (i.e., $\beta \ll 1$), the Newtonian limit applies, and we have

$$\Sigma \approx \frac{4}{3}\pi c^3\eta^3\xi^3 = \frac{4\pi}{3}r^3 \quad (13)$$

Therefore we can define a factor

$$\zeta(\xi) \equiv \frac{3\cosh\xi(\sinh 2\xi - 2\xi)}{4\sinh^3\xi} \quad (14)$$

or equivalently

$$\zeta(\beta) \equiv \frac{3(\gamma^2\beta - 2{\rm artanh}\beta)}{2\gamma^2\beta^3} \quad (15)$$

then we have

$$\Sigma = \frac{4\pi}{3}\zeta c^3\eta^3 \sinh^2\xi \tanh\xi = \frac{4\pi}{3}r^3\zeta(\cosh\xi)^{-1} = \frac{4\pi\zeta}{3\gamma}r^3 \quad (16)$$

It can be checked that $1 \leqslant \zeta(\beta) \leqslant 3/2$ for any $0 \leqslant \beta \leqslant 1$, and $\zeta(0) = 1$, $\zeta(\beta \to 1) = 3/2$.

The total rest mass contained in the sphere (which is a conserved quantity) is

$$M = \rho\Sigma(\xi = \xi_R) = \pi c^3\rho\eta^3(\sinh 2\xi_R - 2\xi_R) \quad (17)$$

where $\rho = \rho(\eta)$ is the rest mass density measured in a frame comoving with the spherical fluid. From Eq. (17) we get

$$\rho(\eta) \propto \frac{1}{\eta^3} \quad (18)$$

It can be derived that

$$\rho\eta^3 = \frac{M}{2\pi c^3(\gamma_R^2\beta_R - {\rm artanh}\beta_R)} = \frac{3M}{4\pi c^3\beta_R^3\gamma_R^2\zeta(\beta_R)} \quad (19)$$

In non-relativistic limit $\beta_R \ll 1$, we get

$$\rho\eta^3 \approx \frac{3M}{4\pi c^3\beta_R^3} \quad (20)$$

While in the ultra-relativistic limit $\gamma_R \gg 1$, we get

$$\rho\eta^3 \approx \frac{M}{2\pi c^3\gamma_R^2} \quad (21)$$

3 Random walks in an expanding sphere

We write the spacetime metric in a more familiar cosmological metric form:

$${\rm d}s^2 = -c^2{\rm d}\eta^2 + a(\eta)^2({\rm d}\xi^2 + \sinh^2\xi{\rm d}\Omega^2) \quad (22)$$

where $a(\eta)$ is the cosmological scale factor, and for the Milne universe

$$a(\eta) = c\eta \quad (23)$$

Consider that a photon takes a random walk in a Milne universe under the interaction of electron scattering. The photon then has a zigzag-shaped world line, with each segment being a null geodesic. The i-th segment of the world line starts at time η_i and ends at time η_{i+1} ($i = 0, 1, \cdots, N-1$), with the proper spatial length

$$l_i = c(\eta_{i+1} - \eta_i) = \frac{1}{\kappa\rho_i}, \quad i = 0, 1, \cdots, N-1 \quad (24)$$

where $\rho_i = \rho(\eta_i)$ is the rest mass density of the medium at time η_i, and κ is the medium opacity for photons.

The length l_i is measured in a frame comoving with a fluid particle at moment η_i. To add all the movement properly, we map each spatial segment onto the spatial hypersurface Σ_0 defined by $\eta = \eta_0$, the moment when the photon starts to move (Fig. 2). On Σ_0, the length of the i-th segment is

$$\hat{l}_i = l_i\frac{a(\eta_0)}{a(\eta_i)} \quad (25)$$

Because the photon is scattered into a random direction during each step of random walk, on Σ_0 the mean square displacement traveled by the photon is

$$\hat{l}_*^2(\eta, \eta_0) = \sum_{i=0}^{N-1}\hat{l}_i^2 = \sum_{i=0}^{N-1}\left(\frac{a_0}{a_i}\right)^2 l_i^2$$

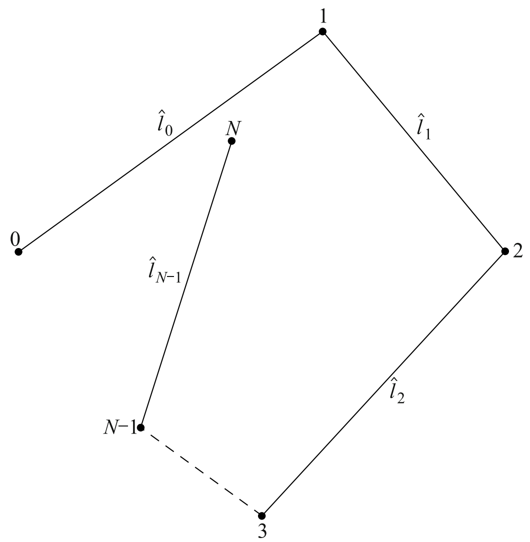

Fig. 2 Random walks of a photon projected onto the spatial hypersurface $\Sigma_0 \equiv \Sigma(\eta = \eta_0)$, starting from the moment $\eta = \eta_0$ and point 0. After a distance $\hat{l}_0$ (proper spatial length on Σ_0), the photon scatters at point 1 to a random direction along $\hat{l}_1$. After another distance $\hat{l}_1$, the photon scatters at point 2 to a random direction along $\hat{l}_2$, then scatters at point $3\cdots$. After N steps of scattering, the photon reaches point N.

$$= \sum_{i=0}^{N-1}\left(\frac{a_0}{a_i}\right)^2 l_i c\Delta\eta_i$$

$$= \sum_{i=0}^{N-1}\left(\frac{a_0}{a_i}\right)^2 \frac{c}{\kappa\rho_i}\Delta\eta_i \tag{26}$$

where $\eta \equiv \eta_N$, $a_0 \equiv a(\eta_0)$, $a_i \equiv a(\eta_i)$, and

$$\Delta\eta_i \equiv \eta_{i+1} - \eta_i \tag{27}$$

Assuming that within each cycle of time evolving of the expansion (e.g., the scale expands by a factor of 2), the photon has taken many steps of random walks (this is necessary for local thermalization to be possible in an expanding medium). Then the summation can be replaced by an integral

$$\hat{l}_*^2(\eta,\eta_0) = \int_{\eta_0}^{\eta}\left[\frac{a_0}{a(\eta)}\right]^2 \frac{c}{\kappa\rho(\eta)}\mathrm{d}\eta \tag{28}$$

Now, mapping the distance $\hat{l}_*$ onto the spatial hypersurface Σ_η defined by $\eta =$ constant at the moment when the photon arrives at a surface sphere at moment η, we get the square mean displacement of the photon measured on Σ_η

$$l_*^2(\eta,\eta_0) = \left[\frac{a(\eta)}{a_0}\right]^2 \hat{l}_*^2(\eta,\eta_0)$$

$$= a(\eta)^2\int_{\eta_0}^{\eta}\frac{c}{\kappa\rho(\eta')a(\eta')^2}\mathrm{d}\eta'$$

$$= \frac{c}{\rho(\eta)a(\eta)}\int_{\eta_0}^{\eta}\frac{a(\eta')}{\kappa(\eta')}\mathrm{d}\eta' \tag{29}$$

where we have used the fact that $\rho \propto a^{-3}$ (conservation of mass).

Now let us calculate the total number of scattering during the whole process of random walks. Differentiation of Eq. (26) with respect to the total number of scattering, N, leads to

$$\Delta\hat{l}_*^2(\eta,\eta_0) = \sum_{i=0}^{N}\hat{l}_i^2 - \sum_{i=0}^{N-1}\hat{l}_i^2 = \hat{l}_N^2\Delta N \tag{30}$$

where $\Delta N = 1$. Hence, we have

$$N = \int\frac{1}{\hat{l}_N^2}\mathrm{d}\hat{l}_*^2(\eta,\eta_0) = \int_{\eta_0}^{\eta}\frac{1}{\hat{l}_N^2}\frac{\mathrm{d}\hat{l}_*^2(\eta,\eta_0)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{31}$$

By Eq. (28),

$$\frac{\mathrm{d}\hat{l}_*^2(\eta,\eta_0)}{\mathrm{d}\eta} = \left[\frac{a_0}{a(\eta)}\right]^2\frac{c}{\kappa\rho(\eta)} \tag{32}$$

Hence, we have

$$N = \int_{\eta_0}^{\eta}\frac{1}{\hat{l}_N^2}\left[\frac{a_0}{a(\eta)}\right]^2\frac{c}{\kappa\rho(\eta)}\mathrm{d}\eta = \int_{\eta_0}^{\eta}\frac{1}{l_N^2}\frac{c}{\kappa\rho(\eta)}\mathrm{d}\eta \tag{33}$$

Since $l_N = 1/[\kappa\rho(\eta)]$, we have

$$N = \int_{\eta_0}^{\eta}\kappa(\eta')\rho(\eta')c\mathrm{d}\eta' = c\rho(\eta)a(\eta)^3\int_{\eta_0}^{\eta}\frac{\kappa(\eta')}{a(\eta')^3}\mathrm{d}\eta' \tag{34}$$

where in the last step we have again used the fact that $\rho \propto a^{-3}$.

It is quite simple to understand Eq. (34). During each time interval $\mathrm{d}\eta$, the photon has in total traveled a distance of $c\mathrm{d}\eta$ along a zigzag-shaped path, during which the total number of scatterings is simply equal to $\kappa\rho(\eta)c\mathrm{d}\eta$ since during the time interval $\mathrm{d}\eta$ the expansion of the sphere can be ignored.

The equations derived above [Eqs. (29) and (34)] apply to any Friedmann cosmological models. For example, Gunn and Peterson [13] used Eq. (34) to calculate the "optical depth" for Lyman-alpha photons emitted by quasars to constrain the density of neutral hydrogen in intergalactic space.

For the expanding sphere discussed in this paper with the Milne metric, we have $a = c\eta$, hence

$$\int_{\eta_0}^{\eta} a(\eta)\mathrm{d}\eta = \frac{c}{2}(\eta^2 - \eta_0^2) \tag{35}$$

and

$$\int_{\eta_0}^{\eta} a(\eta)^{-3}\mathrm{d}\eta = \frac{1}{2c^3\eta_0^2\eta^2}(\eta^2 - \eta_0^2) \tag{36}$$

We assume that the photon scattering process in the expanding sphere is due to the Thompson scattering with free electrons [14], then the opacity is equal to the constant Thompson opacity

$$\kappa = \kappa_{\rm es} = \text{constant} \tag{37}$$

Then we have

$$l_*^2 = \frac{c}{2\kappa_{\rm es}\rho\eta}(\eta^2 - \eta_0^2) = \frac{c\eta^2}{2\kappa_{\rm es}\rho_0\eta_0^3}(\eta^2 - \eta_0^2) \tag{38}$$

and

$$N = \frac{c\kappa_{\rm es}\rho\eta}{2\eta_0^2}(\eta^2 - \eta_0^2) = \frac{c\kappa_{\rm es}\rho_0\eta_0}{2\eta^2}(\eta^2 - \eta_0^2) \tag{39}$$

where $\rho_0 \equiv \rho(\eta_0)$.

Since $\rho\eta^3$ keeps constant as the sphere expands, we can define a critical time scale

$$\eta_c \equiv (c\kappa_{\rm es}\rho\eta^3)^{1/2} = \left[\frac{3\kappa_{\rm es}M}{4\pi c^2\beta_R^3\gamma_R^2\zeta(\beta_R)}\right]^{1/2} \tag{40}$$

Then Eqs. (38) and (39) can be written as

$$l_*^2 = \frac{c^2\eta^2}{2\eta_c^2}(\eta^2 - \eta_0^2) \tag{41}$$

and

$$N = \frac{\eta_c^2}{2\eta^2\eta_0^2}(\eta^2 - \eta_0^2) \tag{42}$$

Define a geometrically mean time $\overline{\eta}$ by

$$\overline{\eta}^2 \equiv \eta\eta_0 \tag{43}$$

Then,

$$\overline{\rho} \equiv \rho(\overline{\eta}) = \rho\left(\frac{\overline{\eta}}{\eta}\right)^{-3} = \rho_0\left(\frac{\overline{\eta}}{\eta_0}\right)^{-3} \tag{44}$$

and

$$\overline{l}_*^2 \equiv l_*^2\left(\frac{\overline{\eta}}{\eta}\right)^2 = \hat{l}_*^2\left(\frac{\overline{\eta}}{\eta_0}\right)^2 \tag{45}$$

Then we get

$$\begin{aligned}\kappa_{\rm es}^2\overline{\rho}^2\overline{l}_*^2 &= \kappa_{\rm es}^2\rho^2 l_*^2\overline{\eta}^{-4}\eta^4 = \kappa_{\rm es}^2\rho^2 l_*^2\eta^2\eta_0^{-2} \\ &= \kappa_{\rm es}^2\rho_0^2\hat{l}_*^2\eta^{-2}\eta_0^2\end{aligned} \tag{46}$$

Then, by Eqs. (38) and (39), we get

$$\kappa_{\rm es}^2\rho^2 l_*^2 = N\frac{\eta_0^2}{\eta^2}, \quad \kappa_{\rm es}^2\rho_0^2\hat{l}_*^2 = N\frac{\eta^2}{\eta_0^2} \tag{47}$$

and

$$\kappa_{\rm es}^2\overline{\rho}^2\overline{l}_*^2 = N \tag{48}$$

Assume that a photon is generated at the center of the sphere when the explosion begins ($\eta = 0$). Now let us calculate the time when it emerges from the surface of the expanding sphere. By setting $\eta_0 = 0$ in Eqs. (41) and (42), we get

$$l_*^2 = \frac{c^2\eta^4}{2\eta_c^2} \tag{49}$$

and

$$N = \frac{\eta_c^2}{2\eta_0^2} = \infty \tag{50}$$

Since l_* is equal to the radius of the sphere measured at the moment of η when the photon appears from the surface of the sphere, measured on the spatial hypersurface Σ_η, we have

$$l_* = c\eta\xi_R \tag{51}$$

where ξ_R is the ξ-value on the surface of the sphere (i.e., the world tube of the sphere surface is defined by $\xi = \xi_R$). The radius of the sphere measured in the rest frame is

$$R = c\beta_R t = \gamma_R c\beta_R\eta \tag{52}$$

Submitting Eq. (51) into Eq. (49), we get

$$\eta = \sqrt{2}\eta_c\xi_R \tag{53}$$

In non-relativistic limit, we have $\xi_R \approx \beta_R \ll 1$, $\gamma_R \approx 1$, $\zeta(\beta_R) \approx 1$, then by Eq. (40)

$$\eta_c \approx \left(\frac{3\kappa_{\rm es}M}{4\pi c^2\beta_R^3}\right)^{1/2} \tag{54}$$

By Eq. (53), the time when the photon emerges from the surface of the expanding sphere is

$$t \approx \eta \approx \sqrt{2}\eta_c\beta_R \tag{55}$$

In ultra-relativistic limit, we have $\beta_R \approx 1$, $\gamma_R \gg 1$, $\xi_R \approx \ln(2\gamma_R)$, $\zeta(\beta_R) \approx 3/2$, then by equation (40)

$$\eta_c \approx \frac{1}{\gamma_R}\left(\frac{\kappa_{\rm es}M}{2\pi c^2}\right)^{1/2} \tag{56}$$

By Eq. (53), the time when the photon emerges from the surface of the expanding sphere is

$$t \approx \eta\gamma_R \approx \sqrt{2}\eta_c\gamma_R\ln(2\gamma_R) \tag{57}$$

4 The photosphere

A photosphere is the surface where a photon emitted at some depth scatters only once before it emerges from the surface of the sphere. If a photon is inside a photosphere, it needs to take $N > 1$ random walks to reach the surface of the sphere. Hence, the photosphere is a boundary separating the opaque (with respect to a remote observer) region from the transparent region. In the 4-dimensional spacetime diagram, a photosphere is a 3-dimensional hypersurface: two space dimensions (θ and ϕ) plus one time dimension (η or t) or one spatial dimension (ξ or r), depending on whether the photosphere surface is timelike or spacelike.

Consider a photosphere defined by $\xi = \xi_{\rm ph}(\eta)$. The outer surface of the expanding sphere is at $\xi = \xi_R$. Assuming a photon is emitted at time η on the photosphere, and arrive the surface at time η_R. Then, by Eqs. (41) and (42), we have

$$l_*^2 \equiv c^2\eta_R^2(\xi_R - \xi_{\rm ph})^2 = \frac{c^2\eta_R^2}{2\eta_c^2}(\eta_R^2 - \eta^2) \tag{58}$$

and

$$N = \frac{\eta_c^2}{2\eta_R^2\eta^2}(\eta_R^2 - \eta_{\rm ph}^2) \tag{59}$$

Here l_* is the spatial distance between the point where the photon is emitted and the point when the photon reaches the surface of the sphere, measured on the 3-dimensional spatial surface Σ_{η_R}, and N is the number of random walks that the photon has taken.

According to the definition of photosphere, $N = 1$. N is the total number of scatterings encountered by a photon emitted at coordinate $(\eta, \xi_{\rm ph})$. Then Eqs. (58) and (59) can be rewritten as

$$(\xi_R - \xi_{\rm ph})^2 = \frac{1}{2\eta_c^2}(\eta_R^2 - \eta^2) \tag{60}$$

and

$$1 = \frac{\eta_c^2}{2}\frac{\eta_R^2 - \eta^2}{\eta_R^2\eta^2} \tag{61}$$

The ratio of the above two equations leads to

$$\eta_R^2 = \frac{\eta_c^4}{\eta^2}(\xi_R - \xi_{\rm ph})^2 \tag{62}$$

Submitting it to Eq. (60), we get

$$(\xi_R - \xi_{\rm ph})^2 = \frac{(\eta/\eta_c)^4}{1 - 2(\eta/\eta_c)^2} \tag{63}$$

Hence, the photosphere is determined by[2)]

$$\xi = \xi_{\rm ph}(\eta) = \xi_R - \frac{(\eta/\eta_c)^2}{[1 - 2(\eta/\eta_c)^2]^{1/2}} \tag{64}$$

When $\xi_R \to \infty$ (i.e., $\beta_R \to 1$), the photosphere equation becomes

$$1 - 2\left(\eta_{\rm ph}/\eta_c\right)^2 = 0, \quad \text{i.e., } \eta = \eta_c/\sqrt{2} \tag{65}$$

which is just the last scattering surface in a Milne universe.

The shape of the photosphere hypersurface for various speed of expansion is shown in Fig. 3 (and Fig. 1; note that, $\beta_R = \tanh\xi_R$). When $\eta_{\rm ph} \ll \eta_c$, we have $\xi_{\rm ph} \approx \xi_R - (\eta_{\rm ph}/\eta_R)^2$. The size of the photosphere shrinks to zero ($\xi_{\rm ph} = 0$) at a maximum η:

$$\eta_{\rm ph,max} = \eta_c\sqrt{-\xi_R^2 + \sqrt{\xi_R^2 + \xi_R^4}} \tag{66}$$

As $\xi_R \to \infty$ ($\beta_R \to 1$), we have $\eta_{\rm ph,max} = \eta_c/\sqrt{2}$. As $\xi_R \to 0$ ($\beta_R \ll 1$), we have $\eta_{\rm ph,max} \approx \eta_c\sqrt{\xi_R} \approx \eta_c\sqrt{\beta_R}$.

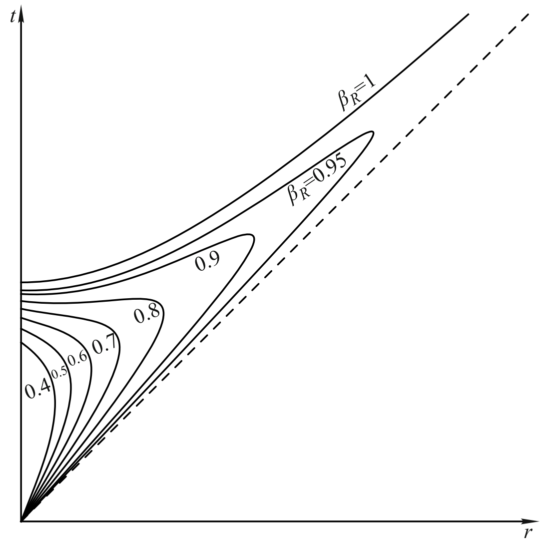

Fig. 3 The photosphere defined by Eq. (64) (each point in the figure represents a 2-sphere). Inner to outer side: $\beta_R = 0.4$, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 1 respectively (η_c = constant as β_R varies). The photosphere with $\beta_R = 1$ corresponds to the last scattering surface in a Milne universe. The diagonal dashed line corresponds to the light cone defined by $r = ct$.

For any $\xi_R > 0.339$ (i.e., $\beta_R > 0.327$), the photosphere hypersurface becomes spacelike when $\eta_{\rm ph} > 0.493\eta_c$.

In the Newtonian limit $\beta_R \ll 1$, we have $\xi_R \approx \beta_R \ll 1$ and thus $\xi_R - \xi_{\rm ph} \ll 1$, hence, by Eq. (63), on the photosphere we must have $(\eta/\eta_c)^2 \ll 1$. Then the photosphere equation becomes

$$\xi_{\rm ph} = \xi_R - \left(\frac{\eta}{\eta_c}\right)^2 \tag{67}$$

Or, in terms of the coordinates t and r,

$$r_{\rm ph} = v_R t - c\eta_c\left(\frac{t}{\eta_c}\right)^3 \tag{68}$$

where $v_R \equiv c\beta_R$ is the expansion speed of the sphere surface. The photosphere shrinks to a zero size at $\eta = t = \eta_c\sqrt{\beta_R}$.

We note that, the photosphere equation derived here only applies to the free expansion phase of the sphere.

[2)] In Ref. [9] a wrong equation for the photosphere was derived due to the wrong definition for the optical depth Eq. (26) in that paper.

In the acceleration phase a photosphere can also exist if the dimensionless entropy in the expanding sphere is large enough, like in some GRB fireball models [15]. In this acceleration regime the treatment presented here is not adequate.

5 Photon diffusion

Assuming that the expanding sphere contains uniform and isotropic blackbody radiation that adiabatically expands with the fluid. On each spatial hypersurface Σ_η, the radiation has a temperature $T(\eta) \propto \eta^{-1}$. Radiation escapes from the surface of the sphere, which contributes a small perturbation to the adiabatic evolution of the radiation.

With the diffusion approximation, which requires that the scale over which the temperatures changes is much larger than the mean-free path of a photon, in the comoving frame at the surface of the sphere the emergent flux vector is [16]

$$q^a = -\frac{c}{\overline{\kappa}\rho}\frac{4}{3}aT^3 h^{ab}(\nabla_b T + a_b T) \tag{69}$$

where $\overline{\kappa}$ is the Rosseland mean opacity, a^a is the four-acceleration vector of the fluid particles, and h^{ab} is the inverted spatial metric on Σ_η

$$h^{ab} = \frac{1}{c^2\eta^2}\left\{\left(\frac{\partial}{\partial\xi}\right)^a\left(\frac{\partial}{\partial\xi}\right)^b + \frac{1}{\sinh^2\xi}\left[\left(\frac{\partial}{\partial\theta}\right)^a\left(\frac{\partial}{\partial\theta}\right)^b + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}\right)^a\left(\frac{\partial}{\partial\phi}\right)^b\right]\right\} \tag{70}$$

For the problem discussed here, the acceleration a^a vanishes, and

$$\nabla_a T = \frac{\partial T}{\partial\eta}\mathrm{d}\eta_a + \frac{\partial T}{\partial\xi}\mathrm{d}\xi_a \tag{71}$$

Of course, relative to $\partial T/\partial\eta$, $\partial T/\partial\xi$ is only a small perturbation. Then we get

$$q^a = -\frac{c}{\overline{\kappa}\rho}\frac{4}{3}aT^3\frac{1}{c^2\eta^2}\frac{\partial T}{\partial\xi}\left(\frac{\partial}{\partial\xi}\right)^a \equiv F\frac{1}{c\eta}\left(\frac{\partial}{\partial\xi}\right)^a \tag{72}$$

where $F = |q^a|$ is the magnitude of the flux density in the frame comoving with the fluid matter. Hence

$$F = -\frac{c}{\overline{\kappa}\rho}\frac{4}{3}aT^3\frac{1}{c\eta}\frac{\partial T}{\partial\xi} \tag{73}$$

Let $4T^3\partial T/\partial\xi \sim -T^4/\xi_R$, then

$$F \sim \frac{acT^4}{3\overline{\kappa}\rho c\eta\xi_R} = \frac{acT^4}{3\tau_\eta(\xi_R)} \tag{74}$$

where the optical depth

$$\tau_\eta(\xi_R) \equiv \overline{\kappa}\rho c\eta\xi_R \tag{75}$$

On the other hand, we have

$$F = \sigma T_{\rm eff}^4 = \frac{ac}{4}T_{\rm eff}^4 \tag{76}$$

where $T_{\rm eff}$ is the effective temperature on the surface of the sphere. By Eqs. (74) and (76), we get

$$T_{\rm eff}^4(\eta) \sim \frac{T^4(\eta)}{\tau_\eta(\xi_R)} \tag{77}$$

Now let us derive the above equation with random walking.

Consider at moment η_0 an amount of thermal radiation of temperature T_0 inside a spherical shell bounded by radii ξ and $\xi + \mathrm{d}\xi$. After many scatterings, the radiation arrives at the surface $\xi = \xi_R$ at moment interval $(\eta - \mathrm{d}\eta)$ — η (photons at larger radii arrive earlier). Setting $l_* = c\eta(\xi_R - \xi_0)$, by Eq. (41) we get

$$(\xi_R - \xi)\mathrm{d}\xi = \frac{\eta}{2\eta_c^2}\mathrm{d}\eta \tag{78}$$

At moment η_0, the total energy of the radiation contained in the spherical shell as measured by a observer comoving with the fluid at ξ and η_0 is

$$\mathrm{d}E_0 = aT_0^4\mathrm{d}\Sigma(\xi,\eta_0) = aT_0^4 4\pi c^3\eta_0^3\sinh^2\xi\mathrm{d}\xi \tag{79}$$

As the radiation approaches the surface $\xi = \xi_R$ at moment η, after many scatterings, the total energy of the radiation as measured by a comoving observer at ξ_R and η is

$$\mathrm{d}E = \mathrm{d}E_0\frac{\eta_0}{\eta} = \frac{aT_0^4\eta_0^4}{\eta}4\pi c^3\sinh^2\xi\mathrm{d}\xi \tag{80}$$

This is because that the number of photons is conserved but the energy of each photon is red-shifted by a factor η_0/η.

The area of the sphere surface at time η is $\mathcal{A}_R = 4\pi c^2\eta^2\sinh^2\xi_R$. Hence, as measured in the comoving frame, the flux density of energy emerging from the sphere surface is

$$F = \frac{\mathrm{d}E}{\mathcal{A}_R\mathrm{d}\eta} = \frac{acT_0^4\eta_0^4}{\eta^3}\frac{\sinh^2\xi}{\sinh^2\xi_R}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} \tag{81}$$

By Eq. (78),

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{\eta}{2\eta_c^2}1\xi_R - \xi = \frac{1}{2\eta}\frac{1}{\tau_\eta(\xi_R,\xi)} \tag{82}$$

where the optical depth

$$\tau_\eta(\xi_R,\xi) \equiv \kappa_{\rm es}\rho c\eta(\xi_R - \xi) = \frac{\eta_c^2}{\eta^2}(\xi_R - \xi) \tag{83}$$

Hence we get

Frontiers of Physics

$$F = \frac{acT^4\eta_0^4}{2\tau_\eta(\xi_R,\xi)} \frac{\sinh^2\xi}{\sinh^2\xi_R} \tag{84}$$

where $T = T_0\eta_0/\eta$ is the temperature of the radiation at time η. Then we get

$$T_{\rm eff}^4 = \frac{4F}{ac} = \frac{T^4}{\tau_\eta(\xi_R,\xi)} \frac{2\sinh^2\xi}{\sinh^2\xi_R} = \frac{T^4}{\tau_\eta(\xi_R,0)} f_\xi \tag{85}$$

where

$$f_\xi = \frac{2\sinh^2\xi}{\sinh^2\xi_R} \frac{\xi_R}{\xi_R - \xi} \tag{86}$$

Let us define a value $\xi_{1/2}$ so that the volume inside a sphere of radius $\xi_{1/2}$ is half of the total volume of the sphere of radius ξ_R. When $\xi_R \ll 1$, we have $\xi_{1/2} = 2^{-1/3}\xi_R \approx 0.79\xi_R$ and $f_\xi \approx 6.1$. When $\gamma_R = \cosh\xi_R = 10^4$, we have $\xi_R = 9.9034$, $\xi_{1/2} = 0.965\xi_R$, and $f_\xi \approx 28.6$. (Indeed, when $\gamma_R = \cosh\xi_R \gg 1$, we have $\xi_{1/2} \approx \xi_R - \ln 2/2$ and $f_\xi \approx 2\xi_R/\ln 2$.) Hence, $T_{\rm eff}^4 \approx T^4/\tau_\eta(\xi_R,0)$ is not a bad approximation. Considering the fact that $\xi_{1/2} \approx \xi_R - \ln 2/2$ when $\gamma_R \gg 1$, perhaps a better approximation is

$$T_{\rm eff}^4 = \frac{(\xi_R/\tanh\xi_R)T^4}{\tau_\eta(\xi_R,0)} \tag{87}$$

where $\xi_R \approx \ln 2\gamma_R$ when $\gamma_R \gg 1$. (This is equivalent to setting $4T^3\partial T/\partial\xi \sim -T^4/1 = -T^4$ in Eq. (73) when $\gamma_R \gg 1$, considering the fact that in this relativistic limit half of radiation is contained in a spherical shell with radius from $\xi = \xi_R - \ln 2/2 = \xi_R - 0.3$ to $\xi = \xi_R$.)

The photosphere is determined by Eq. (64). The whole sphere is opaque (optically thick) when $\eta \ll \eta_{\rm max}$, where $\eta_{\rm max} = \eta_c/\sqrt{2}$ when $\gamma_R \gg 1$, and $\eta_{\rm max} = \eta_c\sqrt{\xi_R}$ when $\xi_R \ll 1$. Then, the opaque condition leads to

$$\frac{\eta^2}{\eta_c^2} \ll \xi_R \tag{88}$$

for any value of ξ_R. This then leads to $\tau_\eta(\xi_R) = \eta_c^2\xi_R/\eta^2 \gg 1$. Hence, when the sphere is opaque (a photon emitted from the center on Σ_η must undergo $N \gg 1$ scatterings before emerging from the surface), we must have $\tau_\eta(\xi_R) \gg 1$. But the reversed statement is not necessarily correct when ξ_R is large.

When $\eta^2/\eta_c^2 \ll 1$, the photosphere Eq. (64) becomes Eq. (67), which is equivalent to the equation $\tau_\eta(\xi_R,\xi) = 1$. [Hence, in non-relativistic case $\xi_R \approx \beta_R \ll 1$, the photosphere is just determined by the equation $\tau_\eta(\xi_R,\xi) = 1$.]

6 The energy flux density on the surface of the sphere

To simplify the expression of equations, let us define

$$\zeta_1 \equiv \frac{\xi_R}{\tanh\xi_R} = \frac{\xi_R}{\beta_R} = \frac{1}{\beta_R}[\ln\gamma_R + \ln(1+\beta_R)] \tag{89}$$

Then Eq. (87) becomes

$$T_{\rm eff}^4 = \frac{\zeta_1 T^4}{\tau_\eta(\xi_R,0)} \tag{90}$$

The quantity $\zeta_1 \to 1$ as $\beta_R \to 0$, and $\zeta_1 \to \ln\gamma_R + \ln 2$ as $\beta_R \to 1$.

With the above notations, the flux density on the surface of the sphere is

$$F = \sigma T_{\rm eff}^4 = \frac{ac}{4}\frac{\zeta_1 T^4}{\tau_\eta(\xi_R,0)} = \frac{\pi acT^4\gamma_R^2c^2\beta_R^2\eta^2\zeta}{3\kappa_{\rm es}M} \tag{91}$$

where ζ is defined in Eq. (15).

But note, the above equations hold only for $\eta \ll \eta_c$, i.e., when the photosphere has a radius almost equal to the sphere radius. When η is not small, the size of the photosphere shrinks and is smaller than the size of the sphere surface, then both $T_{\rm eff}$ and F should be evaluated on the photosphere surface. So, we modify Eq. (90) to

$$T_{\rm eff}^4 = \frac{\zeta_2 T^4}{1+\tau_{\eta,\rm ph}(\xi_R,0)} \tag{92}$$

where

$$\zeta_2 \equiv \frac{\xi_{\rm ph}}{\tanh\xi_{\rm ph}} = \frac{\xi_{\rm ph}}{\beta_{\rm ph}} = \frac{1}{\beta_{\rm ph}}[\ln\gamma_{\rm ph} + \ln(1+\beta_{\rm ph})] \tag{93}$$

where $\xi_{\rm ph}$ is defined by Eq. (64); and

$$\tau_{\eta,\rm ph}(\xi_R,0) \equiv \kappa_{\rm es}\rho c\eta\xi_{\rm ph} = \frac{\eta_c^2}{\eta^2}\xi_{\rm ph} \tag{94}$$

is the optical depth from the sphere center to the photosphere.

In Eq. (92), a unit number "1" is added to the optical depth to make $T_{\rm eff} \to T$ as the size of the photosphere shrinks to zero at $\eta = \eta_{\rm ph,max}$ (then $\xi_2 \to 1$).

Then, the energy flux density on the surface of the photosphere is

$$F = \frac{ac}{4}\frac{\zeta_2 T^4}{1+\tau_{\eta,\rm ph}(\xi_R,0)} \tag{95}$$

In the non-relativistic limit, the luminosity of the emissions from the expanding sphere is simply $L = 4\pi F r_{\rm ph}^2$, i.e. the flux density multiplied by the area of the photosphere surface. However, in the relativistic case, the calculation of the luminosity is somewhat complicated, the following effects must be taken into account:

i) Relativistic beaming [17]. In the frame comoving with the surface of photon emission (the photosphere), the forward-moving photons span a solid angle 2π; however, in the rest frame those forward-moving photons span a solid angle $2\pi(1-\beta_{\rm ph})$ ($\approx \pi/\gamma_{\rm ph}^2$ when $\gamma_{\rm ph} \gg 1$);

ii) Relativistic Doppler effect. For a photon moving toward an observer, with the angle between the direction of photon propagation and the direction of motion of the particle emitting the photon being θ (measured in the observer's frame), the observed frequency of the photon is related to the emitted frequency by $\nu_{\rm obs} = \nu_{\rm em}/\gamma_{\rm ph}(1 - \beta_{\rm ph}\cos\theta)$.

iii) Relativistic time lapse. If the particle emits two photons in the direction of the observer in a time interval $dt_{\rm em} = d\eta_{\rm ph}$ in the frame of the particle, the two photons will arrive at the observer in a time interval $dt_{\rm obs} = dt_{\rm em}\gamma_{\rm ph}(1 - \beta_{\rm ph}\cos\theta)$ in the frame of the observer (just the inverse of the relativistic Doppler effect).

Of course, if the source of emission — the expanding sphere — is at a cosmological distance, cosmological effects must also been taken into account for calculation of the luminosity.

Formulas for computing the luminosity as measured by a distant observer in the relativistic case is presented in Ref. [9] (except that the correct photosphere equation derived in this paper should be used instead of the photosphere equation derived in Ref. [9]). We will not repeat them here.

7 Summary and conclusions

In this paper we have investigated the random walks and the diffusion process of photons in a relativistically expanding sphere, with the assumption that the matter and radiation in the sphere are uniformly distributed and local thermal equilibrium holds. In our treatment, we have considered only the interaction of photons with free electrons through the Thompson scattering, which should be valid when the plasma in the sphere is hot.

It has also been assumed that the internal energy of the radiation is not large enough to affect the dynamics of the sphere and the gravitational potential energy of the sphere is negligible compared to the large kinetic energy, so that the sphere expands freely, and the blackbody radiation inside the sphere expands adiabatically with the emerging energy flux of radiation at the surface of the sphere behaving as a small perturbation to the evolution of the radiation.

We have derived formulas for the total number of scatterings undertaken by a photon when it moves from one place to another inside the sphere, the mean square displacement traveled by a photon for a given number of scatterings, the equation of the photosphere, the relation between the effective temperature on the photosphere and the temperature of the blackbody inside the photosphere, and the energy flux density on the surface of the sphere (or on the photosphere).

These formulas and equations will find their applications in relativistic explosion processes in astrophysics, like GRBs, hypernovae, mergers of compact objects, and the Big Bang universe.

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References

1. L. X. Li and B. Paczyński, Transient events from neutron star mergers, *Astrophys. J.*, 1998, 507(1): L59
2. P. Mészáros, Gamma-ray bursts, *Rep. Prog. Phys.*, 2006, 69(8): 2259
3. S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars, Weinheim: Wiley-VCH, 2004
4. B. Zhang and P. Mészáros, Gamma-ray bursts: Progress, problems & prospects, *Int. J. Mod. Phys. A*, 2004, 19: 2385
5. S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge: Cambridge University Press, 1975
6. R. M. Wald, General Relativity, Chicago: University of Chicago Press, 1984
7. S. Weinberg, Gravitation and Cosmology, New York: John Wiley & Sons, 1972
8. J. Goodman, Are gamma-ray bursts optically thick? *Astrophys. J.*, 1986, 308: L47
9. L. X. Li, Gamma-ray burst precursors as the remnant of the thermal radiation initially trapped in the fireball, *Mon. Not. R. Astron. Soc.*, 2007, 380(2): 621
10. P. A. Mazzali, J. Deng, E. Pian, D. Malesani, N. Tominaga, K. Maeda, K. Nomoto, G. Chincarini, S. Covino, M. Della Valle, D. Fugazza, G. Tagliaferri, and A. Gal-Yam, Models for the Type Ic Hypernova SN 2003lw associated with GRB 031203, *Astrophys. J.*, 2006, 645(2): 1323
11. K. Nomoto, K. Iwamoto, P. A. Mazzali, H. Umeda, et al., A hypernova model for the supernova associated with the γ-ray burst of 25 April 1998, *Nature*, 1998, 395(6703): 672
12. W. Rindler, Essential Relativity: Special, General, and Cosmological, New York: Springer-Verlag, 1977
13. J. E. Gunn and B. A. Peterson, On the density of neutral hydrogen in intergalactic space, *Astrophys. J.*, 1965, 142: 1633
14. B. W. Carroll and D. A. Ostlie, An Introduction to Modern Astrophysics, New York: Addison-Wesley, 1996
15. P. Mészáros and M. J. Rees, Steep slopes and preferred breaks in gamma-ray burst spectra: The role of photospheres and comptonization, *Astrophys. J.*, 2000, 530(1): 292
16. I. D. Novikov and K. S. Thorne, in: Black Holes, edited by C. DeWitt and B. DeWitt, Paris: Gordoan and Breach, 1973
17. M. J. Rees, Appearance of relativistically expanding radio sources, *Nature*, 1966, 211(5048): 468