Front. Phys., 2012, 7(1): 8–15
DOI 10.1007/s11467-011-0213-0

RESEARCH ARTICLE

Polaron, molecule and pairing in one-dimensional spin-1/2 Fermi gas with an attractive Delta-function interaction

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Received September 1, 2011; accepted September 20, 2011

Using solutions of the discrete Bethe ansatz equations, we study in detail the quantum impurity problem of a spin-down fermion immersed into a fully ploarized spin-up Fermi sea with weak attraction. We prove that this impurity fermion in the one-dimensional (1D) fermionic medium behaves like a polaron for weak attraction. However, as the attraction grows, the spin-down fermion binds with one spin-up fermion from the fully-polarized medium to form a tightly bound molecule. Thus it is seen that the system undergos a crossover from a mean field polaron-like nature into a mixture of excess fermions and a bosonic molecule as the attraction changes from weak attraction into strong attraction. This polaron–molecule crossover is universal in 1D many-body systems of interacting fermions. In a thermodynamic limit, we further study the relationship between the Fredholm equations for the 1D spin-1/2 Fermi gas with weakly repulsive and attractive delta-function interactions.

Keywords spin-1/2 Fermi gas, polaron, molecule, Fredholm equations

PACS numbers 03.75.Ss, 03.75.Hh, 02.30.Ik, 05.30.Rt

1 Introduction

The study of one-dimensional (1D) spin-1/2 Delta-function interacting Fermi gas [1, 2] is an active area of research in the field of cold atoms. The fundamental physics of the model with arbitrary spin population imbalance are determined by the set of transcendental equations which were found by Yang [1] using the Bethe ansatz hypothesis in 1967. The model displays a remarkable Fulde–Ferrell–Larkin–Ovchinnikov (FFLO)-like pairing [3–8], quantum phase transitions and quantum critical phenomena [9–11]. It has a novel phase diagram caused by a difference in the number of spin-up and spin-down atoms [12–16]. The key feature of the phase diagram was experimentally confirmed by Liao *et al.* [17] at Rice University in the strongly attractive regime of fermionic ^{6}Li atoms confined to the two lowest sub-hyperfine states.

In 3D, for weak attraction limit, a spin-down fermion propagates almost freely in a spin-up medium. As attraction increases the spin-down atom is dressed with the localized cloud of scattered surrounding fermions constituting the Fermi polaron [18–20]. However, if one considers a small portion of spin-down fermions immersed into a fully polarized spin-up medium with strong attraction, it is seen that the system undergos a phase transition from a fully-polarized Fermi gas into a mixture of excess fermions and bosonic molecules [21, 22].

Using variational ansatz Parish showed that there does not exist a true polaron–molecule transition in the 1D highly polarized Fermionic system [23]. In fact, this variational ansatz is not valid to capture the quasiparticle behaviour for the 1D interacting Fermi gas because it gives a divergent integral. The polaron-like effect can persist in the 1D many-body system [24]. In the present paper, using asymptotic solution of the Bethe ansatz equations, we present an analytical study of the quantum impurity problem in 1D Fermionic medium. We prove that a spin-down fermion immersed into fully polarized spin-up medium with weak attraction is dressed with surrounding fermions and behaves like a Fermi polaron, also see [25]. The spin-down fermions receive a mean field from the fully-polarized Fermi sea. In this limit, decoupling the two spin components gives a polaron-like quasiparticle associating with the spin-down fermion dressed by the particle–hole excitations. The mean field binding energy and the effective mass of the polaron can be analytically calculated from the discrete Bethe ansatz equations. However, as an attractive interaction grows, the spin-down fermion binds with one spin-up fermion from the medium to form a tightly bound molecule. The crossover is evidenced by the changes from a mean field attractive binding energy of the polaron with an effective mass $m^* = m$ to the binding energy of the single molecule with an effective mass $m^* = 2m$ as the attraction grows from $c = 0$ to $c = -\infty$. Here m is the actual mass of the fermions, and c is the interaction strength. Thus the system undergos a cross-over from a mean field polaronic nature into a mixture of excess fermions and a bosonic molecule as the attraction changes from weak into strong attractions. Furthermore, we discuss the relationship between the Fredholm equations for the 1D spin-1/2 Fermi gas with weakly repulsive and attractive Delta-function interactions.

2 The model

The model Hamiltonian [1, 2]

$$\mathcal{H} = \sum_{\sigma=\downarrow,\uparrow} \int \phi_\sigma^\dagger(x)\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu_\sigma\right)\phi_\sigma(x)\mathrm{d}x + g_{1\mathrm{D}}\int \phi_\downarrow^\dagger(x)\phi_\uparrow^\dagger(x)\phi_\uparrow(x)\phi_\downarrow(x)\mathrm{d}x \tag{1}$$

describes 1D δ-function interacting spin-$\frac{1}{2}$ Fermi gas of N fermions with mass m by periodic boundary conditions to a line of length L. The field operators $\phi_\downarrow$ and $\phi_\uparrow$ describe the fermionic atoms in the states $|\uparrow\rangle$ and $\downarrow\rangle$, respectively. We use units of $\hbar = 2m = 1$ and denote coupling constant $g_{1\mathrm{D}} = \hbar^2 c/m$ with $c = -2/a_{1\mathrm{D}}$ where a_{1D} is the effective 1D scattering length [26] $a_{1\mathrm{D}} = -\frac{a_\perp^2}{a_{3\mathrm{D}}} + Aa_\perp$. Here $a_{3\mathrm{D}}$ is the 3D scattering length, $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the transverse oscillator length, and $A \approx 1.0326$ is a numerical constant. For repulsive fermions, $c > 0$ and for attractive fermions, $c < 0$.

For an irreducible representation $R_\psi = [2^{N_\downarrow}, 1^{N_\uparrow - N_\downarrow}]$ [27], where $N_\uparrow$ and $N_\downarrow$ are the numbers of fermions at the two hyperfine levels $|\uparrow\rangle$ and $|\downarrow\rangle$ such that $N_\uparrow \geqslant N_\downarrow$. The energy eigenspectrum is given in terms of the quasimomenta $\{k_i\}$ of the fermions via $E = \frac{\hbar^2}{2m}\sum_{j=1}^N k_j^2$, satisfying the BA equations

$$\exp(\mathrm{i}k_i L) = \prod_{\alpha=1}^{M_1} \frac{k_i - \lambda_\alpha + \mathrm{i}c'}{k_i - \lambda_\alpha - \mathrm{i}c'} \tag{2}$$

$$\prod_{j=1}^{N} \frac{\lambda_\alpha - k_j + \mathrm{i}c'}{\lambda_\alpha - k_j - \mathrm{i}c'} = -\prod_{\beta=1}^{M_1} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}c}{\lambda_\alpha - \lambda_\beta - \mathrm{i}c} \tag{3}$$

$$i = 1, 2, \cdots, N; \quad \alpha = 1, 2, \cdots, M_1$$

with the quantum number $M_1 = N_\downarrow$ and a notation $c' = c/2$. The parameters $\{\lambda_\alpha\}$ are the rapidities for the internal hyperfine spin degrees of freedom.

In thermodynamic limits, and attractive regime, i.e., $c < 0$, quasimomenta $\{k_i\}$ of the fermions with different spins form two-body bound states, i.e., $k_\alpha = \lambda_\alpha \pm \mathrm{i}\frac{1}{2}c$, accompanied by the real spin parameter λ_α [28, 29]. Here $\alpha = 1, \cdots, M_1$. The excess fermions have real quasimomenta $\{k_j\}$ with $j = 1, \cdots, N - 2M_1$. From these root patterns, the BA equations (2) and (3) become

$$\exp(\mathrm{i}k_i L) = \prod_{\alpha=1}^{M_1} \frac{k_i - \lambda_\alpha + \mathrm{i}c'}{k_i - \lambda_\alpha - \mathrm{i}c'} \tag{4}$$

$$\exp(2\mathrm{i}\lambda_\alpha L) = \prod_{\ell=1}^{N-2M_1} \frac{\lambda_\alpha - k_\ell + \mathrm{i}c'}{\lambda_\alpha - k_\ell - \mathrm{i}c'} \prod_{\beta=1}^{M_1} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}c}{\lambda_\alpha - \lambda_\beta - \mathrm{i}c} \tag{5}$$

$$i = 1, 2, \cdots, N - 2M_1; \quad \beta = 1, 2, \cdots, M_1$$

In the above equations $\alpha \neq \beta$.

3 Polaron–molecule crossover

3.1 Polaron-like state

McGuire studied exact eigenvalue problem of $N - 1$ Fermions of the same spin and one fermion of the opposite spin in 1965 and 1966 [30, 31]. He calculated the energy shift caused by this extra spin-down fermion. The highly polarized Fermi system was studied recently by Giraud and Combescot [32] in the context of Fermi polarons. The polaron-like effect in the 1D Fermi–Hubbard model was studied by the variational anastz in Ref. [24]. Here we prove that a single spin-down fermion immersed into the a fully ploarized spin-up Fermi sea with weak attraction is likely to behave like a polaron in such fermionic medium. For the weak coupling limit $L|c| \ll 1$, we find that either the spin-down fermion and a spin-up fermion from the medium form a pair $k_{\downarrow,\uparrow} = p \pm \mathrm{i}\beta$ or the spin-down fermion with a quasimomentum $k_\downarrow = p$ propagates in the medium. Whereas the rest are $N - 2$ real roots $\{k_i\}$ with $i = 1, \cdots, N - 2$.

We consider the quasimomenta of a pair $k_{\downarrow,\uparrow} = p \pm \mathrm{i}\beta$ and $N - 2$ real roots $\{k_i\}$ with $i = 1, \cdots, N - 2$. From the discrete Bethe ansatz equations (2), we have

$$\begin{aligned}
\mathrm{e}^{2\mathrm{i}pL} &= \frac{k_\downarrow - p + \mathrm{i}c'}{k_\downarrow - p - \mathrm{i}c'}\frac{k_\uparrow - p + \mathrm{i}c'}{k_\uparrow - p - \mathrm{i}c'} \\
\mathrm{e}^{-2\beta L} &= \frac{k_\downarrow - p + \mathrm{i}c'}{k_\downarrow - p - \mathrm{i}c'}\frac{k_\uparrow - p - \mathrm{i}c'}{k_\uparrow - p + \mathrm{i}c'} \\
\mathrm{e}^{\mathrm{i}k_i L} &= \frac{k_i - p + \mathrm{i}c'}{k_i - p - \mathrm{i}c'}
\end{aligned} \tag{6}$$

with $i = 1, \cdots, N - 2$. From the second equation in (6), we determine the imaginary part β from

$$\beta L = \mathrm{arctanh}\frac{\beta|c|}{\beta^2 + c^2/4} \tag{7}$$

We see that for weak coupling limit $\beta \to \sqrt{|c|/L}$ whereas for strong coupling limit $\beta \to |c|/2$. From the Bethe ansatz equations (3), we obtain

$$\frac{p-k_\downarrow+\mathrm{i}c'}{p-k_\downarrow-\mathrm{i}c'}\frac{p-k_\uparrow+\mathrm{i}c'}{p-k_\uparrow-\mathrm{i}c'}\prod_{\ell=1}^{N-2}\frac{p-k_\ell+\mathrm{i}c'}{p-k_\ell-\mathrm{i}c'}=1 \tag{8}$$

Using the BA equations (6) and (8), we find that in weak coupling limit the roots satisfy the following polynomial equations

$$k_i \approx \frac{2n_i\pi}{L}-\frac{|c|}{L(k_i-p)} \tag{9}$$

$$p \approx \frac{2n_p\pi}{L}-\frac{|c|}{2L}\sum_{\ell=1}^{N-2}\frac{1}{(p-k_\ell)} \tag{10}$$

with $i=1,\cdots,N-2$. According to the Fermi statistics, here $n_i=\pm1,\pm2,\cdots,(N-2)/2$ and n_p is an integer. The ground state configuration corresponding to $n_p=0$. In the weak coupling limit, we have $\beta=\sqrt{|c|/L}$ so that $p\gg\beta$. Thus we see that the bound pair is not essential in the limit $L|c|\ll 1$. The key feature of the model in this limit is that the spin-down fermion receives a mean field from the fully-polarized Fermi sea. We consider the case $n_p\neq 0$, i.e., $p\neq 0$ for excitations. From Eq. (9), we can calculate the energy of the system with a single spin-down fermion

$$\begin{aligned}E &\approx -2\beta^2+2p^2+\sum_{i=1}^{N-2}k_i^2\\ &\approx -2\beta^2-\frac{2|c|}{L}(N-2)+p^2+\sum_{n=1}^{N_\uparrow/2}\frac{8(n\pi)^2}{L^2}\\ &\quad -4|c|\sum_{i=1}^{\frac{N_\uparrow}{2}-1}\frac{p^2}{L(k_i^2-p^2)}\end{aligned} \tag{11}$$

In the above equations, symmetrization of the $N-2$ quansimomenta in the medium was considered, i.e., the real roots associating with $N-2$ spin-up fermions can be symmetrized by $k_i\approx -k_j$ up to the order of c. This is mainly because the quansimomenta of the spin-up fermions only have an order of $|c|$ deviation from the ones of the free spin-up fermions. For $p=0$, the result (11) coincides with the ground state energy of the Fermi gas with one spin-down fermion given by McGuire [30, 31] (on page 125). If we consider the excitations for the weakly bound pair in the surrounding fully-polarized Fermi sea, i.e., $p\neq 0$, we can find an explicit relation of the energy depending on the total momentum of the system. Defining the total momentum of the system q, thus in weak coupling limit, we find a relation between p and q as

$$p\approx q\Big/\left[1-2|c|\sum_{i=1}^{\frac{N_\uparrow}{2}-1}\frac{1}{L(k_i^2-p^2)}\right] \tag{12}$$

Substituting Eq. (12) into Eq. (11), the last term in Eq. (11) is cancelled out. Then we obtain an energy shift

$$E(q,N,N_\downarrow=1)-E_\uparrow(N_\uparrow,0)\approx\epsilon_{p-b}+\frac{\hbar^2q^2}{2m^*} \tag{13}$$

that behaves like a quasi-particle polaron. Where $E_\uparrow(N_\uparrow,0)=\frac{\hbar^2}{2m}\frac{1}{3L^2}N_\uparrow^3\pi^2$ is the kinetic energy of $N_\uparrow$ spin-up fermions. It is interesting to note that the attractive mean field binding energy

$$\epsilon_{p-b}\approx\frac{\hbar^2}{2m}\left(-2\beta^2-\frac{2|c|}{L}(N-2)\right)\approx-\frac{\hbar^2}{m}n_\uparrow|c| \tag{14}$$

depends on the number of spin-up fermions and interaction strength [18]. In the above equation (13), the polaron-like state with an effective mass $m^*\approx m(1+O(c^2))$ is almost the same as the actual mass of the fermions in the limit $L|c|\ll 1$. This is consistent with the result in Refs. [30–32]. The last term of the equation (11) indicates that an attractive interaction always enhances the effective mass of the polaron. The addressed "binding energy" can be rewritten as

$$\epsilon_{p-b}=-\frac{6}{\pi^2}e_{\mathrm{F}}|\gamma| \tag{15}$$

where the dimensionless interaction strength is defined $\gamma=c/n$. The Fermi energy is $e_{\mathrm{F}}=\frac{\hbar^2}{2m}\frac{1}{3}n^2\pi^2$. This binding energy indicates a mean field effect. This is a 1D analog of Fermi polaron-like state resulted from the weak attraction between the impurity and the fully polarized Fermionic medium. This polaron-like state also exists for weakly repulsive interaction, where the spin-down fermion experiences a repulsive mean field energy shift.

The mean field polaron-like state occurs for a few spin-down fermions immersed into a fully polarized Fermionic sea. In a thermodynamic limit and in a weak coupling regime, i.e., $cL/N\sim 1$, the ground state is the BCS-like pairing state with a pairing correlation length larger than the average interparticle spacing. The correlation function for the single particle Green's function decays exponentially, i.e., $\langle\psi_{x,s}^\dagger\psi_{1,s}\rangle\to \mathrm{e}^{-x/\xi}$ with $\xi=v_{\mathrm{F}}/\varDelta$ and $s=\uparrow,\downarrow$, whereas the singlet pair correlation function decays as a power of distance, i.e., $\langle\psi_{x,\uparrow}^\dagger\psi_{x,\downarrow}^\dagger\psi_{1,\uparrow}\psi_{1,\downarrow}\rangle\to x^{-\theta}$. Here $\varDelta$ is the energy gap, and the critical exponents ξ and θ are both greater than zero. However, once the external field exceeds the critical value, the Cooper pairs are destroyed. Thus both of these correlation functions decay as a power of distance and the pairs lose their dominance, where, molecule and excess fermions form the polarized FFLO pairing-like phase. In this phase the spacial oscillations of pairing correlation are caused by an imbalance in the densities of spin-up and spin-down fermions, i.e., $n_\uparrow-n_\downarrow$, which gives rise to a mismatch in Fermi surfaces between both species of fermions. In 1D, the pair and spin correlations with the spacial oscillation signature are a consequence of the backscattering for bound pairs and unpaired fermions [34]. In the next section, we will see that the polaronic signature is significantly different from the molecule state where a single spin-down fermion and a spin-up fermion from the medium with a strong attraction form a tight bound molecule of two-atom.

3.2 The molecule state

In order to catch the signature of molecule state, we consider a spin-down fermion immersed into fully polarized spin-up medium with strong attraction, i.e., $L|c| \gg 1$. We assume the bound pair $k_{\downarrow,\uparrow} = p \pm \mathrm{i}\beta$ and $N-2$ real roots $\{k_i\}$ with $i = 1, \cdots, N-2$. From (2) and (3) with an odd number $N_\uparrow$, we find the real roots

$$k_i \approx \frac{n_j \pi}{L}\left(1 - \frac{4}{L|c|}\right)^{-1} - \frac{4P}{L|c|}\left(1 - \frac{4}{L|c|}\right)^{-1} \quad (16)$$

with $n_j = \pm 1, \pm 3, \cdots, \pm(N_\uparrow - 1)$. The pair excitations correspond to $p \neq 0$. Then we obtain the energy

$$E = -2\beta^2 + 2p^2 + \sum_{i=1}^{N-2} k_i^2$$
$$= -2\beta^2 + 2p^2 + E_\uparrow(N_\downarrow, 0) + \frac{16p^2(N_\uparrow - 1)}{L^2c^2} \quad (17)$$

In the above equation,

$$E_\uparrow(N_\uparrow, 0) = \frac{\hbar^2}{2m}\frac{1}{3L^2}N_\uparrow^3\pi^2\left(1 - \frac{4}{L|c|}\right)^{-2} \quad (18)$$

is the kinetic energy of $N_\uparrow$ spin-up fermions with $N_\downarrow = 1$. If we consider the pair excitations with total momentum q, we find the relation, i.e., between p and the total momentum of the system q,

$$p \approx q/\left[2\left(1 - \frac{2(N_\uparrow - 2)}{L|c|}\right)\right] \quad (19)$$

Substituting Eq. (19) into Eq. (17) , then we obtain an energy shift

$$E(q, N, N_\downarrow = 1) - E_\uparrow \approx \varepsilon_b^t + \frac{\hbar^2 q^2}{2m^*} - \Delta\mu \quad (20)$$

that behaves like a molecule with a binding energy

$$\varepsilon_b^t = -2\beta^2 + \frac{8\pi^2}{3|\gamma|} \approx -\frac{\hbar^2}{2m}\frac{c^2}{2} + \frac{8\pi^2}{3|\gamma|} \quad (21)$$

in the strong attractive regime $L|c| \gg 1$. In the above equations $E_\uparrow = \frac{\hbar^2}{2m}\frac{1}{3L^2}N_\uparrow^3\pi^2$ is the kinetic energy of $N_\uparrow$ spin-up fermions. However, the effective mass of the molecule

$$m^* \approx 2m\left(1 - \frac{4(N_\uparrow - 2)}{L|c|}\right) \quad (22)$$

is almost twice the actual mass of the fermions. In the above equations, the variation of the chemical potential $\Delta\mu = n_\uparrow^2\pi^2$. We see clearly that the system undergos a cross-over from a mean field polaron-like nature into a mixture of excess fermions and a bosonic molecule as the attraction changes from a weak attraction into a strong one.

4 Ground state energy

4.1 Weak attraction

In order to see physical signature of the ground state energy at vanishing interaction strength, we first focus on weakly attractive interaction in which two fermions with spin-up and spin-down states form a weakly bound pair. In this regime, the weak bound pair is not stable because the kinetic energy of the pair is larger than the binding energy. In the limit $c \to 0^-$, the unpaired fermions sit on two outer wings in the quasimomentum space due to the Fermi statistics. In this weak coupling limit, i.e., $L|c| \ll 1$, the imaginary part of the pseudomomenta for a BCS pair is proportional to $\sqrt{c/L}$. Thus the bound state has a small binding energy $\epsilon_\mathrm{b} = \hbar^2|c|/mL$ that is proportional to $|c|$. For arbitrary polarization, the system is described by M_1 weakly bound pairs with $k_\alpha^\mathrm{p} \approx \lambda_\alpha \pm \mathrm{i}\sqrt{c/L}$ and $N - 2M_1$ unpaired fermions with real k_i. Without losing generality, we assume M_1 is odd and N is even. Substituting this root patterns into the BA equations (2) and (3), we obtain the following equations to determine the positive roots $\{\lambda_\alpha\}$ and the positive real quasimomenta $\{k_j\}$

$$k_j \approx \frac{2n_j\pi}{L} + \frac{c}{Lk_j} + \frac{c}{L}\sum_{\alpha=1}^{\frac{1}{2}(M_1-1)} \frac{2k_j}{k_j^2 - \lambda_\alpha^2} \quad (23)$$

$$\lambda_\alpha \approx \frac{2n_\alpha\pi}{L} + \frac{3c}{2L\lambda_\alpha} + \frac{c}{L}\sum_{\beta=1}^{\frac{1}{2}(M_1-1)} \frac{2\lambda_\alpha}{\lambda_\alpha^2 - \lambda_\beta^2}$$
$$+ \frac{c}{2L}\sum_{j=1}^{\frac{1}{2}(N-2M_1)} \frac{2\lambda_\alpha}{\lambda_\alpha^2 - k_j^2} \quad (24)$$

where $n_j = \frac{M_1+1}{2}, \frac{M+3}{2}, \cdots, \frac{N-M_1-1}{2}$, and $n_\alpha = 1, 2, \cdots, M_1/2$. In the equation (24), $\alpha = \beta$ is excluded. By iteration, we obtain the ground state energy for weakly attractive regime

$$E = -\frac{2M_1|c|}{L} + 4\sum_{\alpha=1}^{\frac{1}{2}(M_1-1)} \lambda_\alpha^2 + 2\sum_{j=\frac{1}{2}(M_1+1)}^{\frac{1}{2}(N-M_1-1)} k_j^2$$
$$= -\frac{2|c|(N - M_1)M_1}{L} + \frac{2\pi^2 M_1(M_1^2 - 1)}{3L^2}$$
$$+ \frac{\pi^2(N - 2M_1)[N^2 + M_1^2 - M_1N - 1]}{3L^2} + O(c^2) \quad (25)$$

We denote the linear density $n = N/L$ and the density of down-spin fermions $n_\downarrow = M_1/L$. In a thermaldynamic limit, the ground state energy per length (25) becomes

$$\frac{E}{L} = \frac{1}{3}n_\uparrow^3\pi^2 + \frac{1}{3}n_\downarrow^3\pi^2 + 2cn_\uparrow n_\downarrow + O(c^2) \tag{26}$$

that agrees with the result given in Ref. [33]. The ground state energy (25) is also valid for weakly repulsive interaction, i.e., for $c > 0$. This gives a mean field effect for the 1D delta-function interacting Fermi gas in weak coupling regime.

4.2 Strong attraction

For strong attraction, i.e., $L|c| \gg 1$, (or say $c \gg k_{\mathrm{F}}$), the discrete BA equations (2) and (3) with the root patterns $k_\alpha^{\mathrm{p}} = \lambda_\alpha \pm \mathrm{i}\frac{1}{2}c$ for pairs and k_j^{u} for unpaired fermions can be linearized. For even $N_\downarrow$, we obtain the momenta of tightly bound pairs and excess fermions [33]

$$k_\alpha^{(\mathrm{p})} \approx \frac{n_\alpha \pi}{2L}\left(1 + \frac{N_\downarrow}{Lc} + \frac{2(N-2N_\downarrow)}{Lc}\right)^{-1} \pm \frac{1}{2}\mathrm{i}c \tag{27}$$

$$k_j^{(\mathrm{u})} \approx \frac{n_j \pi}{L}\left(1 + \frac{4N_\downarrow}{Lc}\right)^{-1} \tag{28}$$

with integers $n_j = \pm 1, \pm 3, \cdots, \pm(N - 2N_\downarrow - 1)$ and $n_\alpha = \pm 1, 2, \cdots, \pm(N_\downarrow - 1)$. In this scenario the bound states behave like hard-core bosons due to Fermi statistics. The per length ground state energy of the model with strong attraction and arbitrary polarization is given by

$$\frac{E}{L} = E_0^u + E_0^b + n_\downarrow \varepsilon_b^t \tag{29}$$

where $n_\downarrow = N_\downarrow/L$ and the binding energy $\varepsilon_b^t = -\frac{c^2}{2}$ and the effective energy for unpaired fermions and pairs

$$E_0^u \approx \frac{(n-2n_\downarrow)^3\pi^2}{3}\left(1 + \frac{8n_\downarrow}{|c|} + \frac{48n_\downarrow^2}{c^2}\right) \tag{30}$$

$$E_0^b \approx \frac{n_\downarrow^3\pi^2}{6}\left[1 + \frac{2(2n_\uparrow - n_\downarrow)}{|c|} + \frac{3(2n_\uparrow - n_\downarrow)^2}{c^2}\right] \tag{31}$$

From the energies (30) and (31), we see that the bound pairs have tails and interfere with each other. But, it is impossible to separate the intermolecular forces from the interference between molecules and single fermions.

5 Relationship between the two sets of the Fredholm equations

The fundamental physics of the model are determined by the set of transcendental equations which can be transformed to the generalised Fredholm equations in the thermodynamic limit. This transformation was found by Yang and Yang in series of papers on the study of spin XXZ model in 1966 (see an insightful article by Yang [35]). For repulsive interaction, the Bethe ansatz quasi-momenta $\{k_i\}$ are real, but all $\{\lambda_\alpha\}$ are real only for the ground state. There are complex roots of λ_α called spin strings for excited states. In the thermodynamic limit, i.e., $L, N \to \infty$, N/L is finite, the above Bethe ansatz equations (2) and (3) can be written as the generalized Fredholm equations

$$r_1(k) = \frac{1}{2\pi} + \int_{-B_2}^{B_2} K_1(k-k')r_2(k')\mathrm{d}k' \tag{32}$$

$$r_2(k) = \int_{-B_1}^{B_1} K_1(k-k')r_1(k')\mathrm{d}k - \int_{-B_2}^{B_2} K_2(k-k')r_2(k')\mathrm{d}k' \tag{33}$$

where the integration boundaries B_1, B_2 are determined by

$$n :\equiv N/L = \int_{-B_1}^{B_1} r_1(k)\mathrm{d}k, \quad M_1/L = \int_{-B_2}^{B_2} r_2(k')\mathrm{d}k' \tag{34}$$

In the above equations, we denote the function

$$K_m(x) = \frac{1}{2\pi}\frac{mc}{(mc/2)^2 + x^2} \tag{35}$$

with $c > 0$ for repulsive regime and $c < 0$ for attractive regime. The ground state energy per unit length is given by

$$E = \int_{-B_1}^{B_1} k^2 r_1(k)\mathrm{d}k \tag{36}$$

The functions $r_m(k)$ denote the Bethe ansatz root distributions in parameter spaces, i.e., $r_1(k)$ stands for quasi-momenta distribution function, whereas $r_2(k)$ are the distribution functions for rapidity parameter λ in the BA equations (2) and (3). The ground state energy and full phase diagram can be obtained by solving analytically the Fredholm equations.

For attractive regime, i.e., $c < 0$, the BA equations (4) and (5) become

$$k_j L = 2\pi I_j - \sum_{l=1}^{M_1} \theta\left(\frac{k_j - \lambda_l}{c'}\right) \tag{37}$$

$$2\lambda_j L = 2\pi J_j - \sum_{l=1}^{N-2M_1} \theta\left(\frac{\lambda_j - k_l}{c'}\right) - \sum_{l=1}^{M_1} \theta\left(\frac{\lambda_j - \lambda_l}{2c'}\right) \tag{38}$$

$$j = 2M+1, \cdots, N; \quad j = 1, \cdots, M_1$$

where $\theta(x) = 2\arctan x$, and $I_j = -(N - 2M_1 - 1)/2, -(N - 2M_1 - 3)/2, \cdots, (N - 2M_1 - 1)/2$ and $J_j = -(M_1 - 1)/2, \cdots, (M_1 - 3)/2, (M_1 - 1)/2$. In the thermodynamic limit, we introduced the density of unpaired fermions $\rho_1(k) = dI_j(k)/(Ldk)$ and the density of pairs $\rho_2(k) = \mathrm{d}J_j(k)/(L\mathrm{d}k)$. They satisfy the following Fredholm equations [28, 29]

$$\rho_1(k) = \frac{1}{2\pi} + \int_{-Q_2}^{Q_2} K_1(k-k')\rho_2(k')\mathrm{d}k' \tag{39}$$

$$\rho_2(k) = \frac{2}{2\pi} + \int_{-Q_1}^{Q_1} K_1(k-k')\rho_1(k')\mathrm{d}k' + \int_{-Q_2}^{Q_2} K_2(k-k')\rho_2(k')\mathrm{d}k' \tag{40}$$

The linear densities are defined by

$$\frac{N}{L} = 2\int_{-Q_2}^{Q_2} \rho_2(k)\mathrm{d}k + \int_{-Q_1}^{Q_1} \rho_1(k)\mathrm{d}k$$
$$\frac{M_1}{L} = \int_{-Q_2}^{Q_2} \rho_2(k)\mathrm{d}k \tag{41}$$

The ground state energy per length is given by

$$E = \int_{-Q_2}^{Q_2} \left(2k^2 - c^2/2\right)\rho_2(k)\mathrm{d}k + \int_{-Q_1}^{Q_1} k^2\rho_1(k)\mathrm{d}k \tag{42}$$

We will investigate the relationship between the two sets of the Fredholm equations for the Fermi gas with repulsive and attractive delta-function interactions.

In the light of Takahashi's unification of the ground state energy of the spin-1/2 weakly interacting Fermi gas [36], we first examine the relationship between the Fredholm equations for 1D Fermions with repulsive and attractive delta-function interactions. It is convenient to use Yang's operator notations [28] for the Fredholm equations. Here we denote the integral operator

$$k_n :\equiv \langle k|k_n|k'\rangle = \frac{1}{2\pi}\frac{nc}{n^2c^2/4 + (k-k')^2} \tag{43}$$

which is a symmetric function. We define the projection operators A_i and its dual projection operators $\bar{A}_i$

$$\begin{cases} \langle k|A_i r_i\rangle = 0, & \text{for } |k| > B_i \\ \langle k|A_i r_i\rangle = r_i(k), & \text{for } |k| \leqslant B_i \end{cases} \tag{44}$$

$$\begin{cases} \langle k|\bar{A}_i r_i\rangle = r_i(k), & \text{for } |k| > B_i \\ \langle k|\bar{A}_i r_i\rangle = 0, & \text{for } |k| \leqslant B_i \end{cases} \tag{45}$$

where $i = 1, 2$. Similar notations are carried out for attractive interaction regime.

The Fredholm equations (32) and (33) for repulsive interaction regime can be rewritten in terms of these operators

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} \\ \frac{1}{2\pi} \end{pmatrix} + \begin{pmatrix} 0 & K_1 \\ -K_1 & 0 \end{pmatrix}\begin{pmatrix} \bar{A}_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \tag{46}$$

where the integration boundaries B_i satisfy the following conditions

$$\frac{N^1}{L} = \frac{B_1}{\pi} - \frac{1}{\pi}\int \langle k|A_2 r_2\rangle G_+(B_1,k)\mathrm{d}k \tag{47}$$

$$\frac{N^2}{L} = \frac{B_2}{\pi} - \frac{1}{\pi}\int \langle k|\bar{A}_1 r_1\rangle G_-(k,B_2)\mathrm{d}k \tag{48}$$

for repulsive regime. Here we denote

$$G_\pm(x,y) = \arctan\frac{c}{2(x-y)} \pm \arctan\frac{c}{2(x+y)} \tag{49}$$

For attractive regime the Fredholm equations are rewritten as

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} \\ \frac{1}{2\pi} \end{pmatrix} + \begin{pmatrix} 0 & K_1 \\ -K_1 & 0 \end{pmatrix}\begin{pmatrix} \bar{A}_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \tag{50}$$

where Q_1 and Q_2 are determined by

$$\frac{N^1}{L} = \frac{Q_1}{\pi} - \frac{1}{\pi}\int \langle k|A_2\rho_2\rangle G_+(Q_1,k)\mathrm{d}k \tag{51}$$

$$\frac{N^2}{L} = \frac{Q_2}{\pi} - \frac{1}{\pi}\int \langle k|\bar{A}_1\rho_1\rangle G_-(k,Q_2)\mathrm{d}k \tag{52}$$

We prove that under a mapping

$$r_1(k) \longleftrightarrow \rho_1(k), \quad r_2(k) \longleftrightarrow \rho_2(k) \tag{53}$$

the Fredholm equations (46) with (47), (48) for repulsive regime and the Fredholm equations (50) with (51), (52) for attractive regime are identical. In the above equations it is implied that $c > 0$ for repulsive interaction regime and $c < 0$ for attractive interaction regime.

Furthermore, in a repulsive regime, the Fredholm equations (32) and (33) with the Fermi boundaries conditions (47) and (48) exhibit a symmetry

$$r_1(k) \longleftrightarrow r_2(k), \quad A_1 \longleftrightarrow \bar{A}_2$$
$$\bar{A}_1 \longleftrightarrow A_2, \quad c \longleftrightarrow -c \tag{54}$$

This symmetry relates to the spin-up and spin-down reversal symmetry of the model. This transformation maps the eigenstates with $N_\downarrow$ down-spin atoms and $N_\uparrow$ up-spin atoms one-to-one onto the eigenstates with $N_\downarrow$ up-spin atoms and $N_\uparrow$ down-spin atoms for the gas. Similarly, in attractive regime, the Fredholm equations (39) and (40) with the Fermi boundaries conditions (51) and (52) preserve the spin reversal symmetry

$$\rho_1(k) \longleftrightarrow \rho_2(k), \quad A_1 \longleftrightarrow \bar{A}_2$$
$$\bar{A}_1 \longleftrightarrow A_2, \quad c \longleftrightarrow -c \tag{55}$$

As a consequence of the mapping (53), the ground state energy smoothly connects at vanishing interaction strength. To see this point, we need to unify the ground state energy for weakly repulsive and weakly attractive regimes. For weakly repulsive regime, the ground state energy is given by

$$E = \int_{-B_1}^{B_1} k^2\langle k|A_1 r_1\rangle\mathrm{d}k = \frac{B_1^3}{3\pi} + \int_{-B_2}^{B_2}\langle k'|A_2 r_2\rangle\left[\int_{-B_1}^{-B_1} k^2\langle k|K_1|k'\rangle\mathrm{d}k\right]\mathrm{d}k' \tag{56}$$

Substituting (46) into the above equations, we obtain the ground state energy per length

$$E=\frac{B_1^2}{3\pi}+\frac{1}{2\pi}\int_{-B_2}^{B_2}H(k,B_1)\mathrm{d}k-\int_{-B_2}^{B_2}\left[\int_{|k'|>B_1}\langle k|K_1|k'\rangle\langle k'|\bar{A}_1 r_1\rangle\mathrm{d}k'\right]H(k,B_1)\mathrm{d}k \tag{57}$$

where

$$H(x,y)=\frac{1}{\pi}\left[(x^2-\frac{c^2}{4})\pi g_y(x)+yc+\frac{1}{2}xc\ln\frac{4(x-y)^2+c^2}{4(x+y)^2+c^2}\right]$$
$$g_y(x)=1-G_+(B_1,x)$$

For attractive regime, the ground state energy (42) is rewritten as

$$\frac{E}{L}=\int_{-Q_2}^{Q_2}\left(2k^2-\frac{c^2}{2}\right)\langle k|A_2\rho_2\rangle\mathrm{d}k+\int_{-Q_1}^{Q_1}k^2\left[\frac{1}{2\pi}-\int_{-Q_2}^{Q_2}\langle k|K_1|k'\rangle\langle k'|A_2\rho_2\rangle\mathrm{d}k'\right]\mathrm{d}k \tag{58}$$

Substituting Eq. (50) into the above equation (58), we obtain the ground state energy for weakly attractive regime

$$E=\frac{Q_1^2}{3\pi}+\frac{1}{2\pi}\int_{-Q_2}^{Q_2}H(k,Q_1)\mathrm{d}k-\int_{-Q_2}^{Q_2}\left[\int_{|k'|>Q_1}\langle k|K_1|k'\rangle\langle k'|\bar{A}_1\rho_1\rangle\mathrm{d}k'\right]H(k,Q_1)\mathrm{d}k \tag{59}$$

We see that the ground state of the gas with a weakly repulsive interaction and a weakly attractive interaction can be unified through (57) and (59). The ground state energy Eq. (57) and Eq. (59) can be calculated in a straight forward way. In a weak coupling regime, it covers the result (26) (also see Ref. [33]). The energy smoothly connects at vanishing interaction strength (but not analytically connects at $c=0$). Takahashi [36] proved that the energy is infinitely differentiable at $c=0$ for a real value of c. But the two sets of the Fredholm equations turn to be divergent in the region $c\to \mathrm{i}0$, see a discussion in Ref. [37].

6 Conclusions

In conclusion, we have studied the polaron–molecule crossover in the 1D spin-1/2 Fermi gas with an attractive delta-function interaction. We have found that a spin-down fermion immersed into a fully ploarized spin-up Fermi sea with weak attraction is dressed to form a Fermi polaron-like quasiparticle in the 1D fermionic medium. The spin-down fermion receives a mean field attraction from the fully-polarized Fermi sea. However, as the attraction grows, the spin-down fermion binds with one spin-up fermion from the fully-polarized medium to form a tightly bound molecule. We have presented the mean field binding energy and an effective mass of the polaron in the weak attraction limit and also presented the binding energy and an effective mass of the molecule in the strong attraction regime. The system undergos a cross-over from a mean field polaron-like nature into a mixture of excess fermions and a bosonic molecule as the attraction changes from a weak attraction into a strong one. The asymptotic solutions of the discrete Bethe ansatz equations provides insight into the understanding of the mean field nature of the polaron-like state and the novel pairing in the interacting Fermi gas. For both weak and strong coupling regimes, we have obtained the ground state energy from the Bethe ansatz roots of bound pairs and excess fermions, where they interfere with each other. Furthermore, we have proved that the two sets of the Fredholm equations for the 1D spin-1/2 Fermi gas with repulsive and attractive delta-function interactions are identical. The result we obtained for weak and strong attractions opens to experimental study of such mean field nature of polaron-like state and molecule signature in 1D trapped cold atoms. The current experiment is capable of catching the polaronic signature of 1D interacting quantum gases of cold atoms. This can be possibly achieved by using a species selective dipole potential (see a recent experiment on the polaronic dynamics of the 1D Bose gas [38]).

Acknowledgements This work was supported by the Australian Research Council. Thanks also go to Prof. Tin-Lun Ho, Prof. Zhong-Qi Ma and Prof. Chen-Ning Yang for helpful discussions and encouragements. He also acknowledges the Sun Yat-Sen University for their kind hospitality.

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