Review article

Entanglement and nonlocality in multi-particle systems

Margaret D. Reid[†], **Qiong-Yi He**, **Peter D. Drummond**

ARC Centre of Excellence for Quantum-Atom Optics, Centre for Atom Optics and Ultrafast Spectroscopy,
Swinburne University of Technology, Melbourne 3122, Australia
E-mail: [†] *MDREID@groupwise.swin.edu.au*

Entanglement, the Einstein–Podolsky–Rosen (EPR) paradox and Bell's failure of local-hidden-variable (LHV) theories are three historically famous forms of "quantum nonlocality". We give experimental criteria for these three forms of nonlocality in multi-particle systems, with the aim of better understanding the transition from microscopic to macroscopic nonlocality. We examine the nonlocality of N separated spin J systems. First, we obtain multipartite Bell inequalities that address the correlation between spin values measured at each site, and then we review spin squeezing inequalities that address the degree of reduction in the variance of collective spins. The latter have been particularly useful as a tool for investigating entanglement in Bose–Einstein condensates (BEC). We present solutions for two topical quantum states: multi-qubit Greenberger–Horne–Zeilinger (GHZ) states, and the ground state of a two-well BEC.

Contents

1 Introduction

Nonlocality in quantum mechanics has been extensively experimentally investigated. Results to date support the quantum prediction, first presented by Bell, that quantum theory is inconsistent with a combination of premises now generally called "local realism" [1, 2].

However, the extent that quantum mechanics is inconsistent with local realism at a more mesoscopic or macroscopic level is still not well understood. Schrödinger presented the case that loss of realism macroscopically would be a concern, and raised the question of how to link the loss of local realism with macroscopic superposition states [3–6].

The advent of entangled Bose–Einstein condensate (BEC) states leads to new possibilities for testing mesoscopic and macroscopic quantum mechanics. With this in mind, the objective of this article is to give an overview of

a body of work that explores nonlocality in multi-particle or multi-site systems.

Three types of nonlocality are reviewed: entanglement [3], the Einstein–Podolsky–Rosen (EPR) paradox [7], and Bell's nonlocality [1, 2]. Examples of criteria to demonstrate each of these nonlocalities is presented, first for multi-site "qubits" (many spin 1/2 particles) and then for multi-site "qudits" (many systems of higher dimensionality such as high spin particles).

The criteria presented in this paper are useful for detecting the nonlocality of the N-qubit (or N-qudit) Greenberger–Horne–Zeilinger (GHZ) states [8, 9]. These states are extreme superpositions that were shown by GHZ to demonstrate a very striking "all or nothing" type of nonlocality. This nonlocality can manifest as a violation of a Bell inequality, and at first glance these violations, because they increase exponentially with N, appear to indicate a more extreme nonlocality as the size N of the system increases [10].

We point out, however, that the detection of *genuine* N-body nonlocality, as first discussed by Svetlichny [11, 12], requires much higher thresholds. Genuine n-party nonlocality (e.g., genuine entanglement) requires that the nonlocality is shared among *all* N parties, or particles. The violations in this case do not increase with N, and the detection over many sites is very sensitive to loss and inefficiencies.

Finally, we review and outline how to detect entanglement [12] and the EPR paradox using collective spin measurements. This approach has recently been employed to establish a genuine entanglement of many particles in a BEC [13, 14].

2 Three famous types of nonlocality

The earliest studies of nonlocality concerned bipartite systems. Einstein–Podolsky–Rosen (EPR) [7] began the debate about quantum nonlocality, by pointing out that for some quantum states there exists an inconsistency between the premises we now call "local realism" and the completeness of quantum mechanics.

Local realism (LR) may be summarized as follows. EPR argued [7, 15] first for "locality", by claiming that there could be no "action-at-a-distance". A measurement made at one location cannot instantaneously affect the outcomes of measurements made at another distant location. EPR also argued for "reality", which they considered in the following context. Suppose one can predict with certainty the result of a measurement made on a system, without disturbing that system. Realism implies that this prediction is possible, only because the outcome for that measurement was a predetermined property of the system. EPR called this predetermined property an "element of reality", though most often the element of

reality is interpreted as a "hidden variable". The essence of EPR's local realism assumption is that results of measurements made on a system at one location come about because of predetermined properties of that system, and because of their local interactions with the measurement apparatus, not because of measurements that are made simultaneously at a distant locations.

2.1 EPR paradox

EPR argued that for states such as the spin 1/2 singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right) \tag{1}$$

there arises an inconsistency of the LR premises with the quantum predictions. Here, we define $|\uparrow\rangle_{A/B}$ and $|\downarrow\rangle_{A/B}$ as the spin "up" and "down" eigenstates of $J^Z_{A/B}$ for a system at location A/B. For the state (1), the prediction of the spin component J^Z_A can be made by measurement of the component J^Z_B at B. From quantum theory, the two measurements are perfectly anticorrelated. According to EPR's Local Realism premise (as explained above), there must exist an "element of reality" to describe the predetermined nature of the spin at A. We let this element of reality be symbolized by the variable λ_z, and we note that λ_z assumes the values $\pm 1/2$ (1).

Calculation shows that there is a similar prediction of a perfect anti-correlation for the other spin component pairs. Therefore, according to LR, each of the spin components J^Y_A and J^X_A can also be represented by an element of reality, which we denote λ_x and λ_y respectively. A moment's thought tells us that if there is a state for which all three spins are completely and precisely predetermined in this way, then this "state" cannot be a quantum state. Such a "state" is generally called a "local hidden variable (LHV) state", and the set of three variables are "hidden", since they are not part of standard quantum theory. Hence, EPR argued, quantum mechanics is incomplete.

Since perfect anticorrelation is experimentally impossible, an operational criterion for an EPR paradox can be formulated as follows. Consider two observables X and P, with commutators like position and momentum. The Heisenberg Uncertainty Principle is $\Delta X \Delta P \geqslant 1$, where ΔX and ΔP are the variances of the outcomes of measurements for X and P respectively. The EPR paradox criterion is [16]

$$\Delta_{\inf} X \Delta_{\inf} P < 1 \tag{2}$$

where $\Delta_{\inf} X \equiv V(X|O_B)$ is the "variance of inference", i.e., the variance of X conditional on the measurement of an observable O_B at a distant location B. The $\Delta_{\inf} P \equiv V(P|Q_B)$ is defined similarly where Q_B is a second observable for measurement at B. This criterion

reflects that the combined uncertainty of inference is reduced below the Heisenberg limit. Of course, the reduced uncertainty applies over an ensemble of measurements, where only one of the conjugate measurements is made at a time. This criterion is also applicable to optical quadrature observables, where it has been experimentally violated, although without causal separation. With spin commutators, other types of uncertainty principle can be used to obtain analogous inferred uncertainty limits.

The demonstration of an EPR paradox through the measurement of correlations satisfying Eq. (2) is a proof that local realism is inconsistent with the completeness of quantum mechanics (QM). Logically, one must: discard local realism, the completeness of QM, or both. However, it does not indicate which alternative is correct.

2.2 Schrödinger's entanglement

Schrödinger [3–5] noted that the state (1) is a special sort of state, which he called an *entangled* state. An entangled state is one which cannot be factorized: for a pure state, we say there is entanglement between A and B if we cannot write the composite state $|\psi\rangle$ (that describes all measurements at the two locations) in the form $|\psi\rangle = |\psi\rangle_A |\psi\rangle_B$, where $|\psi\rangle_{A/B}$ is a state for the system at A/B only.

For mixed states, there is entanglement when the density operator for the composite system cannot be written as a mixture of factorizable states [17]. A mixture of factorizable states is said to be a *separable* state, which where there are just two sites, is written as

$$\rho = \sum_R P_R \rho_A^R \rho_B^R \qquad (3)$$

If the density operator cannot be written as Eq. (3), then the mixed system possesses *entanglement* (between A and B). More generally, for N sites, full separability implies

$$\rho = \sum_R P_R \rho_1^R \cdots \rho_N^R \qquad (4)$$

If the density operator cannot be expressed in the fully separable form (4), there is entanglement between at least two of the sites.

We consider measurements $\hat{X}_k$, with associated outcomes X_k, that can be performed on the k-th system $(k = 1, \cdots, N)$. For a separable state (4), it follows that the joint probability for outcomes is expressible as

$$P(X_1, \cdots, X_N) = \int_\lambda P(\lambda) P_Q(X_1|\lambda) \cdots P_Q(X_N|\lambda) \mathrm{d}\lambda \qquad (5)$$

where we have replaced for convenience of notation the index R by λ, and used a continuous summation symbolically, rather than a discrete one, so that $P(\lambda) \equiv P_R$. The subscript Q represents "quantum", because

there exists the quantum density operator $\rho_k^\lambda \equiv \rho_k^R$ for which $P(X_k|\lambda) \equiv \langle X_k|\rho_k^\lambda|X_k\rangle$. In this case, we write $P(X_k|\lambda) \equiv P_Q(X_k|\lambda)$, where the subscript Q reminds us that this is a quantum probability distribution. The model (5) implies (3) [18, 19], and has been studied in Ref. [20], in which it is referred to as a *quantum separable model* (QS).

We can test nonlocality when each system k is spatially separated. We will see from the next section that LR implies the form (5), but without the subscripts "Q", that is, without the underlying local states designated by λ necessarily being quantum states. If the quantum separable QS model can be shown to fail where each k is spatially separated, one can only have consistency with Local Realism if there exist underlying local states that are *non-quantum.* This is an EPR paradox, since it is an argument to complete quantum mechanics, based on a requirement that LR be valid.

The EPR paradox necessarily requires entanglement [15, 21]. The reason for this is that for separable states (3)–(5), the uncertainty relation that applies to each of the states $|\psi\rangle_A$ and $|\psi\rangle_B$ will imply a minimum level of local uncertainty, which means that the noncommuting observables cannot be sufficiently correlated to obtain an EPR paradox. In other words, the entangled state (1) can possess a greater correlation than possible for (3).

Schrödinger also pointed to two paradoxes [3–5] in relation to the EPR paper. These gedanken-experiments strengthen the apparent need for the existence of EPR "elements of reality", in situations involving macroscopic systems, or spatially separated ones. The first is famously known as the Schrödinger's cat paradox, and emphasizes the importance of EPR's "elements of reality" at a *macroscopic* level. Reality applied to the state of a cat would imply a cat to be either dead or alive, prior to any measurement that might be made to determine its "state of living or death". We can define an "element of reality" λ_{cat}, to represent that the cat is *predetermined* to be dead (in which case $\lambda_{\mathrm{cat}} = -1$) or alive (in which case $\lambda_{\mathrm{cat}} = +1$). Thus, the observer looking inside a box, to make a measurement that gives the outcome "dead" or "alive", is simply uncovering the value of λ_{cat}. Schrödinger's point was that the element of reality specification is not present in the quantum description $|\Psi\rangle = \frac{1}{\sqrt{2}}(|dead\rangle + |alive\rangle)$ of a superposition of two macroscopically distinguishable states.

The second paradox raised by Schrödinger concerns the apparent "action at-a-distance" that seems to occur for the EPR entangled state. Unless one identifies an element of reality for the outcome A, it would seem to be the action of the measurement of B that immediately enables prediction of the outcome for the measurement at A. Schrödinger thus introduced the notion of "*steering*".

While all these paradoxes require entanglement, we emphasize that entanglement *per se* is a relatively com-

mon situation in quantum mechanics. It is necessary for a quantum paradox, but does not by itself demonstrate any paradox.

2.3 Bell's nonlocality: Failure of local hidden variables (LHV)

EPR claimed as a solution to their EPR paradox that hidden variables consistent with local realism would exist to further specify the quantum state. It is the famous work of Bell that proved the impossibility of finding such a theory. This narrows down the two alternatives possible from a demonstration of the EPR paradox, and shows that local realism itself is invalid.

Specifically, Bell considered the predictions of a Local Hidden Variable (LHV) theory, to show that they would be different from the predictions of the spin-half EPR state (1). Following Bell [1, 2], we have a *local hidden variable model* (LHV) if the joint probability for outcomes of simultaneous measurements performed on the N spatially separated systems is given by

$$P(X_1, \cdots, X_N) = \int_\lambda P(\lambda)P(X_1|\lambda) \cdots P(X_N|\lambda)\mathrm{d}\lambda \quad (6)$$

Here λ are the "local hidden variables" and $P(X_k|\lambda)$ is the probability of X_k given the values of λ, with $P(\lambda)$ being the probability distribution for λ. The factorization in the integrand is Bell's locality assumption, that $P(X_k|\lambda)$ depends on the parameters λ, and the measurement choice made at k only. The hidden variables λ describe a local state for each site, in that the probability distribution $P(X_k|\lambda)$ for the measurement at k is given as a function of the λ. The form of (6) is formally similar to (4) except in the latter there is the additional requirement that the local states are quantum states. If (6) fails, then we have proved a failure of all LHV theories, which we refer to as a *Bell violation* or *Bell nonlocality* [20].

The famous Bell–Clauser–Horne–Shimony–Holt (CHSH) inequalities follow from the LHV model, in the $N = 2$ case. Bell considered measurements of the spin components $J_A^\theta = \cos\theta J_A^X + \sin\theta J_A^Y$ and $J_B^\theta = \cos\phi J_B^X + \sin\phi J_B^Y$. He then defined the spin product $E(\theta, \phi) = \langle J_A^\theta J_B^\phi \rangle$ and showed that for the LHV model, there is always the constraint

$$B = E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi') \leqslant 2 \quad (7)$$

The quantum prediction for an entangled Bell state (1) is $E(\theta, \phi) = \cos(\theta - \phi)$ and the inequality is violated for the choice of angles

$$\theta = 0, \quad \theta' = \pi/2, \quad \phi = \pi/4, \quad \phi' = 3\pi/4 \quad (8)$$

for which the quantum prediction becomes $B = 2\sqrt{2}$. Tsirelson's theorem proves the value of $B = 2\sqrt{2}$ to be the maximum violation possible for any quantum state [22]. We note that experimental inefficiencies mean that

violation of the CHSH inequalities for causally separated detectors is difficult, and has so far always required additional assumptions in the interpretation of experimental data.

2.4 Steering as a special nonlocality

Recently, Wiseman *et al.* (WJD) [18, 19] have constructed a hybrid separability model, called the Local Hidden State Model (LHS), the violation of which is confirmation of Schrödinger's "steering" (Fig. 1). The bipartite local hidden state model (LHS) assumes

$$P(X_A, X_B) = \int_\lambda P(\lambda)P(X_A|\lambda)P_Q(X_B|\lambda)\mathrm{d}\lambda \quad (9)$$

Thus, for one site A which we call "Alice", we assume a local hidden variable (LHV) state, but at the second site B, which we call "Bob", we assume a local quantum state (LQS). The violation of this model occurs iff there is a "*steering*" of Bob's state by Alice [23].
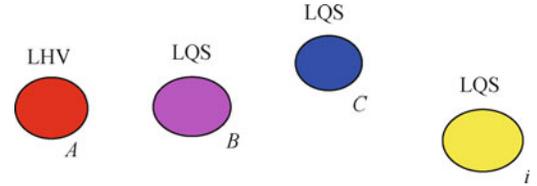
Fig. 1 The LHS model: Multiple "local" systems at are different spatial locations; the separability model is assumed. Some local systems are constrained only to be described by LHV, while others are constrained to be quantum systems (LQS).

WJD pointed out the association of steering with the EPR paradox [18]. The EPR criterion is also a criterion for steering, as defined by the violation of the LHS model. An analysis of the EPR argument when generalized to allow for imperfect correlation and arbitrary measurements reveals that violation of the LHS model occurs iff there is an EPR paradox [15, 20]. As a consequence, the violation of the LHS model is referred to as demonstration of a type of nonlocality called "*EPR steering*" [20]. EPR steering confirms the incompatibility of local realism with the *completeness* of quantum mechanics, just as with the approach of EPR in their original paper [7].

The notion of steering can be generalized to consider N sites, or observers [24]. The multipartite LHS model is (Fig. 1)

$$P(X_1, \cdots, X_N) =$$

$$\int \mathrm{d}\lambda P(\lambda) \prod_{j=1}^T P_Q(X_j|\lambda) \prod_{j=T+1}^N P(X_j|\lambda) \quad (10)$$

where we have T quantum states, and $N - T$ LHV local states. We use the symbol T to represent the quantum states, since these are the "trusted sites" in the picture put forward by WJD [18]. This refers to an application of this generalized steering to a type of quantum cryp-

tography in which an encrypted secret is being shared between sites. At some of the sites, the equipment and the observers are trusted, while at other sites this is not the case.

In this picture, which is an application of the LHS model, an observer C wishes to establish entanglement between two observers Alice and Bob. The violation of the QS model is sufficient to do this, provided each of the two observers Alice and Bob can be trusted to report the values for their local measurements. It is conceivable however that they report instead statistics that can give a violation of LHS model, so it seems as if there is entanglement when there is not. WJD point out the extra security present if instead there is the stronger requirement of violation of the LHV model, in which the untrusted observers are identified with an LHV state.

Cavalcanti *et al.* [24] have considered the multipartite model (10), and shown that violation of (10) where $T = 1$ is sufficient to imply that an *EPR steering* paradox exists between at least two of the sites. Violation where $T = 0$ is proof of Bell's nonlocality, and violation where $T = N$ is a confirmation of entanglement (quantum inseparability).

2.5 Hierarchy of nonlocality

WJD established formally the concept of a hierarchy of nonlocality [18, 19]. Werner [25] showed that some classes of entangled state can be described by Local Hidden Variable theories and hence cannot exhibit a Bell nonlocality. WJD showed that not all entangled states are "steerable" states, defined as those that can exhibit EPR steering. Similarly, they also showed that not all EPR steerable states exhibit Bell nonlocality. However, we see from the definitions that all EPR steering states must be entangled, and all Bell-nonlocal states (defined as those exhibiting Bell nonlocality) must be EPR steering states. Thus, the Bell-nonlocal states are a strict subset of EPR steering states, which are a strict subset of entangled states, and a hierarchy of nonlocality is established.

3 Multiparticle nonlocality

Experiments that have been performed on many microscopic systems support quantum mechanics. Those that test Bell's theorem [1, 2], or the equivalent, are the most useful, since they directly refute the assumption of local realism. While these experiments still require additional assumptions, it is generally expected that improved technology will close the remaining loopholes.

There remains however the very important question of whether reality will hold macroscopically. Quantum mechanics predicts the possibility of superpositions of two macroscopically distinguishable states [6], like a cat in a superposition of dead and alive states. Despite the apparent paradox, there is increasing evidence for the existence of mesoscopic and macroscopic quantum superpositions.

As with microscopic systems, there is a need to verify the loss of reality for macroscopic superpositions in an objective sense, by following Bell's example and comparing the predictions of quantum mechanics with those based on premises of local realism. The first steps toward this have been taken, through theoretical studies of nonlocality for multi-particle systems. Two limits have been rather extensively examined. The first is that of bipartite qudits. The second is multipartite qubits. Surprisingly, while it may have been thought that the violation of LR would diminish or vanish at a critical number of particles, failure of local realism has been shown possible according to quantum mechanics, for arbitrarily large numbers of particles. The third possibility of multipartite qudits has not been treated in as much detail.

3.1 Bipartite qudits

The simplest mesoscopic extension of the Bell case (1) is to consider bipartite qudits: two sites of higher dimensionality. The maximally entangled state in this case is

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle \tag{11}$$

where $|jj\rangle$ is abbreviation for $|j\rangle_A |j\rangle_B$, and d is the dimensionality of the systems at A and B. In this case at each site A and B the possible outcomes are $j = 0, \cdots, d - 1$. This system can be realized by two spin J systems, for which the outcomes are x given by $-J, -J+1, \cdots, J-1, J$, so that $d = 2J+1$, and j of Eq. (11) is $j \equiv x + J$ where x is the outcome of spin. It can also be realized by multi-particle systems.

It was shown initially by Mermin, Garg and Drummond, and Peres and others [26–32] that quantum systems could violate local realism for large d. The approach was to use the classic Bell inequalities derived for binary outcomes.

Later, Kaszlikowski *et al.* [33] showed that for maximally entangled states (11), the strength of violation actually becomes stronger for increasing d. A new set of Bell inequalities for bipartite qudits was presented by Collins *et al.* (CGLMP) [34, 35] and it was shown subsequently by Acin *et al.* [36] that greater violations can be obtained with non-maximally entangled states, and that the violations increase with d. Chen *et al.* [37] have shown that the violation of CGLMP inequalities increases as $d \to \infty$ to a limiting value.

We wish to address the question of how the entanglement and EPR steering nonlocalities increase with d. Since Bell nonlocality implies both EPR steering and

entanglement, these nonlocalities also increase with d. However, since there are distinct nested classes of nonlocality, the violation could well be greater, for an appropriate set of measures of the nonlocalities, and this problem is not completely solved for the CGLMP approach. We later investigate alternative criteria that show differing levels of violation for the different classes of nonlocality.

3.2 Multipartite qubits: MABK Bell inequalities

The next mesoscopic — macroscopic scenario that we will consider is that of many distinct single particles — the multi-site qubit system. The interest here began with the Greenberger–Horne–Zeilinger (GHZ) argument [8], which revealed a more extreme "all-or-nothing" form of nonlocality for the case of three and four spin 1/2 particle (three or four qubits), prepared in a so-called GHZ state. The N qubit GHZ state is written

$$|\Psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}}\{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\} \qquad (12)$$

where $|0\rangle$ and $|1\rangle$ in this case are spin up/ down eigenstates. Mermin then showed that for this extreme superposition, there corresponds a greater violation of LR, in the sense that the new "Mermin" Bell inequalities were violated by an amount that increased exponentially with N [10]. These new multipartite Bell inequalities of Mermin were later generalized by Ardehali, Belinski and Klyshko, to give a set of MABK Bell inequalities [38, 39].

The MABK inequalities define moments like $\langle J_A^+ J_B^+ J_C^-\rangle$, where $J^\pm = J^X \pm \mathrm{i}J^Y$ and J^X, J^Y, J^Z, J^2 are the standard quantum spin operators. In the MABK case of qubits, Pauli operators are used, so that the spin outcomes $\pm 1/2$ are normalized to ± 1. The $J^{X/Y}$ are redefined accordingly. The moments are defined generally by

$$\prod_N = \langle\prod_{k=1}^N J_k^{s_k}\rangle \qquad (13)$$

where $s_k = \pm 1$ and $J^{s_1} \equiv J^+$ and $J^{s_{-1}} \equiv J^-$. A LHV theory expresses such moments as the integral of a complex number product:

$$\prod_N = \int \mathrm{d}\lambda P(\lambda)\Pi_{N,\lambda} \qquad (14)$$

where $\Pi_{N,\lambda} = \Pi_{k=1}^N\langle J_k^{s_k}\rangle_\lambda$, and $\langle J_k^\pm\rangle_\lambda = \langle J_k^X\rangle \pm \mathrm{i}\langle J_k^Y\rangle$ where $\langle J_k^{X/Y}\rangle_\lambda$ is the expected value of outcome for measurement $J^{X/Y}$ made at site k given the local hidden state λ. The $\Pi_{N,\lambda}$ is a complex number product, which Mermin [10] showed has the following extremal values: for N odd, a magnitude $2^{N/2}$ at angle $\pi/4$ to real axis; for N even, magnitude $2^{N/2}$ aligned along the real or imaginary axis. With this algebraic constraint, LR will imply the following inequalities, for odd N:

$$\mathrm{Re}\prod_N, \ \mathrm{Im}\prod_N \leqslant 2^{(N-1)/2} \qquad (15)$$

For even N, the inequality $\mathrm{Re}\prod_N, \ \mathrm{Im}\prod_N \leqslant 2^{N/2}$ will hold. However, it is also true, for even N, that [38]

$$\mathrm{Re}\prod_N + \mathrm{Im}\prod_N \leqslant 2^{N/2} \qquad (16)$$

The Eqs. (15)–(16) are the MABK Bell inequalities [39]. Maximum violation of these inequalities is obtained for the N-qubit Greenberger–Horne–Zeilinger (GHZ) state (12) [40]. For optimal angle choice, a maximum value

$$\langle\mathrm{Re}\Pi_N\rangle, \langle\mathrm{Im}\Pi_N\rangle = 2^{N-1} \qquad (17)$$

can be reached for the left-side of (15), while for a different optimal angle choice, the maximum value

$$\langle\mathrm{Re}\Pi_N\rangle + \langle\mathrm{Im}\Pi_N\rangle = 2^{N-1/2} \qquad (18)$$

can be reached for the left-side of (16). MABK Bell inequalities became famous for the prediction of exponential gain in violation as the number of particles (sites), N, increases. The size of violation is most easily measured as the ratio of left-side to right-side of the inequalities (15, 16), which is $2^{(N-1)/2}$ for the MABK inequalities. Werner and Wolf [40] showed the quantum prediction to be maximum for two-setting inequalities.

3.3 MABK-type EPR steering and entanglement inequalities for multipartite qubits

Recently, MABK-type inequalities have been derived for EPR steering and entanglement [24]. Entanglement is a failure of quantum separability, where each of the local states in (10) are quantum states ($T = N$). EPR steering occurs when there is failure of the LHS model with $T = 1$. To summarize the approach of Ref. [24], we note the statistics of each *quantum* state must satisfy a quantum uncertainty relation

$$\Delta^2 J^X + \Delta^2 J^Y \geqslant 1 \qquad (19)$$

As a consequence, for every *quantum* local state λ,

$$\langle J^X\rangle^2 + \langle J^Y\rangle^2 \leqslant 1 \qquad (20)$$

which implies the complex number product can have arbitrary phase, leading to the new nonlocality inequalities, which apply for all N, even or odd, and $T > 0$:

$$\langle\mathrm{Re}\Pi_N\rangle, \langle\mathrm{Im}\Pi_N\rangle \leqslant 2^{(N-T)/2} \qquad (21)$$

$$\langle\mathrm{Re}\Pi_N\rangle + \langle\mathrm{Im}\Pi_N\rangle \leqslant 2^{(N-T+1)/2} \qquad (22)$$

For $T = N$, these inequalities, if violated, will imply entanglement, as shown by Roy [41]. If they are violated for $T = 1$, there is EPR steering. As pointed out in Ref. [24], the exponential gain factor of the violation with the number of particles N increases for increasing T: the strength of violation as measured by left to right side

ratio is $2^{(N+T-2)/2}$, but for both inequalities (21, 22).

3.4 CFRD Multipartite qudit Bell, EPR steering and entanglement inequalities

We now summarize an alternative approach to nonlocality inequalities, developed by Cavalcanti, Foster, Reid and Drummond (CFRD) [42–46]. These hold for any operators, and are not restricted to spin-half or qubits. We shall apply this approach to the case of a hierarchy of inequalities, with some quantum and some classical hidden variable states. Consider

$$|\prod_N| \leqslant \int d\lambda P(\lambda) \prod_{k=1}^{N} |\langle J_k^{s_k} \rangle_\lambda| \tag{23}$$

$$= \int d\lambda P(\lambda) \prod_{k=1}^{N} \{\langle J_k^X \rangle_\lambda^2 + \langle J_k^Y \rangle_\lambda^2\}^{1/2} \tag{24}$$

We can see that for any LHV, because the variance is always positive, one can derive an inequality for any operator

$$\langle J_k^X \rangle_\lambda^2 + \langle J_k^Y \rangle_\lambda^2 \leqslant \langle (J_k^X)^2 \rangle_\lambda + \langle (J_k^Y)^2 \rangle_\lambda \tag{25}$$

but then for a quantum state in view of the uncertainty relation (19), it is the case that for qubits (spin-1/2)

$$\langle J_k^X \rangle_\lambda^2 + \langle J_k^Y \rangle_\lambda^2 \leqslant \langle (J_k^X)^2 \rangle_\lambda + \langle (J_k^Y)^2 \rangle_\lambda - 1 \tag{26}$$

For the particular case of qubits, the outcomes are ± 1 so that simplification occurs, to give final bounds based on local realism that are identical to Eqs. (21) and (22). We note that at $T = 0$, there is also a CFRD Bell inequality, but it is weaker than that of MABK, in the sense that the violation is not as strong as is not predicted for $N = 2$. Since this approach holds for any operator, we now can generalize to arbitrary spin.

The expression (25) and (26) also holds for arbitrary spin, for which case we revert to the usual spin outcomes (rather than the Pauli spin outcomes of ± 1). The LHV result for arbitrary spin is constrained by Eq. (25). The quantum result, however, requires a more careful uncertainty relation that is relevant to higher spins. In fact, for systems of fixed dimensionality d, or fixed spin J, the "qudits", the following uncertainty relation holds

$$\Delta^2 J^X + \Delta^2 J^Y \geqslant C_J \tag{27}$$

where the C_J has been derived and presented in Ref. [47]. The use of the more general result (27) gives the following higher-spin (qudit) nonlocality inequalities derived in Ref. [48]:

$$|\langle \prod_{k=1}^{N} J_k^{s_k} \rangle|^2 \leqslant \int d\lambda P(\lambda) \prod_{k=1}^{N} |\langle J_k^{s_k} \rangle_\lambda|^2$$

$$\leqslant \langle \prod_{k=1}^{T} (J_k^X)^2 + (J_k^Y)^2 - C_k \prod_{k=T+1}^{N} (J_k^X)^2 + (J_k^Y)^2 \rangle \tag{28}$$

Thus,

(i) Entanglement is verified if ($T = N$)

$$|\langle \prod_{k=1}^{N} J_k^{s_k} \rangle|^2 > \langle \prod_{k=1}^{N} [(J_k)^2 - (J_k^Z)^2 - C_J] \rangle \tag{29}$$

(ii) An EPR-steering nonlocality is verified if ($T = 1$)

$$|\langle \prod_{k=1}^{N} J_k^{s_k} \rangle|^2 > \langle [(J_1)^2 - (J_1^Z)^2 - C_J]$$

$$\times \prod_{k=2}^{N} [(J_k^X)^2 + (J_k^Y)^2] \rangle \tag{30}$$

(iii) Bell inequality ($T = 0$). The criterion to detect failure of the LHV theories is

$$|\langle \prod_{k=1}^{N} J_k^{s_k} \rangle|^2 > \langle \prod_{k=0}^{N} [(J_k^X)^2 + (J_k^Y)^2] \rangle \tag{31}$$

These criteria will be called the "C_J" CFRD nonlocality criteria, and allow investigation of nonlocality in multisite qudits, where the spin J is fixed.

We investigate predictions for quantum states that are maximally entangled, or not so, according to measures of entanglement that are justified for pure states. Maximally-entangled, highly correlated states for a fixed spin J are written as

$$|\Psi\rangle_{\max} = \frac{1}{\sqrt{d}} \sum_{m=-J}^{J} |m\rangle_1 |m\rangle_2 \cdots |m\rangle_N$$

$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_1 |j\rangle_2 \cdots |j\rangle_N \tag{32}$$

where $|m\rangle_k \equiv |J, m\rangle_k$ is the eigenstate of J_k^2 and J_k^Z (eigenvalue m for J_k^Z), defined at site k, and the dimensionality is $d = 2J + 1$. This state is the extension of Eq. (11) for multiple sites. We follow [36] however and consider more generally the non-maximally entangled but highly correlated spin states of form

$$|\psi\rangle_{\text{non}} = \frac{1}{\sqrt{n}} [r_{-J}|J, -J\rangle^{\otimes N} + r_{-J+1}|J, -J+1\rangle^{\otimes N}$$

$$+ \cdots + r_J|J, +J\rangle^{\otimes N}] \tag{33}$$

where $|J, m\rangle^{\otimes N} = \Pi_{k=1}^{N}|J, m\rangle_k$, $n = \sum_{m=-J}^{J} r_m^2$. Here we will restrict to the case of real parameters symmetrically distributed around $m = 0$. The amplitude r_m can be selected to optimize the nonlocality result. It is known, for example, with N sites and a spin-1 system that the optimized state

$$|\psi\rangle = \frac{1}{\sqrt{r^2 + 2}} (|1, -1\rangle^{\otimes N} + r|1, 0\rangle^{\otimes N} + |1, +1\rangle^{\otimes N}) \tag{34}$$

will give improved violation over the maximally entangled state (for which the amplitudes are uniform) for

some Bell inequalities [36].

With the optimization described above, we summarize the results explained in Ref. [48] that a growth of the violation of the nonlocality inequalities for increasing number N of spin sites is maintained with arbitrary d. This is shown in Fig. 2 for qudits $d = 2$ and $d = 3$ (spin $J = 1/2$ and $J = 1$), but means for higher d that one can obtain in principle a violation of inequalities for arbitrary d by increasing N. Thus, quantum mechanics predicts that at least for some states, increasing contradiction with separable theories is possible, as the number of sites increases, even where one has at each site a system of high spin. These results are consistent with those obtained by other authors [49–51].
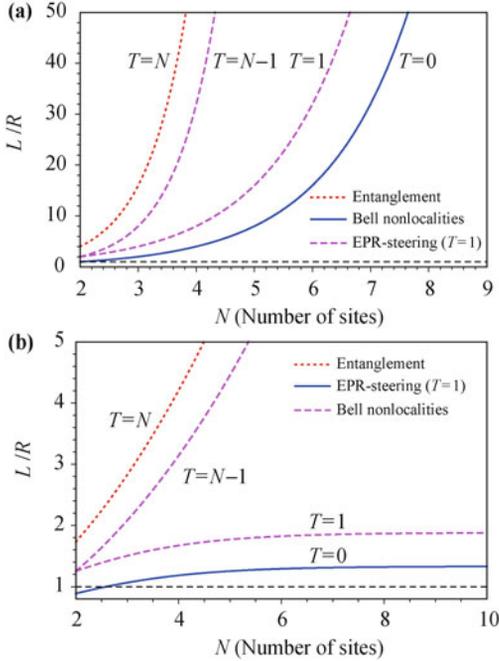
Fig. 2 Showing nonlocality to be possible for large numbers N of spin systems. The violation of the CFRD Bell ($T = 0$), steering ($T = 1$) and entanglement ($T = N$) inequalities of (28), as measured by the ratio of left side to right side (L/R), for N spin-1/2 systems **(a)**, and N spin-1 systems **(b)**. Nonlocality is demonstrated when $L/R > 1$.

4 Genuine multiparticle nonlocality: Qubit example

Svetlichny [11] addressed the following question. How many particles are *genuinely* entangled? The above nonlocality inequalities can fail if separability/locality fails between a single *pair* of sites. To prove that *all* N sites are entangled, or that the Bell nonlocality is shared between *all* N sites, is a more challenging task, and one that relates more closely to the question of multi-particle quantum mechanics.

To detect genuine nonlocality, one needs to construct different criteria. For example where $N = 3$, to show genuine tripartite entanglement, we need to exclude that

the statistics can be described by bipartite entanglement, i.e., by the models

$$\rho = \sum_R P_R \rho_{AB}^R \rho_C^R, \; \rho = \sum_R P_R \rho_A^R \rho_{BC}^R$$

$$\rho = \sum_R P_R \rho_B^R \rho_{AC}^R \tag{35}$$

where ρ_{IJ}^R can be *any* density operator for composite system I and J. These models can fail *only* if there is genuine tripartite entanglement. Thus, to show there is a genuine tripartite Bell nonlocality, one needs to falsify all models encompassing bipartite Bell nonlocality, i.e.,

$$P(x_\theta, x_\phi, x_\vartheta) = \int d\lambda P(\lambda) P_{AB}(x_\theta, x_\phi|\lambda) P_C(x_\vartheta|\lambda) \tag{36}$$

and the permutations. In the expansion (36), locality is not assumed between A and B, but is assumed between a composite system AB, and C. This model allows bipartite entanglement between A and B, but not tripartite entanglement. To test genuine nonlocality or entanglement, it is therefore useful to consider hybrid local-nonlocal models. What is a condition for genuine N partite entanglement?

Consider again the N-qubit system. A recent analysis [24] follows Svetlichny [11] and Collins *et al.* (CGPRS) [12], to consider a hybrid local-nonlocal model in which Bell nonlocality *can* exist, but only if shared among $k = N - 1$ or fewer parties. Separability must then be retained between any two groups A and B of k and $N - k$ parties respectively, if $k > N/2$, and one can write:

$$\langle \prod_{j=1}^N F_j^{s_j} \rangle = \int_\lambda d\lambda P(\lambda) \langle \prod_{j=1}^k F_j^{s_j} \rangle_{A,\lambda} \langle \prod_{j=k+1}^N F_j^{s_j} \rangle_{B,\lambda} \tag{37}$$

Violation of all such "k-nonlocality" models then implies the nonlocality to be genuinely "$k + 1$ partite". We summarize Ref. [24] by considering the consequences of the hybrid model (37) for the three different types of nonlocality. Multiplying out $\prod_{j=1}^N F_j^{s_j} = \mathrm{Re}\Pi_N + i\mathrm{Im}\Pi_N$ reveals recursive relations $\mathrm{Re}\Pi_N = \mathrm{Re}\Pi_{N-1}\sigma_x^N - \mathrm{Im}\Pi_{N-1}\sigma_y^N$, $\mathrm{Im}\Pi_N = \mathrm{Re}\Pi_{N-1}\sigma_y^N + \mathrm{Im}\Pi_{N-1}\sigma_x^N$ which imply algebraic constraints that must hold for all theories [10]

$$\langle \mathrm{Re}\Pi_N \rangle, \langle \mathrm{Im}\Pi_N \rangle \leqslant 2^{N-1} \tag{38}$$

$$\langle \mathrm{Re}\Pi_N \rangle + \langle \mathrm{Im}\Pi_N \rangle \leqslant 2^N \tag{39}$$

These recursive relations and the CHSH lemma summarized by Ardehali [38] gives the Svetlichny-CGPRS inequality [11, 12]

$$\langle \mathrm{Re}\Pi_N \rangle + \langle \mathrm{Im}\Pi_N \rangle \leqslant 2^{N-1}$$

the violation of which confirms genuine N partite Bell-nonlocality. The quantum prediction maximizes at (18) to predict violation by a *constant* amount ($S_N = \sqrt{2}$)

[52, 53].

In order to investigate the other nonlocalities, for example, that of genuine multipartite steering, the authors of Ref. [24] suggest the hybrid approach of *quantizing B*, the group of $N - k$ qubits, but not group A. In this case, the extremal points of the hidden variable product $\langle \Pi_k^A \rangle_\lambda = \langle \prod_{j=1}^k F_j^{s_j} \rangle_{A,\lambda}$ of A is constrained only by the *algebraic* limit (38), whereas the product $\langle \Pi_{N-k}^B \rangle_\lambda \equiv \langle \prod_{j=k+1}^N F_j^{s_j} \rangle_\lambda$ for group B is constrained by the *quantum* result (18). We note that a criterion for genuine N-qubit entanglement is obtained by constraining *both* A and B to be quantum, leading to the condition

$$\langle \mathrm{Re}\Pi_N \rangle, \langle \mathrm{Im}\Pi_N \rangle \leqslant 2^{N-2} \tag{40}$$

(as derived in Ref. [54]), and $\langle \mathrm{Re}\Pi_N \rangle + \langle \mathrm{Im}\Pi_N \rangle \leqslant 2^{N-3/2}$. These are violated by Eqs. (17) and (18) to confirm genuine Nqubit entanglement ($S_N = 2$).

In short, genuine N particle nonlocality can be confirmed using MABK Bell inequalities for N qubits, but a higher threshold is required. The threshold is reached by the quantum prediction of the GHZ states, but the higher bound implies the level of violation is *no longer* exponentially increasing with N. As a related consequence, the higher threshold also implies a much higher bound for efficiency, which makes multi-particle nonlocality difficult to detect for increasingly larger systems.

5   Investigating entanglement using collective measurements: Spin squeezing inequalities

While detection of individual qubits could be fulfilled in many systems, the demonstration of a large multi-particle nonlocality would likely require exceptional detection efficiencies if one is to detect a *genuine* multi-particle nonlocality for large N. We thus review and outline a complementary approach, which is the measurement of the *collective* spin of a system.

5.1   Spin squeezing entanglement criterion

Consider N identical spin-J particles (Fig. 3). One defines the collective spin operator

$$J^X = \sum_{k=1}^N J_k^X \tag{41}$$

and similarly a J^Y and J^Z. Entanglement between the spin J particles can be inferred via measurements of these collective operators. The concept of spin squeezing was pioneered by Kitagawa and Ueda [55], and Wineland *et al.* [56].

To investigate entanglement, we note that for each particle, or quantum site k, the Heisenberg uncertainty relation holds
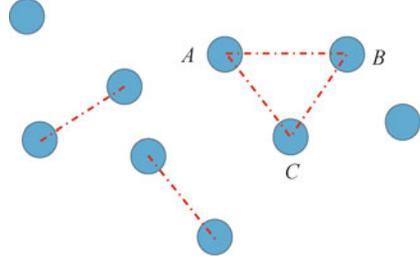
Fig. 3 Genuine nonlocality and entanglement: Distinguishing entanglement that may occur between two atoms, and a larger scale entanglement that necessarily involves more than two atoms. Here is a depiction of a group of atoms, in which the three atoms A, B and C are *genuinely* entangled. We summarize some conditions which are sufficient to demonstrate an N-body nonlocality, whether it be entanglement, EPR steering, or Bell nonlocality.

$$\Delta J_k^X \Delta J_k^Y \geqslant |\langle J_k^Z \rangle|/2 \tag{42}$$

If the system is fully separable (no entanglement) then

$$\rho = \sum_R P_R \rho_1^R \cdots \rho_k^R \cdots \rho_N^R \tag{43}$$

For a mixture, the variance is greater than the average of the variances of the components, which for a product state is the sum of the individual variances [57]. Thus, separability implies

$$\Delta^2 J^X \geqslant \sum_R P_R \sum_{k=1}^N \Delta^2 J_k^X \tag{44}$$

The next point to note is that for a fixed dimensionality spin-J system, there is a constraint on the *minimum* value for the variance of spin. The constraint on the minimum arises because of the constraint on the *maximum* variance, which for fixed spin J must be bounded by

$$\Delta^2 J^Y \leqslant J^2 \tag{45}$$

This implies, by the uncertainty relation, the lower bound on the minimum variance for a spin J system

$$\Delta^2 J^X \geqslant \langle J^Z \rangle^2/(4J^2) \tag{46}$$

Then we can prove, using Eq. (44) to get the first line,

$$\begin{aligned}
\Delta^2 J^X &\geqslant \frac{1}{4J^2} \sum_{k=1}^N \sum_R P_R \langle J_k^Z \rangle_R^2 \\
&\geqslant \frac{1}{4J^2} \sum_{k=1}^N |\sum_R P_R \langle J_k^Z \rangle_R|^2 \\
&= \frac{1}{4J^2} \sum_{k=1}^N |\langle J_k^Z \rangle|^2
\end{aligned} \tag{47}$$

and the Cauchy Schwarz inequality to get the second to last line [use $(\sum x^2)(\sum y^2) \geqslant |\sum xy|^2$ where $x = \sqrt{P_R}$ and $y = \sqrt{P_R}\langle J_k^Z \rangle_R$]. We can rewrite and use the Cauchy-Schwarz inequality again (this time, $x = 1/\sqrt{N}$ and $y = \langle J_k^{Z} \rangle/\sqrt{N}$), to obtain

$$\Delta^2 J^X = \frac{N}{4J^2} \sum_{k=1}^{N} \frac{1}{N} |\langle J_k^Z \rangle|^2$$

$$\geqslant \frac{N}{4J^2} |\sum_{k=1}^{N} \frac{1}{N} \langle J_k^Z \rangle|^2$$

$$= \frac{1}{4NJ^2} |\langle J^Z \rangle|^2 \tag{48}$$

We can express the result as

$$y = x^2/(4J) \tag{49}$$

where $y = \Delta^2 J^X / J$ and $x = |\langle J^Z \rangle|/J$. For $J = 1/2$, we obtain the result that for a fully separable state,

$$\Delta^2 J^X \geqslant |\langle J^Z \rangle|^2 / N \tag{50}$$

($y = x^2/2$). This result for spin 1/2 was first derived by Sørensen *et al.* [58], and is referred to as the "spin squeezing criterion" to detect entanglement. Failure of (50) reflects a reduction in variance (hence "squeezing"), and is confirmation that there is entanglement between at least two particles (sites). The criterion is often expressed in terms of the parameter defined by Wineland *et al.* [56], that is useful measure of interferometric enhancement, as

$$\xi = \frac{\sqrt{N}\Delta J^X}{|\langle J^Z \rangle|} < 1 \tag{51}$$

The spin squeezing criterion has been used to investigate entanglement within a group of atoms in a BEC by Esteve *et al.*, Gross *et al.* and Riedel *et al.* [13, 14, 59]. In fact, spin squeezing is predicted for the ground state of the following two-mode Hamiltonian

$$H = \kappa(a^\dagger b + ab^\dagger) + \frac{g}{2}(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) \tag{52}$$

which is a good model for a two-component BEC. Here κ denotes the conversion rate between the two components, and g is a self interaction term. More details on one method of solution of this Hamiltonian and some other possible entanglement criteria are given in Ref. [60]. To summarize, collective spin operators can be defined in the Schwinger representation:

$$J^Z = (a^\dagger a - b^\dagger b)/2$$
$$J^X = (a^\dagger b + ab^\dagger)/2$$
$$J^Y = (a^\dagger b - ab^\dagger)/(2\mathrm{i})$$
$$J^2 = \hat{N}(\hat{N} + 2)/4$$
$$\hat{N} = a^\dagger a + b^\dagger b \tag{53}$$

The system is viewed as N atoms, each with two-levels (components) available to it. For each atom, the spin is defined in terms of boson operators $J_i^Z = (a_i^\dagger a_i - b_i^\dagger b_i)/2$ where the total number for each atom is $N_i = 1$, and the outcomes for $a_i^\dagger a_i$ and $b_i^\dagger b_i$ are 0 and 1. The collective spin defined as $J^Z = \sum_i J_i^Z = \sum_i (a_i^\dagger a_i - b_i^\dagger b_i)/2$ can

then be re-expressed in terms of the total occupation number sums of each level. Figure 4 shows predictions for the variance of the collective spins J^Z or J^Y, where the mean spin is aligned along direction J^X, as a function of ratio Ng/κ, for a fixed number of atoms N and a fixed intercomponent coupling. In nonlinear regimes, indicated by $g \neq 0$, we see $\xi < 1$ is predicted, which is sufficient to detect entanglement. Heisenberg relations imply $\xi \geqslant 1/\sqrt{N}$.
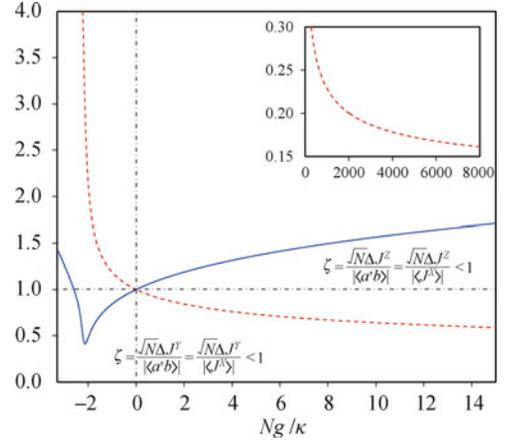
Fig. 4 Detecting entanglement within the atoms of a two-component BEC using the spin squeezing criterion (51) of Sorenson *et al.* [58]. Here, there are $N = 100$ atoms in total, with a fixed intercomponent coupling of $\kappa/k_B = 50$ nK and increasing intracomponent interaction g. Two states are available to each atom, and the Schwinger spin observables can be constructed as spin operators, according to Eq. (53). The spin squeezing parameter is plotted according to the prediction of the ground state solution of the Hamiltonian (52).

Sørensen and Mølmer [61] have evaluated the exact minimum variance of the spin squeezing for a fixed J. Their result for $J = 1/2$ agrees with Eq. (46) and Eq. (50), but for $J \geqslant 1$ there is a tighter lower bound for the minimum variance, which can be expressed as

$$\Delta^2 J^X / J \geqslant F_J(\langle J^Z \rangle / J) \tag{54}$$

where the functions F_J are given in Ref. [61].

The above criteria have been applied to particles that are effectively indistinguishable [62]. It is of interest to detect entanglement when the particles involved are distinguishable, or, even better, causally separated. We ask how to detect entanglement between spatially-separated or at least distinguishable groups of spin J. We examined this question in Section 3, and considered criteria that were useful for superposition states with mean zero spin amplitude.

Another method put forward by Sorenson and Molmer (SM) is as follows. The separability assumption (43) will imply [61]

$$\Delta^2 J^Z \geqslant NJF_J(\langle J^X \rangle / NJ) \tag{55}$$

where for convenience we have exchanged the notation of X and Z directions [compared to Eq. (54)]. The expres-

sion applies when considering N states ρ_k^R which have a fixed spin J, and could be useful where the mean spin is nonzero.

5.2 Depth of entanglement and genuine entanglement

We note from Eq. (54) that the minimum variance (maximum spin squeezing) reduces as J increases. Sorenson and Molmer (SM) showed how this feature can be used to demonstrate that a minimum number of particles or sites are genuinely entangled [61]. If

$$\Delta^2 J^Z/(NJ) < F_{J_0}[\langle J^X\rangle/(NJ)] \qquad (56)$$

then we must have $J > J_0$ and so a minimum number N_0 of particles (where the maximum spin for a block of N atoms is $J = N/2$, we must have $N_0 = 2J_0$) are involved, to allow the higher spin value.

It will be useful to summarize the proof of this result giving some detail as follows. Consider a system with the density matrix

$$\rho = \sum_R P_R \rho^R$$

$$= \sum_R P_R \prod_{i=1}^{N_R} \rho_i^R \qquad (57)$$

We will consider *that the overall system has a fixed number of atoms N_T and a fixed total spin J_{tot}*. The density operator (57) describes a system in a mixture of states ρ_R, with probability P_R. For each possibility R, there are N_R blocks each with $N_{R,i}$ atoms and a total spin $J_{R,i}$ (note that $J_{R,i} \leqslant N_{R,i}/2$) (Fig. 5).
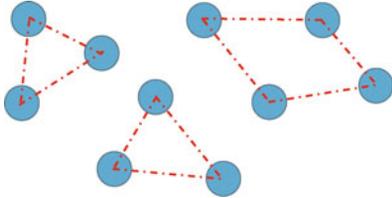
Fig. 5 Genuine multi-particle entanglement: The SM approach is to consider that the N_T atoms may be described by product states of blocks of up to N_0 entangled atoms. Here $N_0 = 4$.

We note that if the maximum number of atoms in each block is N_0 then the *maximum* spin for the block is $J_0 = N_0/2$. Also, if the total number of atoms is fixed, at N_T, then $N_T = \sum_{i=1}^{N_R} N_{R,i}$, which implies that each ρ_i^R has a definite number $N_{R,i}$, meaning it cannot be a superposition of state of different numbers. Similarly, for a product state the total spin must be the sum of the individual spins (as readily verified on checking Clebsch–Gordan coefficients), which implies that if the total spin is fixed, then each ρ_i^R has a fixed spin (that is, it cannot be in a superposition state of different spins).

Using again that the variance of the mixture cannot be less than the average of its components, and that the

variance of the product state ρ^R is the sum of the variances $(\Delta^2 J^Z)_{R,i}$ of each factor state ρ_i^R, we apply Eq. (54), that the variance has a lower bound determined by the spin $J_{R,i}$. Thus we can write

$$\Delta^2 J^Z \geqslant \sum_R P_R \sum_{i=1}^{N_R} (\Delta^2 J^Z)_{R,i}$$

$$\geqslant \sum_R P_R \sum_{i=1}^{N_R} J_{R,i} F_{J_{R,i}}(|\langle J^X\rangle|_{R,i}/J_{R,i}) \qquad (58)$$

Now we can use the fact that the curves F_J are nested to form a decreasing sequence at each value of their domain, as J increases, as explained by Sorenson and Molmer. We then apply the steps based on the SM proof [lines (6)–(8) of their paper], which uses the convexity of the functions F_J. We cannot exclude that the total spin of a block can be zero, $J_{R,i} = 0$, for which $(\Delta^2 J^Z)_{R,i} \geqslant 0$, but such blocks do not contribute to the summation and can be formally excluded. We define the total spin $\sum_{i=1}^{N_R} J_{R,i} = J_{\text{tot}}^R$ for each ρ^R, but note that for fixed total spin this is equal to J_{tot}, and we also note that $J_{\text{tot}}^R \leqslant J_0$. In the later steps below, we define the total spin as $J_{\text{tot}} = \sum_R P_R J_{\text{tot}}^R$ and the collective spin operator J^Z.

$$\Delta^2 J^Z \geqslant \sum_R P_R \sum_{i=1}^{N_R} J_{R,i} F_{J_0}(\langle J^X\rangle_{R,i}/J_{R,i})$$

$$= \sum_R P_R J_{\text{tot}}^R \sum_{i=1}^{N_R} \frac{J_{R,i}}{J_{\text{tot}}^R} F_{J_0}(\langle J^X\rangle_{R,i}/J_{R,i})$$

$$\geqslant \sum_R P_R J_{\text{tot}}^R F_{J_0}\left(\frac{\sum_{i=1}^{N_R}\langle J^X\rangle_{R,i}}{J_{\text{tot}}^R}\right)$$

$$= J_{\text{tot}} \sum_R P_R \frac{J_{\text{tot}}^R}{J_{\text{tot}}} F_{J_0}\left(\frac{\sum_{i=1}^{N_R}\langle J^X\rangle_{R,i}}{J_{\text{tot}}^R}\right)$$

$$\geqslant J_{\text{tot}} F_{J_0}\left(\sum_R P_R \frac{1}{J_{\text{tot}}}\sum_{i=1}^{N_R}\langle J^X\rangle_{R,i}\right)$$

$$\geqslant J_{\text{tot}} F_{J_0}\left(\frac{1}{J_{\text{tot}}}\sum_R P_R \sum_{i=1}^{N_R}\langle J^X\rangle_{R,i}\right)$$

$$= J_{\text{tot}} F_{J_0}(\langle J^X\rangle/J_{\text{tot}}) \qquad (59)$$

The total spin J_{tot} is maximum at $J_{\text{tot}} = N/2$ where N is total number of atoms over both systems, but is assumed measurable. Thus, if the maximum number of atoms in each block does not exceed N_0, then the inequality (59) must always hold. The violation of (59) is a demonstration of a group of atoms that are genuinely entangled [61].

The predictions of the model (52) are given in Fig. 6, for a range of values of N (the total number of atoms). In each case, there is a constant total spin, $J \equiv J_{\text{tot}}$, given by $\langle (J^X)^2 + (J^Y)^2 + (J^Z)^2\rangle = J(J+1)$ where $J = N/2$. We keep N and the interwell coupling κ fixed, and note that the variance of J^Z decreases with increasing g, while

the variance in J^X increases. Evaluation of the normalized quantities of the SM inequality (59) are given in the second plot of Fig. 6. Examining the functions F_{J_0} reveals the prediction of a full N particle entanglement, where N is an integer value.
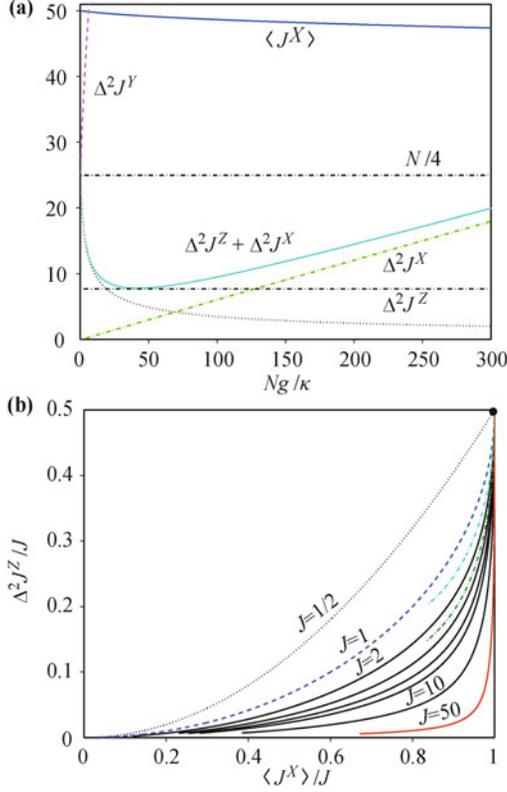
Fig. 6 Detecting multi-particle entanglement in ground state of a two-component BEC, as modeled by Eq. (52). **(a)** The predictions according to Eq. (52) for the spin moments for 100 atoms ($N = 100$), where there is a fixed conversion rate of $\kappa/k_{\mathrm{B}} = 50$ nK, but an increasing intrawell interaction g. **(b)** The corresponding prediction for the ratio $\Delta^2 J^Z/J$ as a function of normalized mean spin amplitude $\langle J^X \rangle/J$ for different N, so that the SM inequality (59) can be tested.

We note this treatment does not itself test nonlocality, or even the quantum separability models [Eqs. (4), (5)] because measurements are not taken at distinct locations. However, it can reveal, *within a quantum framework*, an underlying entanglement, of the type that could give nonlocality if the individual spins could be measured at different locations. The great advantage of the collective criteria is the reduced sensitivity to efficiency, since it is no longer necessary to measure the spin at each site. The depth of spin squeezing has been used recently and reported at this conference to infer blocks of entangled atoms in BEC condensates [13, 14].

To test nonlocality between sites, the criteria will need to involve measurements made at the different spatial locations. How to detect entanglement between two-modes using spin operators [63–70], and how to detect a true Einstein–Podolsky–Rosen (EPR) entanglement [15, 16, 71–74] in BEC [60, 73, 74] are topics of much current

interest.

5.3 EPR steering nonlocality with atoms

An interesting question is whether one derive criteria, involving collective operators, to determine whether there are stronger underlying nonlocalities. How can we infer whether the one group of atoms A can "steer" a second group B, as shown in schematic form in Fig. 7? This would confirm an EPR paradox between the two groups, that the correlations imply inconsistency between Local Realism (LR) and the completeness of quantum mechanics. This is an interesting task since very little experimental work has been done on confirming EPR paradoxes between even single atoms. Steering paradoxes between groups of atoms raise even more fundamental questions about mesoscopic quantum mechanics.
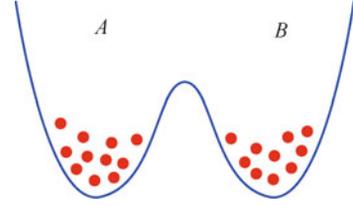
Fig. 7 Is "EPR steering" of one group of atoms by another group possible? How can we detect such steering?

As an example, we thus consider the following. EPR steering is demonstrated between N sites when the LHS model (9) with $T = 1$ fails. The system (which we will call B) at the one site corresponding to $T = 1$ is described by a local quantum state LQS, which means it is constrained by the uncertainty principle. All other groups are described by a Local Hidden Variable Theory (LHV), and thus are constrained to have only a non-negative variance. For this first group (only) there is the SM minimum variance (implied by quantum mechanics):

$$\Delta^2 J_B^X \geqslant J_B F_J(\langle J^Z \rangle/J_B) \qquad (60)$$

Hence, with this assumption, we follow the approach of Section 5. B, to write (where we assume the maximum spin of the steered group B is J_0)

$$\Delta^2 J^X \geqslant \sum_R P_R\{J_{R,B} F_{J_0}(\langle J^Z \rangle_{R,B}/J_{R,B})\}$$
$$= \sum_R P_R J_{\mathrm{tot}}^R, \frac{J_{R,B}}{J_{\mathrm{tot}}^R} F_{J_0}(\langle J^Z \rangle_{R,B}/J_{R,B})$$
$$\geqslant \sum_R P_R J_{\mathrm{tot}}^R F_{J_0}(\frac{\langle J^Z \rangle_{R,B}}{J_{\mathrm{tot}}^R})$$
$$= J_{\mathrm{tot}} \sum_R P_R \frac{J_{\mathrm{tot}}^R}{J_{\mathrm{tot}}} F_{J_0}(\frac{\sum_{i=1}^{N_R} \langle J^Z \rangle_{R,i}}{J_{\mathrm{tot}}^R})$$
$$\geqslant J_{\mathrm{tot}} F_{J_0}(\sum_R P_R \frac{1}{J_{\mathrm{tot}}} \langle J^Z \rangle_{R,B})$$
$$\geqslant J_{\mathrm{tot}} F_{J_0}(\frac{1}{J_{\mathrm{tot}}} \sum_R P_R \langle J^Z \rangle_{R,B})$$

$$= J_{\text{tot}} F_{J_0}(\langle J_B^Z \rangle / J_{\text{tot}}) \tag{61}$$

If the inequality is violated, a "steering" between the two groups is confirmed possible: group A "steers" group B. In this case, the spins of spatially separated systems B would need to be measured. Such "EPR" systems have been proposed, with a view to this sort of experiment in the future [60, 71, 72].

6 Conclusion

We have examined a strategy for testing multi-particle nonlocality, by first defining three distinct levels of non-locality: (i) entanglement, (ii) EPR paradox/ steering, or (ii) failure of local hidden variable (LHV) theories (which we call Bell's nonlocality). We then focused on two types of earlier studies that yielded information about nonlocality in systems of more than two particles.

The first study originated with Greenberger, Horne and Zeilinger (GHZ) and considers N spatially separated in 1/2 particles, on which individual spin measurements are made. The study revealed that nonlocality involving N spatially separated (spin 1/2) particles can be more extreme. Mermin showed that the deviation of the quantum prediction from the classical LHV boundaries can grow exponentially with N for this scenario. Here we have summarized some recent results by us that reveal similar features for entanglement and EPR steering non-localities. Inequalities are presented that enable detection of these nonlocalities in this multipartite scenario, for certain correlated quantum states. The results are also applicable to N spin J particles (or systems), and thus reveal nonlocality can survive for N systems even where these systems have a higher dimensionality.

We then examined the meaning of "multi-particle non-locality", in the sense originated by Svetlichny, that there is an "$N-$body" nonlocality, necessarily shared among *all N*systems. For example, three-particle entanglement is defined as an entanglement that cannot be modeled using two-particle entangled or separable states only. Such entanglement, generalized to N parties, is called genuine N partite entanglement. We present some recent inequalities that detect such genuine nonlocality for the GHZ/ Mermin scenario of N spin 1/2 particles, and show a higher threshold is required that will imply a much greater sensitivity to inefficiencies η. In other words, the depth of violation of the Bell or nonlocality inequalities determines the level of *genuine* multi-particle nonlocality.

This led to the final focus of the paper, which examined criteria that employ collective spin measurements. For example, the spin squeezing entanglement criterion of Sørensen *et al.* enables entanglement to be confirmed between N spin 1/2 particles, based on a reduction in the overall variance ("squeezing") of a single collective spin component. The criterion works because of the finite dimensionality of the spin Hilbert space, which means only higher spin systems − as can be formed from entangled spin 1/2 states − can have larger variances in one spin component, and hence smaller variances in the other. As shown by Sørensen and Mølmer, even greater squeezing of the spin variances will imply larger entanglement, between more particles. Hence the depth of spin squeezing, as with the depth of Bell violations in the GHZ Mermin example above, will imply genuine entanglement between a minimum number of particles. This result has recently been used to detect experimental multi-particle entanglement in BEC systems. We present a model of the ground state BEC for the two component system, calculating the extent of such multi-particle squeezing.

We make the final point that, while collective spin measurements are useful in detecting multi-particle entanglement and overcoming problems with detection inefficiencies, the method does not address tests of non-locality unless the measured systems can be at least in principle spatially separated. This provides motivation for studies of entanglement and EPR steering between groups of atoms in spatially distinct environments.

References

1. J. S. Bell, Physics, 1964, 1: 195
2. J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett., 1969, 23(15): 880
3. E. Schrödinger, Naturwiss., 1935, 23: 807
4. E. Schrödinger, Math. Proc. Camb. Philos. Soc., 1935, 31(04): 555
5. E. Schrödinger, Math. Proc. Camb. Philos. Soc., 1936, 32(03): 446
6. A. J. Leggett, Supplement of the Progress of Theoretical Physics, 1980, 69: 80
7. A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev., 1935, 47(10): 777
8. D. M. Greenberger, M. Horne, and A. Zeilinger, in Bell's Theorem, Quantum Theory and Conceptions of the Universe, edited by M. Kafatos, Dordrecht, The Netherlands: Kluwer Academic, 1989
9. N. D. Mermin, Phys. Today, 1990, 43: 9
10. N. D. Mermin, Phys. Rev. Lett., 1990, 65(15): 1838
11. G. Svetlichny, Phys. Rev. D, 1987, 35(10): 3066
12. D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett., 2002, 88(17): 170405
13. C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature, 2010, 464(7292): 1165
14. M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature, 2010, 464(7292): 1170

15. M. D. Reid, P. D. Drummond, W. P. Bowen, P. K. Lam, H. A. Bachor, U. L. Andersen, and G. Leuchs, Rev. Mod. Phys., 2009, 81(4): 1727

16. M. D. Reid, Phys. Rev. A, 1989, 40(2): 913

17. A. Peres, Phys. Rev. Lett., 1996, 77(8): 1413

18. H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett., 2007, 98(14): 140402

19. S. J. Jones, H. M. Wiseman, and A. C. Doherty, Phys. Rev. A, 2007, 76(5): 052116

20. E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A, 2009, 80(3): 032112

21. M. J. Mallon, M. D. Reid, and M. K. Olsen, J. Phys. B, 2008, 41(1): 015501

22. B. S. Cirel'son, Lett. Math. Phys., 1980, 4: 83

23. D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Nat. Phys., 2010, 6(11): 845

24. E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, Phys. Rev. A, 2011, 84(3): 032115

25. R. F. Werner, Phys. Rev. A, 1989, 40(8): 4277

26. A. Garg and N. Mermin, Phys. Rev. D, 1983, 27(2): 339

27. N. D. Mermin, Phys. Rev. D, 1980, 22(2): 356

28. P. D. Drummond, Phys. Rev. Lett., 1983, 50(19): 1407

29. A. Peres, Found. Phys., 1999, 29(4): 589

30. A. Peres, Phys. Rev. A, 1992, 46(7): 4413

31. N. Gisin and A. Peres, Phys. Lett. A, 1992, 162(1): 15

32. M. D. Reid, W. J. Munro, and F. De Martini, Phys. Rev. A, 2002, 66(3): 033801

33. D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett., 2000, 85(21): 4418

34. D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett., 2002, 88(4): 040404

35. L. B. Fu, Phys. Rev. Lett., 2004, 92(13): 130404

36. A. Acin, T. Durt, N. Gisin, and J. I. Latorre, Phys. Rev. A, 2002, 65(5): 052325

37. J. L. Chen, C. F. Wu, L. C. Kwek, C. H. Oh, and M. L. Ge, Phys. Rev. A, 2006, 74(3): 032106

38. M. Ardehali, Phys. Rev. A, 1992, 46(9): 5375

39. A. V. Belinskii and D. N. Klyshko, Physics-Uspekhi, 1993, 36: 654

40. R. F. Werner and M. M. Wolf, Phys. Rev. A, 2001, 64(3): 032112

41. S. M. Roy, Phys. Rev. Lett., 2005, 94(1): 010402

42. E. G. Cavalcanti, C. J. Foster, M. D. Reid, and P. D. Drummond, Phys. Rev. Lett., 2007, 99(21): 210405

43. E. Shchukin and W. Vogel, Phys. Rev. A, 2008, 78(3): 032104

44. A Salles, D. Cavalcanti, and A. Acin, Phys. Rev. Lett., 2008, 101(4): 040404

45. Q. Y. He, E. G. Cavalcanti, M. D. Reid, and P. D. Drummond, Phys. Rev. Lett., 2009, 103(18): 180402

46. Q. Y. He, E. G. Cavalcanti, M. D. Reid, and P. D. Drummond, Phys. Rev. A, 2010, 81(6): 062106

47. Q. Y. He, Shi-Guo Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A, 2011, 84(2): 022107

48. Q. Y. He, P. D. Drummond, and M. D. Reid, Phys. Rev. A, 2011, 83(3): 032120

49. A. Cabello, Phys. Rev. A, 2002, 65(6): 062105

50. W. Son, J. Lee, and M. S. Kim, Phys. Rev. Lett., 2006, 96(6): 060406

51. J. L. Chen and D. L. Deng, Phys. Rev. A, 2009, 79(1): 012111

52. S. Ghose, N. Sinclair, S. Debnath, P. Rungta, and R. Stock, Phys. Rev. Lett., 2009, 102(25): 250404

53. A. Ajoy and P. Rungta, Phys. Rev. A, 2010, 81(5): 052334

54. G. Toth and O. Guhne, Phys. Rev. A, 2005, 72(2): 022340

55. M. Kitagawa and M. Ueda, Phys. Rev. A, 1993, 47(6): 5138

56. D. J. Wineland, J. Bollinger, W. M. Itano, and D. J. Heinzen, Phys. Rev. A, 1994, 50(1): 67

57. H. F. Hofmann and S. Takeuchi, Phys. Rev. A, 2003, 68(3): 032103

58. A. Sørenson, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature, 2001, 409: 63

59. J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature, 2008, 455: 1216

60. Q. Y. He, M. D. Reid, C. Gross, M. Oberthaler, and P. D. Drummond, Phys. Rev. Lett., 2011, 106(12): 120405

61. A. S. Sørensen and K. Mølmer, Phys. Rev. Lett., 2001, 86(20): 4431

62. F. Benatti, R. Floreanini, and U. Marzolino, J. Phys. B, 2011, 44(9): 091001

63. M. Hillery and M. S. Zubairy, Phys. Rev. Lett., 2006, 96(5): 050503

64. E. Shchukin and W. Vogel, Phys. Rev. Lett., 2005, 95(23): 230502

65. G. Toth, C. Knapp, O. Guhne, and H. Briegel, Phys. Rev. Lett., 2007, 99(25): 250405

66. A. Bednorz and W. Belzig, Phys. Rev. B, 2011, 83(12): 125304

67. J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett., 2005, 95(12): 120502

68. E. G. Cavalcanti, P. D. Drummond, H. A. Bachor, and M. D. Reid, Opt. Express, 2009, 17(21): 18693

69. G. A. Durkin and C. Simon, Phys. Rev. Lett., 2005, 95(18): 180402

70. V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A, 2003, 67(2): 022320

71. K. Dechoum, S. Chaturvedi, P. D. Drummond, and M. D. Reid, Phys. Rev. A, 2004, 70(5): 053807

72. E. G. Cavalcanti and M. D. Reid, J. Mod. Opt., 2007, 54: 2373

73. A. J. Ferris, M. K. Olsen, E. G. Cavalcanti, and M. J. Davis, Phys. Rev. A, 2008, 78(6): 060104

74. N. Bar-Gill, C. Gross, I. Mazets, M. Oberthaler, and G. Kurizki, Phys. Rev. Lett., 2011, 106(12): 120404