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RESEARCH ARTICLE

Dynamics of Bose–Einstein condensates in a one-dimensional optical lattice with double-well potential

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We study dynamical behaviors of the weakly interacting Bose–Einstein condensate in the one-dimensional optical lattice with an overall double-well potential by solving the time-dependent Gross–Pitaevskii equation. It is observed that the double-well potential dominates the dynamics of such a system even if the lattice depth is several times larger than the height of the double-well potential. This result suggests that the condensate flows without resistance in the periodic lattice just like the case of a single particle moving in periodic potentials. Nevertheless, the effective mass of atoms is increased, which can be experimentally verified since it is connected to the Josephson oscillation frequency. Moreover, the periodic lattice enhances the nonlinearity of the double-well condensate, making the condensate more "self-trapped" in the π-mode self-trapping regime.

Keywords Bose–Einstein condensate, double-well potential, optical lattice, dynamical behavior

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1 Introduction

The experimental realization of confining cold atoms in optical lattices (OL) provides great opportunities to understand quantum nature of matter waves inside periodic potentials which is one of the typical topics in condensed matter physics [1]. As is well-known in solid-state physics, the perfect periodic potential does not impede the movement of electrons, but gives rise to Bloch band structures. Consequently, dynamics of Bloch electrons greatly differ from free electrons in aspects such as effective mass, Bloch oscillation and Landau–Zener tunneling [2]. The motion of atoms in OL somehow resembles that of electrons in solids. Band theory for cold atoms in periodic potentials has been developed for a long time [3, 4]. The Bloch oscillation and Landau–Zener tunneling were observed in ultracold atoms before the achievement of condensate [5–7], confirming the band structure of atoms in periodic potentials. The experiment with Bose–Einstein condensates (BECs) in OL was first carried out by Anderson and Kasevich [8] and it stimulated considerable interests in this regard [9–11].

In most experiments, BECs are first prepared in a magnetic trap described as a harmonic potential. A standing wave is then created using laser beam. In order to observe the Bloch oscillation or Landau–Zener tunneling, the overall harmonic potential needs to be switched off [8, 9] since it destroys translational invariance of the lattice and alters band structures as a consequence. On the other hand, the overall harmonic potential has some advantages in its own right. For a condensate in the harmonic potential, it exhibits oscillation if its initial position deviates from the bottom of the trap. It is observed that the oscillation behaviors can be maintained in the presence of the OL, but the oscillation frequency decreases, corresponding to an increase in effective mass [10, 11]. This method can also be used to identify different dynamical regimes by varying the initial displacement of the BEC from the trap bottom.

In this paper we study dynamics of BECs inside a double-well (DW) potential in the presence of a one-dimensional (1D) optical lattice. An experimental situation may be realized by first loading BEC into a double-well and then switching on the OL suddenly. A question arises: how does the periodic lattice affect dynamics of the DW condensate?

Before exploring the answer to this question, we briefly summarize some already well-established features of dynamics of the DW condensate in absence of the OL. First of all, it exhibits the well-known dc, ac Josephson effects [12–18] and the Shapiro effect [18], analogous to a superconducting Josephson junction. Secondly, it displays some new fascinating phenomena different from the superconducting Josephson junction, such as the π-phase oscillations and macroscopic quantum self-trapping (MQST) [17–19]. Moreover, a number of novel phenomena arising from quantum fluctuations are predicted in theory, e.g., collapses and revivals of quantum oscillation [13], coherence and decoherence [20], and the breaking of MQST state [21]. After the first double-well experiment, which proved the coherence of BEC by interference patterns [22], Albiez *et al.* achieved a single DW condensate and conducted a direct observation of the Josephson oscillation and MQST [23]. The ac and dc Josephson effects were also observed [24]. So far, the static, thermal and dynamical properties of the DW condensate have been systematically investigated in experiments [25].

The paper is organized as follows. The theoretical model is presented in Section 2. We employ the time-dependent Gross–Pitaevskii equation (GPE) to describe the double-well condensate in the optical lattice. Section 3 gives a discussion of the numerical results of the GPE. A brief summary is given in the last section.

2 The model

Our study is based on the time-dependent Gross–Pitaevskii equation. Numerical studies of the GPE have been carried out to visualize dynamics of BECs in a one-dimensional optical lattice with harmonic potential and to account for the main features of experimental observations [10, 11].

The time-dependent GPE for BECs in a 1D optical lattice with a DW potential is given by

$$\mathrm{i}\hbar\frac{\partial\psi(x;t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V(x)+gN|\psi(x;t)|^2\right]\psi(x;t) \tag{1}$$

where $\psi(x;t)$ is the macroscopic wave function at position x and time t, g represents the short-range interaction. The potential has the form $V(x)=m\omega^2x^2/2+V_\mathrm{b}\exp(-x^2/p^2)+V_{ol}\cos^2(2\pi x/\lambda_L)$, where the first two terms describe the DW potential with ω being the trap frequency [23] and V_b denoting the barrier height, and the third term describes the OL which is supposed to be created by a laser beam. V_{ol} describes the lattice depth and $\lambda_L/2$ denotes the lattice spacing with λ_L being the wave length of the laser beam. If the width of the double-well is set to be $2x_0$, the number of lattice sites that each well contains is $k=2x_0/\lambda_L$. Benefiting from the characteristic length $l=\sqrt{\hbar/(m\omega)}$ and the characteristic energy $\hbar\omega$ correspondingly, we have $x'=x/l$, $\tau=t\omega/2$, then the 1D GPE can be reduced to a dimensionless one,

$$\mathrm{i}\frac{\partial\bar\psi(x';t)}{\partial\tau}=\left[-\frac{\partial^2}{\partial x'^2}+v(x')+\frac{g'}{\pi}|\bar\psi(x';\tau)|^2\right]\bar\psi(x';\tau) \tag{2}$$

where $v(x')=x'^2+2v_\mathrm{b}\exp(-x'^2/p'^2)+2v_{ol}\cdot\cos^2(k\pi x'/x'_0)$, with $v_{ol}=V_{ol}/(\hbar\omega)$, $v_b=V_b/(\hbar\omega)$, $p'=p/l$ and $x'_0=x_0/l$. The dimensionless interaction parameter $g'=gNm/(l\hbar^2)$ where N is the number of atoms.

The wave function must satisfy the condition $\int_{-\infty}^{\infty}\mathrm{d}x'|\bar\psi(x';\tau)|^2=1$. $n_L(\tau)=\int_{-\infty}^{0}\mathrm{d}x'|\bar\psi(x';\tau)|^2$ is the fraction of the number of atoms in the left well and $n_R(\tau)=\int_0^{\infty}\mathrm{d}x'|\bar\psi(x';\tau)|^2$ in the right well [26]. $\theta_L(\tau)=\arctan\frac{\int_{-\infty}^{0}\mathrm{d}x'\mathrm{Im}[\bar\psi(x';\tau)]\rho(x';\tau)}{\int_{-\infty}^{0}\mathrm{d}x'\mathrm{Re}[\bar\psi(x';\tau)]\rho(x';\tau)}$ and $\theta_R(\tau)=\arctan\frac{\int_0^{\infty}\mathrm{d}x'\mathrm{Im}[\bar\psi(x';\tau)]\rho(x';\tau)}{\int_0^{\infty}\mathrm{d}x'\mathrm{Re}[\bar\psi(x';\tau)]\rho(x';\tau)}$ are the phases in the left well and the right well, respectively, with the density $\rho(x';\tau)=\bar\psi^*(x';\tau)\bar\psi(x';\tau)$.

For the condensates in the double-well, $\phi_+(x')$ and $\phi_-(x')$ represent the ground state and the first excited state wave functions. Their linear combinations are defined as the left (right) well mode: $\psi_{L,R}(x')=\frac{\phi_+(x')\pm\phi_-(x')}{2}$. They satisfy the orthogonal condition $\int\mathrm{d}x'\psi_L(x')\psi_R(x')=0$. The trial wave function for obtaining the initial state can be chosen as the superposition of $\psi_L(x')$ and $\psi_R(x')$ as in Ref. [26] $\bar\psi(x';\tau)=\psi_L(\tau)\phi_L(x')+\psi_R(\tau)\phi_R(x')$, where $\psi_{L(R)}(\tau)=\sqrt{n_{L(R)}(\tau)}\mathrm{e}^{\mathrm{i}\theta_{L(R)}(\tau)}$. At time τ, the population imbalance and relative phase can be defined as $\Delta n=n_L-n_R$ and $\Delta\theta=\theta_L-\theta_R$, separately. A destructive technique to measure $\Delta\theta$ is to release the BECs from the double-well potential after different evolution times to gain the interfere patterns [23]. Moreover, based on stimulated light scattering, a newly developed method can be used to detect $\Delta\theta$ nondestructively [27]. An initially given trial wave function at $\tau=0$ is constructed as $\bar\psi(x';0)=\mathrm{e}^{i\Delta\theta(0)}\sqrt{n_L(0)}\psi_L(x')+\sqrt{n_R(0)}\psi_R(x')$. $\Delta n(0)=n_L(0)-n_R(0)$ is the initial population imbalance and $\Delta\theta(0)$ the phase difference of condensates in the double-well.

The time-dependent GPE is numerically solved using a commonly accepted Split-Step Crank-Nicolson discretization scheme.

3 Results and discussion

In our calculation, the height of the barrier of the double-well is fixed, $v_b = 5$, while the lattice depth, v_{ol}, is tunable. One can imagine that the double-well feature predominates the dynamics when v_{ol} is small. The phase space diagram has been systematically calculated for the DW condensate in the absence of the OL [28]. At the parameters $p' = 1$ and $g' = 0.1$, three typical dynamical regimes can be present in the phase diagram, including Josephson oscillation, π-mode MQST and running-phase MQST. In the following, emphasis is placed on the temporal evolutions of two typical states; one in a Josephson oscillation orbit around the point ($\Delta n = 0$, $\Delta\theta = 0$) and one in a π-mode MQST orbit around a point ($\Delta n > 0$, $\Delta\theta = \pi$).

Figures 1 and 2 illustrate the density and the phase distributions for the Josephson oscillation case. The condensate is oscillating between the two wells in the absence of the OL, as shown clearly in Fig. 1(a). As the optical lattice is switched on, the density peak is split into several narrow peaks, each of which resides in one lattice site. The narrow peaks become more and more protrudent with increasing the lattice depth, as shown in Fig. 1(b), (c) and (d). Here we choose $k = 7$ so there are 7 sites in each well [29]. Four peaks are clearly observed in each well but the peaks near the edge of the double-well are not high enough to be seen. Supplementary, the lattice feature is quite obvious in the phase distribution near the double-well edge. Without optical lattice, the phase at a given time is almost constant in each well. In the presence of the lattice, the phase varies slowly with position at each site, but shows abrupt change between the neighboring sites, as shown in Fig. 2(b), (c) and (d).

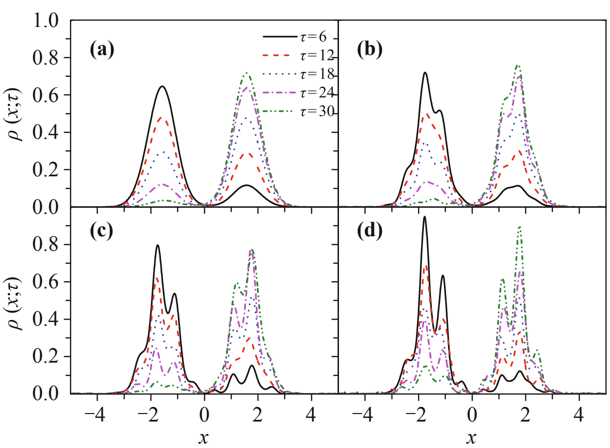

Fig. 1 Snapshots of particle density distributions in the double-well at different given times. The condensate is initially in the Josephson oscillation state ($\Delta n = 0.9$, $\Delta\theta = 0$). The height of the double-well barrier is fixed $v_b = 5$, but the lattice depth is tuned to be $v_{ol} = 0$ **(a)**, $v_{ol} = 5$ **(b)**, $v_{ol} = 10$ **(c)**, $v_{ol} = 15$ **(d)**.

Figures 3 and 4 plot the density and the phase distributions for the π-mode MQST case. Most atoms are trapped in the left well where the density peaks are

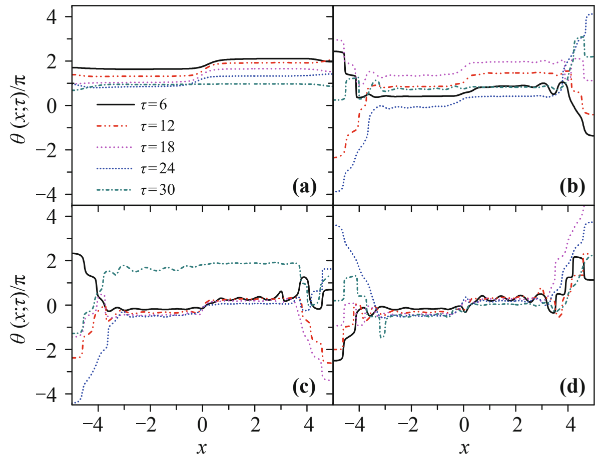

Fig. 2 Snapshots of phase distributions in the double-well at different given times. The initial state is the same as in Fig. 1. **(a)** $v_b = 5$ and $v_{ol} = 0$, **(b)** $v_{ol} = 5$, **(c)** $v_{ol} = 10$, **(d)** $v_{ol} = 15$.

clearly visible. On the other hand, the variation in the phase is more pronounced in the right well. The larger the lattice depth, the more lumpy the phase gets.

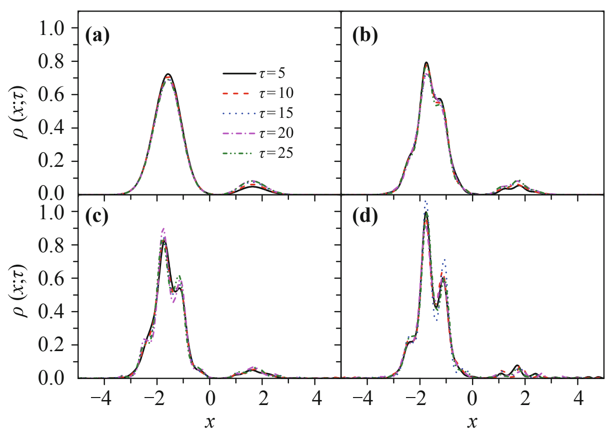

Fig. 3 Snapshots of particle density distributions in the double-well at different given times. The condensate evolves initially from a π-mode MQST state ($\Delta n = 0.9$, $\Delta\theta = \pi$). **(a)** $v_b = 5$ and $v_{ol} = 0$, **(b)** $v_{ol} = 5$, **(c)** $v_{ol} = 10$, **(d)** $v_{ol} = 15$.

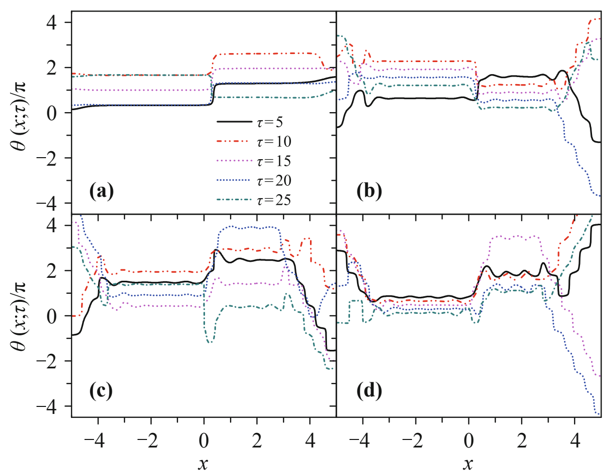

Fig. 4 Snapshots of phase density distributions in the double-well at different given times. The initial state is the same as in Fig. 3. **(a)** $v_b = 5$ and $v_{ol} = 0$, **(b)** $v_{ol} = 5$, **(c)** $v_{ol} = 10$, **(d)** $v_{ol} = 15$.

Although the optical lattice brings about remarkable changes in density and phase distributions, the essential characters of the double-well dynamics are not significantly affected. This phenomenon is very surprising, because, for example, in the $v_{ol} = 10$ and 15 cases, the DW potential is very weak comparing to the deep OL. To illustrate this point further, we calculate temporal evolutions of the population imbalance $\Delta n(\tau)$ and the relative phase $\Delta\theta(\tau)$, as shown in Fig. 5. For the Josephson oscillation, amplitudes of both the population imbalance and the relative phase are just changed slightly, but their periods have apparently been prolonged. For a pure double-well occasion, the Josephson oscillation frequency $\omega \sim K$ where K is the tunneling constant [17]. It is approximately inversely proportion to the mass, m. Therefore, the decrease of the oscillation frequency corresponds to the increase of the effective mass of atoms. This result is consistent with that obtained by studying oscillation of a BEC in the hamornic trap with the OL [10, 11].

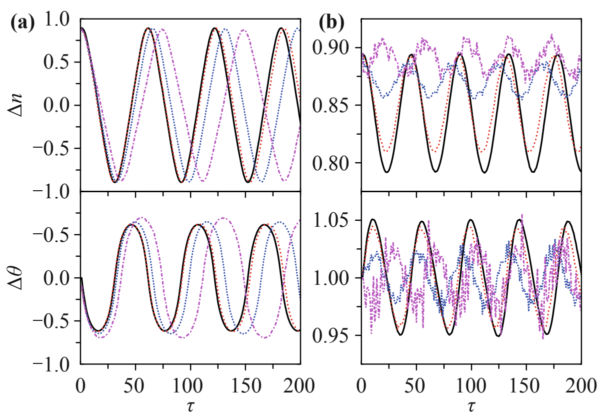

Fig. 5 Evolution of $\Delta\theta$ and Δn versus time starting from **(a)** a Josephson oscillation state ($\Delta n = 0.9$, $\Delta\theta = 0$) and **(b)** a π-mode MQST state ($\Delta n = 0.9$, $\Delta\theta = \pi$). The lattice depth is $v_{ol} = 0$ (*solid lines*), $v_{ol} = 5$ (*short-dashed lines*), $v_{ol} = 10$ (*short-dotted lines*) and $v_{ol} = 15$ (*short-dash-dotted lines*).

The influence of OL on the π-mode MQST seems more notable than that on the Josephson oscillation. As shown in Fig. 5(b), the evolution loses its perfect periodicity quickly with increased lattice depth. Both Δn–τ and $\Delta\theta$–τ lines become jagged at $v_{ol} = 10$ and 15 and the oscillation amplitudes diminish significantly. In spite of this, the self-trapping feature remains.

At last, we attempt to demonstrate the two typical dynamical processes in the phase space diagram. These can be obtained by combining Δn–τ and $\Delta\theta$–τ relations. Figure 6(a) shows the phase diagram for an initial state of Josephson oscillation (Δn=0.9, $\Delta\theta$=0), which lies in a closed orbit circling the lowest energy point ($\Delta n = 0, \Delta\theta = 0$). With the OL imposed, the orbit circle becomes slightly enlarged, meantime, the orbit line develops into a narrow belt. If the lattice depth increases further, the orbit will become fuzzy and the double-well picture will finally break down. One can expect that the system may undergo a transition from the superfluid state into a Mott-insulator region [30].

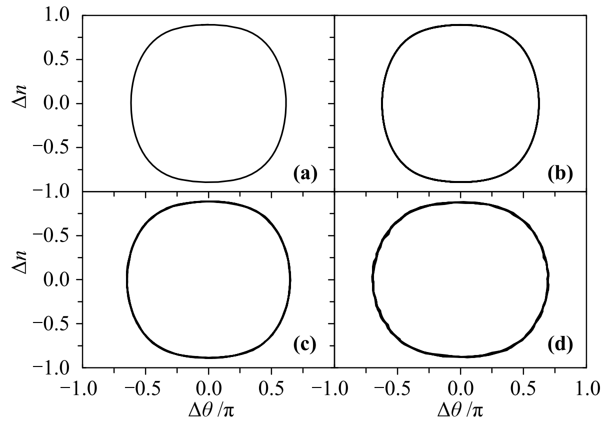

Fig. 6 The $\Delta\theta$–Δn phase space diagram of a system initially in the Josephson oscillation state ($\Delta n = 0.9$, $\Delta\theta = 0$) with the frequency of optical lattice $k = 7$ fixed while optical lattice amplifications are different. **(a)** $v_{ol} = 0$, **(b)** $v_{ol} = 5$, **(c)** $v_{ol} = 10$, **(d)** $v_{ol} = 15$.

For the π-mode MQST case, the orbit circle apparently shrinks when the lattice switches on, contrary to the Josephson oscillation case, as shown in Fig. 7. Meanwhile the orbit-path becomes obscured dramatically. The "orbit" has already lost its definition at $v_{ol} = 10$. However, the lattice compels the condensate more "self-trapped", and the average value of Δn tends to grow. MQST is a nonlinear effect from the interaction between atoms. With the interaction strength growing, the MQST regime shrinks and moves to larger Δn regions in the phase space diagram [17, 28]. In this sense, increasing the lattice depth yields similar results as increasing the interaction strength. According to the tow-mode model

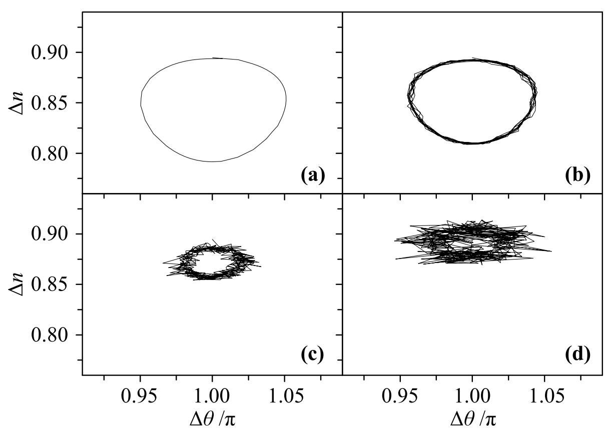

Fig. 7 The $\Delta\theta$–Δn phase space diagram of a system initially in the π-mode MQST state ($\Delta n = 0.9$, $\Delta\theta = \pi$) with the frequency of optical lattice $k = 7$ fixed while optical lattice amplifications are different. **(a)** $v_{ol} = 0$, **(b)** $v_{ol} = 5$, **(c)** $v_{ol} = 10$, **(d)** $v_{ol} = 15$.

[17], the interaction strength $\Lambda = UN/(2K)$ grows with the effective mass since K decreases with it.

4 Conclusion

In conclusion, we investigate the transport of BECs in a periodic optical lattice with an overall double-well potential. We find that the dynamics of the condensate is mainly governed by the double-well potential even if the lattice depth is several times larger than the height of the double-well barrier. For all that, the periodic lattice brings about nontrivial influence. For the Josephson oscillation state which is in the linear dynamics regime of double-well condensates, the periodic lattice enhances the effective atom mass which is related to the oscillation frequency and thus the effect can be detected in experiments. For the π-mode MQST state located in the nonlinear regimes, the lattice effect is quite pronounced. It results in the loss of the perfect periodicity of the π-mode oscillation and obscures the orbit path in the phase space diagram. But, still, the "self-trapped" feature is strengthened.

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