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RESEARCH ARTICLE

The entanglement of several graph states

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We exactly evaluate the entanglement of a six vertex and a nine vertex graph states which correspond to non "two-colorable" graphs. The upper bound of entanglement for five vertex ring graph state is improved to 2.9275, less than the upper bound determined by local operations and classical communication. An upper bound of entanglement is proposed based on the definition of graph state.

Keywords graph state, closest separable state, multipartite entanglement

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1 Introduction

Entanglement is one of the most important concepts and resources in quantum information theory. However, the quantification of the entanglement of a given quantum state is difficult except for bipartite pure state, where the distance-like measure of the entanglement (Relative Entropy of Entanglement) [1, 2] and the operational measures of entanglement (Entanglement of Formation, Distillable Entanglement) [3] are all equal to the entropy of the reduced state obtained by tracing out one part of the pure state. In a bipartite system, apart from pure state, the entanglement of a mixed state is not easy to calculate in general if not impossible. The situation becomes even worse for a multipartite system where the basic states corresponding to Bell basis are not clearly recognized [4, 5]. Thus the extension of operational entanglement measures to a multipartite system is not available. Nevertheless, a variety of different entanglement measures have been proposed for the multipartite setting. Among them are the (Global) Robustness of Entanglement [6], the Relative Entropy of Entanglement and the Geometric Measure [7]. The robustness measures the minimal noise (arbitrary state) that needs to be added to make the state separable. The geometric measure is the distance of the state to the closest product state in terms of the fidelity. The relative entropy of entanglement is the relative entropy of the state to the closest fully separable state. It is a valid entanglement measure for multipartite state.

The quantification of multipartite entanglement is usually difficult as most measures are defined as the solutions to difficult variational problems. Even for pure multipartite state, entanglement can be obtained only for some special scenarios. Fortunately, due to the inequalities on the logarithmic robustness, relative entropy of entanglement and geometric measure of entanglement [8–10], these entanglement measures are all equal for stabilizer states [11]. Thus for a stabilizer state $|S\rangle$, the entanglement can be written as

$$E = \min_{\phi}(-\log_2 |\langle S|\phi\rangle|^2) \tag{1}$$

where $\phi = \bigotimes_j (\sqrt{p_j}|0\rangle + \sqrt{1-p_j}\mathrm{e}^{\mathrm{i}\varphi_j}|1\rangle)$ is the separable pure state.

The entanglement is upper bounded by the local operations and classical communication (LOCC) bound E_{LOCC}, and lower bounded by some bipartite entanglement deduced from the state, that is, the "matching" bound E_{bi} [12]. It is well known that all graph states are stabilizer states [13], so the inequalities for the entanglement of a graph state are

$$E_{bi} \leqslant E \leqslant E_{\mathrm{LOCC}} \tag{2}$$

If the lower bound coincides with the upper bound, then the entanglement of the graph state can be obtained directly. This is true for "two-colorable" graph states such as multipartite GHZ states, Steane codewords, and states of ring graph with even vertices. For a state of ring graph with odd n vertices, we have $\lfloor \frac{n}{2} \rfloor \leqslant E \leqslant \lceil \frac{n}{2} \rceil$ [12]. Recently, progress has been made on the study of seperability of graph diagonal states [14, 15]. Graph state has also been applied to the question of quantum capacity [16], which is closely related to the entanglement problem.

In this paper we will be concerned with the entanglement of a graph state whose graph is not "two-colorable". A new upper bound based directly on the definition of graph state is proposed. The symmetry of the graph is utilized to further reduce the upper bounds for some highly symmetric graph states, including those of five vertex ring and Peterson graph.

2 Graph state

A graph $G = (V;\Gamma)$ is composed of a set V of n vertices and a set of edges specified by the adjacency matrix Γ, which is an $n \times n$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if vertices a, b are connected and $\Gamma_{ab} = 0$ otherwise. The neighborhood of a vertex a is denoted by $N_a = \{v \in V | \Gamma_{av} = 1\}$, i.e., the set of all the vertices that are connected to a. Graph states [17, 18] are useful multipartite entangled states that are essential resources for one-way computing [19] and can be experimentally demonstrated [20, 21]. To associate the graph state to the underlying graph, we assign each vertex with a qubit, each edge representing the interaction between the two corresponding qubits. More physically, the interaction may be Ising interaction of spin qubits. Let us denote the Pauli matrices at the qubit a by X_a, Y_a, Z_a and the identity by I_a. The graph state related to the graph G is defined as

$$|G\rangle = \prod_{\Gamma_{ab}=1} U_{ab}|+\rangle^V = \frac{1}{\sqrt{2^n}} \sum_{\mu=\mathbf{0}}^{\mathbf{1}} (-1)^{\frac{1}{2}\mu\Gamma\mu^{\mathrm{T}}} |\mu\rangle \tag{3}$$

where $|\mu\rangle$ is the joint eigenstate of Pauli operators Z_a $(a \in V)$ with eigenvalues $(-1)^{\mu_a}$, $|+\rangle^V$ being the joint $+1$ eigenstate of Pauli operators X_a $(a \in V)$, and U_{ab} ($U_{ab} = \mathrm{diag}\{1, 1, 1, -1\}$ in the Z basis) being the controlled phase gate between qubits a and b. Graph state can also be viewed as the result of successively performing 2-qubit Control-Z operations U_{ab} on the initially unconnected n qubit state $|+\rangle^V$. It can be shown that graph state is the joint $+1$ eigenstate of the n vertices stabilizers

$$K_a = X_a \prod_{b\in N_a} Z_b := X_a Z_{N_a}, \quad a \in V \tag{4}$$

Meanwhile, the graph state basis are $|G_{k_1,k_2,\cdots,k_n}\rangle = \prod_{a\in V} Z_a^{k_a}|G\rangle$, with $k_a = 0, 1$. Since all of the graph basis states are local unitary equivalent, they all have equal entanglement, so we need only to determine the entanglement of graph state $|G\rangle$. Once the entanglement of a graph state is obtained, the entanglement of all the graph basis states are obtained.

3 The upper bound of graph state

The (squared) fidelity $F_\phi = |\langle G|\phi\rangle|^2$ plays a crucial rule in calculating the entanglement. For a graph state, we have

$$E = \min_\phi -\log_2 |\langle G|\phi\rangle|^2 = -\log_2(\max_\phi F_\phi) \tag{5}$$

Denote $F = \max_\phi F_\phi$ as the fidelity between the graph state and the closest pure separable state. One of the ways to obtain the upper bound of entanglement is to relax the maximization. For two-colorable graph, the set with majority vertices is colored with Amber, and the set with minority vertices is colored with Blue. Without loss of generality, the Amber colored vertices are labelled as $a = 1, \cdots, |A|$, and the Blue vertices are labelled as $b = |A| + 1, \cdots, n$. Since all Amber vertices are not adjacent with each other, we can perform X_a $(a = 1, \cdots, |A|)$ measurements to all Amber qubits simultaneously. Applying Z_b $(b = |A| + 1, \cdots, n)$ measurements to all Blue qubits at the same time. Thus all Amber stabilizers K_a can be measured simultaneously by LOCC. The maximal number of states that can be discriminated by LOCC then is $2^{|A|}$ according to the theory of graph state basis [12]. Applying the inequality on the relationship of LOCC discrimination of states and the entanglement [9], one has $|A| \leqslant n - E$, that is,

$$E \leqslant n - |A| \tag{6}$$

This upper bound of LOCC may be extended to graphs that are not two-colorable by some modification. However, it is possible to obtain the upper bound without the LOCC state discrimination.

We will obtain the upper bound of the entanglement with the definition of the graph state. The graph state may not be two-colorable. Suppose the maximal non-adjacent vertices set A has $|A|$ vertices. As before, we label these vertices with $a = 1, \cdots, |A|$. The other vertices are in the set $B = V - A$, and the vertices are labelled with $b = |A| + 1, \cdots, n$. Note that the vertices within the set B may be connected with each other, for the graph may not be two-colorable. The adjacency matrix Γ now is

$$\Gamma = \begin{pmatrix} \Gamma_A & \Gamma_{AB} \\ \Gamma_{AB}^{\mathrm{T}} & \Gamma_B \end{pmatrix} \tag{7}$$

Since any vertices pairs are not adjacent in the set A, the adjacency matrix of the set A is an all zero $|A| \times |A|$ matrix,

$$\Gamma_A = \mathbf{0} \tag{8}$$

Denote $\mu = (\mu_A, \mu_B)$, where the binary vectors $\mu_A = (\mu_1, \cdots, \mu_{|A|})$, $\mu_B = (\mu_{|A|+1}, \cdots, \mu_n)$, then the graph state can be written as $|G\rangle = |G_1\rangle + |G_2\rangle$, the unnormalized states

$$|G_1\rangle = \frac{1}{\sqrt{2^n}}\sum_{\mu_A=\mathbf{0}}^{\mathbf{1}}(-1)^{\frac{1}{2}(\mu_A,\mathbf{0})\Gamma(\mu_A,\mathbf{0})^{\mathrm{T}}}|\mu_A,\mathbf{0}\rangle = \frac{1}{\sqrt{2^n}}\sum_{\mu_A=\mathbf{0}}^{\mathbf{1}}|\mu_A,\mathbf{0}\rangle \tag{9}$$

where we have used the fact that

$$\frac{1}{2}(\mu_A,\mathbf{0})\begin{pmatrix}\mathbf{0} & \Gamma_{AB}\\ \Gamma_{AB}^{\mathrm{T}} & \Gamma_B\end{pmatrix}(\mu_A,\mathbf{0})^{\mathrm{T}} = 0 \tag{10}$$

And

$$|G_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{\mu_A=\mathbf{0}}^{\mathbf{1}}\sum_{\mu_B\neq\mathbf{0}}(-1)^{\frac{1}{2}(\mu_A,\mu_B)\Gamma(\mu_A,\mu_B)^{\mathrm{T}}}|\mu_A,\mu_B\rangle \tag{11}$$

To obtain a lower bound of the extremal fidelity F, we can choose

$$|\phi\rangle = \bigotimes_{a=1}^{|A|}(\sqrt{p_a}|0\rangle + \sqrt{1-p_a}\mathrm{e}^{\mathrm{i}\varphi_a}|1\rangle)\otimes|0\rangle^{\otimes(n-|A|)} \tag{12}$$

Since $\mu_B \neq \mathbf{0}$ in the state $|G_2\rangle$ and the last $(n-|A|)$ qubits of $|\phi\rangle$ are all in $|0\rangle$, we have $\langle G_2|\phi\rangle = 0$. Thus one has

$$\langle G|\phi\rangle = \langle G_1|\phi\rangle = \frac{1}{\sqrt{2^n}}\sum_{\mu_A=\mathbf{0}}^{\mathbf{1}}\langle\mu_A|\bigotimes_{a=1}^{|A|}(\sqrt{p_a}|0\rangle + \sqrt{1-p_a}\mathrm{e}^{\mathrm{i}\varphi_a}|1\rangle) = \frac{1}{\sqrt{2^n}}\prod_{a=1}^{|A|}(\sqrt{p_a}+\sqrt{1-p_a}\mathrm{e}^{\mathrm{i}\varphi_a}) \tag{13}$$

For all the separable states of type (12), the maximal fidelity is

$$F_0 = \max_{p_a,\varphi_a}|\langle G|\phi\rangle|^2 = 2^{-(n-|A|)} \tag{14}$$

which is achieved when $p_a = \frac{1}{2}$, $\varphi_a = 0 (a = 1,\cdots,|A|)$. The upper bound of the entanglement is

$$-\log_2 F_0 = n - |A| \tag{15}$$

where $|A|$ is the maximal number of non-adjacent vertices of the graph. Note that we obtain the upper bound without the knowledge of LOCC, and the result can be applied to any graph state.

As an application, we consider the graph in Fig. 1(a). The stabilizer code based on this graph is [[6, 1, 3]] quantum error correction code. The maximal number of non-adjacent vertices of the graph is 3 (i.e., vertices $1,3,6$ or vertices $2,4,6$). Thus we have the entanglement upper bound $6-3=3$. Meanwhile, the lower "matching" bound [12] is also 3. This is because of the fact that multipartite entanglement is no less than the corresponding bipartite entanglement [12]. We have $E \geqslant E_{bi}$. The lower bound E_{bi} is obtained with a bipartition of the graph into subgraph $C = \{V_C, \Gamma_C\}$ and $D = \{V_D, \Gamma_D\}$,with $V_C = \{1,4,5\}$ and $V_D = \{2,3,6\}$. Removing the local edges in both parties by local Control-Z operations U_{ij}, we obtain 3 Bell pairs between the two parties. So $E_{bi} = 3$. The detailed process is to apply U_{15} and U_{45} operations which are local in subgraph C, and apply U_{23} operation which is local in subgraph D. The edges of the remaining graph are $(1,2),(3,4)$ and $(5,6)$. The upper bound and the lower bound coincide, thus the entanglement of the graph state is 3.

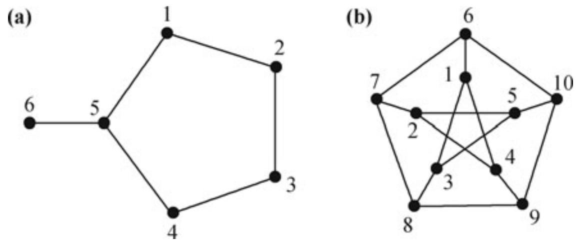

Fig. 1 **(a)** The graph for [[6, 1, 3]] code; **(b)** Peterson graph.

4 Improving the upper bound with symmetry

For a graph state of ring graph with odd n vertices, we have $\lfloor\frac{n}{2}\rfloor \leqslant E \leqslant \lceil\frac{n}{2}\rceil$. The upper bound is obtained by LOCC and also by our non-adjacent vertices set method. We will show that this upper bound can be further improved for five vertex ring graph state by making use of the symmetry of the graph. Our symmetrical consideration also shows that the Peterson graph [13] state has entanglement upper bound of 5, while LOCC and non-adjacent vertices set method can only give an upper bound of 6.

4.1 Five vertex ring

The graph of five vertex ring is the underlying graph of the famous [[5, 1, 3]] stabilizer code. Denote the codewords of [[5, 1, 3]] as $|\bar{0}\rangle$ and $|\bar{1}\rangle$ [3], and the graph state will be $|G\rangle = -\frac{1}{\sqrt{2}}(|\bar{0}\rangle - |\bar{1}\rangle)$. We simply suppose the separable state to be

$$|\phi\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle)^{\otimes 5} \tag{16}$$

where we have considered the symmetry of the graph. Denote $x = \sqrt{p}, y = \sqrt{1-p}\mathrm{e}^{\mathrm{i}\varphi}$, the fidelity $F_G = |\langle G|\phi\rangle|^2$ will be

$$F_G = \frac{1}{32}|x^5 - y^5 + 5x^4y - 5xy^4|^2 \tag{17}$$

With a numerical calculation, the entanglement upper bound can be found to be

$$\min_{p,\varphi}(-\log_2 F_G) \approx 2.9275 \tag{18}$$

A more precise condition for the maximum of F_G can be obtained. Let us consider the fidelity $F_{\overline{0}} = |\langle\overline{0}|\phi\rangle|^2$,

$$F_{\overline{0}} = \frac{1}{16}|x^5 - 5xy^4|^2 \tag{19}$$

The fidelity $F_{\overline{0}}$ can be rewritten as $F_{\overline{0}} = \frac{1}{16}[p^5 + 25p(1-p)^4 - 10p^3(1-p)^2\cos 4\varphi]$. The maximal will be achieved when $\cos 4\varphi = -1$, so $F_{\overline{0}} = \frac{p}{16}[p^2 + 5(1-p)^2]^2$. The derivative $\frac{\mathrm{d}F_{\overline{0}}}{\mathrm{d}p} = 0$ reduces to $6p^2 - 6p + 1 = 0$, which is $p = \frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})$. For $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$, $\frac{\mathrm{d}^2F_{\overline{0}}}{\mathrm{d}p^2} = -\frac{5}{4}(1+\sqrt{3}) < 0$; For $p = \frac{1}{2}(1 + \frac{1}{\sqrt{3}})$, $\frac{\mathrm{d}^2F_{\overline{0}}}{\mathrm{d}p^2} = \frac{5}{4}(\sqrt{3}-1) > 0$. Thus the fidelity reaches its maximal $F_{\overline{0}\max} = \frac{3+\sqrt{3}}{36}$ when $\varphi = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, and $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$. It is interesting that at $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$, $\varphi = \pm\frac{\pi}{4}$, we have the maximal fidelity $F_{G\max} = \frac{3+\sqrt{3}}{36}$. We may calculate the derivatives of F_G at points $(p,\varphi) = (\frac{1}{2}(1 - \frac{1}{\sqrt{3}}), \pm\frac{\pi}{4})$. The first order derivatives are $\frac{\partial F_G}{\partial p} = 0, \frac{\partial F_G}{\partial \varphi} = 0$. The second order derivatives are $\frac{\partial^2 F_G}{\partial p^2} = -\frac{5}{4} < 0$, $\frac{\partial^2 F_G}{\partial p \partial \varphi} = \pm\frac{5}{12}$, $\frac{\partial^2 F_G}{\partial \varphi^2} = -\frac{5}{108}(3 + 2\sqrt{3}) < 0$. The Jacobian is

$$J = \begin{vmatrix} \dfrac{\partial^2 F_G}{\partial p^2} & \dfrac{\partial^2 F_G}{\partial p \partial \varphi} \\ \dfrac{\partial^2 F_G}{\partial p \partial \varphi} & \dfrac{\partial^2 F_G}{\partial \varphi^2} \end{vmatrix} = \frac{25\sqrt{3}}{216} > 0 \tag{20}$$

Thus $(p,\varphi) = (\frac{1}{2}(1 - \frac{1}{\sqrt{3}}), \pm\frac{\pi}{4})$ are the points of maximal fidelity F_G. We can also prove that $(p,\varphi) = (\frac{1}{2}(1 + \frac{1}{\sqrt{3}}), \pm\frac{\pi}{4})$ are the points of maximal fidelity F_G.

The entanglement upper bound is

$$\min_{p,\varphi} -\log_2 F_G = -\log_2 \frac{3+\sqrt{3}}{36} \approx 2.9275 \tag{21}$$

It is less than 3, the best upper bound by LOCC or non-adjacent vertices set method.

Although we use the identical product state to obtain the upper bound of the entanglement, and this upper bound is far from the lower bound, which is 2, a random search calculation indicates that this upper bound is possibly the entanglement itself.

4.2 Peterson graph

For Peterson graph G_P in Fig. 1(b), the lower bipartite bound for the entanglement of the graph state is easily obtained as 5, which is the number of Bell pairs between the subgraph C with $V_C = \{1, 2, 3, 4, 5\}$ and subgraph D with $V_D = \{6, 7, 8, 9, 10\}$. The number of maximal non-adjacent vertices set is 4, thus the entanglement upper bound is $10 - 4 = 6$.

Suppose the separable state to be

$$|\phi\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}\,\mathrm{e}^{\mathrm{i}\varphi}|1\rangle)^{\otimes 10} \tag{22}$$

Denote $x = \sqrt{p}, y = \sqrt{1-p}\,\mathrm{e}^{\mathrm{i}\varphi}$. We have

$$\langle G_P|\phi\rangle = 2^{-5}\sum_{j=0}^{10} c_j x^{10-j} y^j \tag{23}$$

The coefficient $c_j = \sum_{\mu\in\Lambda_j}(-1)^{\frac{1}{2}\mu\Gamma\mu^{\mathrm{T}}}$, where $\Lambda_j = \{\mu|\sum_{k=1}^{10}\mu_k = j\}$. The coefficient vector is

$$\boldsymbol{c} = (1, 10, 15, 0, -50, 108, 50, 0, -15, 10, -1) \tag{24}$$

A special closest separable state is with $p = \frac{1}{2}, \varphi = \frac{\pi}{2}$. The maximal fidelity is

$$F = |2^{-3}(-1+\mathrm{i})|^2 = \frac{1}{32} \tag{25}$$

The entanglement upper bound coincides with its lower bound. The entanglement of the Peterson graph state is 5.

5 Conclusions

We have proposed an upper bound for the entanglement of a graph state. The bound is based on the definition of the graph state. We obtain the bound by calculating the fidelity of the graph state with respect to some separable state. The vertices of the graph are divided into two subsets, one with all its vertices that are not adjacent with each other. We make this subset as large as possible and it has $|A|$ vertices. Then the entanglement of the n vertices graph state is upper bounded by $n - |A|$. The entanglement measure can be the (Global) Robustness of Entanglement, the Relative Entropy of Entanglement, and the Geometric Measure. These measures are all equal for graph states. Using this bound, we have found the entanglement of graph state which [[6, 1, 3]] code is based on to be 3. The upper bound of the graph state has been further improved for some highly symmetric states. These states are five vertices ring graph state and Peterson graph state. With the product of identical qubit states, we find that the entanglement upper bound for five vertices ring graph state is about 2.9275, which is less than 3, the bound given by LOCC and our non-adjacent vertices set method. We also determine the entanglement of Peterson graph state to be 5 (less than 6 given by LOCC) by using the product of identical qubit state as the closest separable state.

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