

Correspondence between quantum-optical transform and classical-optical transform explored by developing Dirac's symbolic method

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By virtue of the new technique of performing integration over Dirac's ket–bra operators, we explore quantum optical version of classical optical transformations such as optical Fresnel transform, Hankel transform, fractional Fourier transform, Wigner transform, wavelet transform and Fresnel–Hadamard combinatorial transform etc. In this way one may gain benefit for developing classical optics theory from the research in quantum optics, or vice-versa. We cannot only find some new quantum mechanical unitary operators which correspond to the known optical transformations, deriving a new theorem for calculating quantum tomogram of density operators, but also can reveal some new classical optical transformations. For examples, we find the generalized Fresnel operator (GFO) to correspond to the generalized Fresnel transform (GFT) in classical optics. We derive GFO's normal product form and its canonical coherent state representation and find that GFO is the loyal representation of symplectic group multiplication rule. We show that GFT is just the transformation matrix element of GFO in the coordinate representation such that two successive GFTs is still a GFT. The *ABCD* rule of the Gaussian beam propagation is directly demonstrated in the context of quantum optics. Especially, the introduction of quantum mechanical entangled state representations opens up a new area in finding new classical optical transformations. The complex wavelet transform and the condition of mother wavelet are studied in the context of quantum optics too. Throughout our discussions, the coherent state, the entangled state representation of the two-mode squeezing operators and the technique of integration within an ordered product (IWOP) of operators are fully used. All these have confirmed Dirac's assertion: "...for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory".

Keywords Dirac's symbolic method, IWOP technique, entangled state of continuum variables, entangled Fresnel transform, Collins formula, Generalized Fresnel operator, complex wavelet transform, complex Wigner transform, complex fractional Fourier transform, symplectic wavelet transform, entangled symplectic wavelet transform, Symplectic-dilation mixed wavelet transform, fractional Radon transform, new eigenmodes of fractional Fourier transform

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1 Introduction

The history of quantum mechanics records that from

the very beginning the founders of the quantum theory realized that there might exist formal connection between classical optics and quantum mechanics. For example, Schrödinger considered that classical dynamics of a point particle should be the “geometrical optics” approximation of a linear wave equation, in the same way as ray optics is a limiting approximation of wave optics; Schrödinger also searched for some quantum mechanical state which behaves like a classical ‘particle’, and this state was later recognized as the coherent state [1–6], which plays an essential role in quantum optics theory and laser physics; As Dirac wrote in his famous book “Principles of Quantum Mechanics” [7]: “... for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory”. According to Dirac, there should exist a formal correspondence between quantum optics unitary-transform operators and classical optics transformations. Indeed, in the last century physicists also found some rigorous mathematical analogies between classical optics and quantum mechanics, i.e., the similarity between the optical Helmholtz equation and the time-independent Schrödinger equation; Since 1960s, the advent of a laser and the appearance of coherent state theory of radiation field [1–5, 8], quantum optics has experienced rapid development and achieved great success in revealing and explaining the quantum mechanical features of optical field and non-classical behavior (for instance, Hanbury–Brown–Twiss effect, photon antibunching, squeezing, sub-Poissonian photon statistics) of photons in various photon–atom interactions [9–11]. The relationship between classical and quantum coherence has been discussed in the book of Mandel and Wolf [9]; The Hermite–Gauss or the Laguerre–Gauss modes of a laser beam are described using the bosonic operator algebra by Nienhuis and Allen [12]. In addition, displaced light beams refracted by lenses according to the law of geometrical optics were found to be the paraxial optics analog of a coherent state. Besides, phase space correspondence between classical optics and quantum mechanics, say for example, the Wigner function theory, is inspected in the literature [13].

On the other hand, classical optics, which tackles vast majority of physical-optics experiments and is based on Maxwell’s equations, has never ceased its own evolving steps, physicists have endeavored to develop various optical transforms in light propagation through lens systems and various continuous media. The two research fields, quantum optics and classical optics, have their own physical objects and conceptions. From the point of view of mathematics, classical optics is framed in the group transform and associated representations on appropriate function space, while quantum optics deals with operators and state vectors, and their overlap seems little at

first glance. It seems to us that if one wants to further relate them to each other, one needs some new theoretical method to “bridge” them. For example, what is the quantum mechanical unitary operator corresponding to the Fresnel transform in Fourier optics? Is there any so-called Fresnel operator as the image of classical generalized Fresnel transform? Since generalized Fresnel transforms are very popularly used in optical instrument design and optical propagation through lenses and various media, it is worthy to study these transforms in the context of quantum optics theory, especially based on coherent state, squeezed state [14–17] and the newly invented entangled state theory [18–27].

Fortunately, the recently invented technique of integration within an ordered product (IWOP) of operators [28, 29], which present a novel method for performing integration over Dirac’s ket-bra operators, is of great aid to studying quantum optical version of classical optical transformations. Using the IWOP technique one may gain benefit for classical optics from quantum optics’ research, or vice-versa.

Our present review is arranged as follows: in Section 2 we briefly recall the classical diffraction theory [30, 31]. This prepares for later sections in which we shall show that most frequently employed classical optical transforms have their counterparts in quantum optics theory. In Section 3 we introduce the IWOP technique and demonstrate that the completeness relation of fundamental quantum mechanical representations can be recast into normally ordered Gaussian operator form. Using the IWOP technique we can directly perform the asymmetric ket–bra integration $\mu^{-1/2} \int_{-\infty}^{\infty} dq |q/\mu\rangle \langle q|$ in the coordinate representation, which leads to the normally ordered single-mode squeezing operator, which seems to be a direct way of understanding the squeezing mechanism as a mapping from the classical scaling $q \rightarrow q/\mu$. In Section 4 with the help of IWOP technique and based on the concept of quantum entanglement of Einstein–Podolsky–Rosen [32] we construct two mutually conjugate entangled states of continuum variables, $|\eta\rangle$ versus $|\xi\rangle$, and their *deduced entangled states* (or *named* correlated-amplitude — number-difference entangled states), which are all qualified to make up quantum mechanical representations. It is remarkable that using the IWOP technique in performing the asymmetric ket–bra integration $\mu^{-1} \int d^2\eta |\eta/\mu\rangle \langle \eta|$ leads to the normally ordered two-mode squeezing operator, which implies that the two-mode squeezed state is simultaneously an entangled state. We point out that the entangled state $|\eta\rangle$ also embodies entanglement in the aspect of correlative amplitude and the phase. We felt encouraged from that the overlap between two mutually conjugate *deduced entangled states* is just the Bessel function — the optical Hankel transform kernel [33], which again shows that the new representations in the con-

text of physics theory match beautiful mathematical formalism exactly. We then employ the *deduced* entangled states to derive quantum optical version of classical circular harmonic correlation. Section 5 is devoted to finding a quantum operator which corresponds to the optical Fresnel transform, with use of the coherent state representation and by projecting the classical symplectic transform $z \rightarrow sz - rz^*$ ($|s|^2 - |r|^2 = 1$) in phase space onto the quantum mechanical Hilbert space. We are able to recognize which operator is the single-mode Fresnel operator (FO). It turns out that the one-dimensional (1D) optical Fresnel transform is just the matrix element of the Fresnel operator F in the coordinate eigenstates. Besides, the coherent state projection operator representation of FO constitutes a loyal realization of symplectic group, which coincides with the fact that two successive optical Fresnel transforms make up a new Fresnel transform. Then in Section 6, based on the coherent state projection representation of FO, we prove *ABCD* rule for optical propagation in the context of quantum optics. In Section 7 the quadratic operator form of FO is also presented and the four fundamental optical operators are derived by decomposing the FO. In Section 8 we discuss how to apply the Fresnel operator to quantum tomography theory. By introducing the Fresnel quadrature phase $FXF^\dagger = X_F$, we point out that Wigner operator's Radon transformation is just the pure state projection operator $|x\rangle_{s,r} \langle x|$, where $|x\rangle_{s,r} = F|x\rangle$ and $|x\rangle$ is the position eigenstate, so the probability distribution for the Fresnel quadrature phase is the Radon transform of the Wigner function. Moreover, the tomogram of quantum state $|\psi\rangle$ is just the squared modulus of the wave function ${}_{s,r}\langle x|\psi\rangle$. This new relation between quantum tomography and optical Fresnel transform may provide experimentalists to figure out new approach for testing tomography. In addition, we propose another new theorem for calculating tomogram, i.e., the tomogram of a density operator ρ is equal to the marginal integration of the classical Weyl correspondence function of $F^\dagger \rho F$. In Section 9, by virtue of the coherent state and IWOP method we propose two-mode generalized Fresnel operator (GFO). In this case we employ the entangled state representation to relate the 2-mode GFO to classical transforms, since the 2-mode GFO is not simply the direct product of two 1-mode GFOs. The corresponding quantum optics *ABCD* rule for two-mode case is also proved. The 2-mode GFO can also be expressed in quadratic operators form in entangled way. The relation between optical Fresnel transform and quantum tomography in the two-mode case is also revealed. In Section 10 we propose a kind of integration transformation, $\iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} h(p, q) \equiv f(x, y)$, which is invertible and obeys Parseval theorem. Remarkably, it can convert chirplet function to the kernel of fractional Fourier transform (FrFT). This transformation can also serve for solving some operator ordering

problems. In Section 11 we employ the entangled state representation to introduce the complex FrFT (CFrFT), which is not the direct product of two independent 1D FrFT transform. The eigenmodes on CFrFT is derived. New eigen-modes for light propagation in graded-index medium and the fractional Hankel transform are presented. The Wigner transform theory is extended to the complex form and its relation to CFrFT is shown; The integration transformation in Section 10 is also extended to the entangled case. In Section 12 we shall treat the adaption problem of Collins diffraction formula to the CFrFT with the use of 2-mode (3 parameters) squeezing operator and in the entangled state representation of continuous variables, in so doing the quantum mechanical version of associated theory of classical diffraction and classical CFrFT is obtained, which connects classical optics and quantum optics in this aspect. In Section 13 we discuss quantum optical version of classical wavelet transforms (WTs), including how to recast the condition of mother wavelet into the context of quantum optics. Some properties, such as Parseval theorem, Inversion formula, and orthogonal property, the relation between WT and Wigner–Husimi distribution function are also discussed. In Section 14, we generalize the usual wavelet transform to symplectic wavelet transformation (SWT) by using the coherent state representation and making transformation $z \rightarrow s(z - \kappa) - r(z^* - \kappa^*)$ ($|s|^2 - |r|^2 = 1$) in phase space. The relation between SWT and optical Fresnel transformation is revealed. Then the SWT is extended to the entangled case by mapping the classical mixed transformation $(z, z') \rightarrow (sz + rz'^*, sz' + rz^*)$ in 2-mode coherent state $|z, z'\rangle$ representation. At the end of this section, we introduce a new symplectic-dilation mixed WT by employing a new entangled-coherent state representation $|\alpha, x\rangle$. The corresponding classical optical transform is also presented. In the last section, we introduce the Fresnel–Hadamard combinatorial operator by virtue of the IWOP technique and $|\alpha, x\rangle$. This unitary operator plays the role of both Fresnel transformation for mode $\frac{a_1 - a_2}{\sqrt{2}}$ and Hadamard transformation for mode $\frac{a_1 + a_2}{\sqrt{2}}$, respectively, and the two transformations are combinatorial. All these sections are used to prove the existence of a one-to-one correspondence between quantum optical operators that transform state vectors in Hilbert space and the classical optical transforms that change the distribution of optical field.

2 Some typical classical optical transformations

Here we briefly review some typical optical transforms based on light diffraction theory. These transformations, as one can see in later sections, are just the correspondence of some representation transformations between

certain quantum mechanical states of which some are newly constructed.

It was Huygens who gave a first illustrative explanation to wave theory by proposing every point in the propagating space as a sub-excitation source of a new sub-wave. An intuitive theory mathematically supporting Huygens' principle is the scalar diffraction approximation, so named because optical fields (electromagnetic fields) actually are vector fields, whereby the theory is valid approximately. This theory is based on the superposition of the combined radiation field of multiple re-emission sources initiated by Huygens. Light diffraction phenomena has played an important role in the development of the wave theory of light, and now underlies the Fourier optics and information optics. The formulation of a diffraction problem essentially considers an incident free-space wave whose propagation is interrupted by an obstacle or mask which changes the phase and/or amplitude of the wave locally by a well determined factor [34]. A more rigorous, but still in the scheme of scalar wave, derivation has been given by Kirchhoff who reformulated the diffraction problem as a boundary-value problem, which essentially justifies the use of Huygens principle. The Fresnel–Kirchhoff (or Rayleigh–Sommerfeld) diffraction formula is practically reduced to the Fresnel integral formula in paraxial and far-field approximation [30, 31] that reads

$$\begin{aligned}
 U_2(x_2, y_2) &= \frac{\exp(ikz)}{i\lambda z} \iint_{-\infty}^{\infty} U_1(x_1, y_1) \\
 &\times \exp\left\{i\frac{k}{2z} [(x_2 - x_1)^2 + (y_2 - y_1)^2]\right\} \\
 &\times dx_1 dy_1 \quad (1)
 \end{aligned}$$

where $U_1(x_1, y_1)$ is the optical distribution of a 2D light source and $U_2(x_2, y_2)$ is its image on the observation plane, λ is the optical wavelength, $k = \frac{2\pi}{\lambda}$ is the wave number in the vacuum and z is the propagation distance. When $z^2 \gg \frac{k}{2}(x_1^2 + y_1^2)_{\max}$ is satisfied, Eq. (1) is reduced to

$$\begin{aligned}
 U_2(x_2, y_2) &= \frac{\exp(ikz) \exp[i\frac{k}{2z}(x_1^2 + y_1^2)]}{i\lambda z} \\
 &\times \iint_{-\infty}^{\infty} U_1(x_1, y_1) \\
 &\times \exp\left[-i\frac{2\pi}{\lambda z}(x_1 x_2 + y_1 y_2)\right] dx_1 dy_1 \quad (2)
 \end{aligned}$$

which is named the Fraunhofer diffraction formula.

The Fresnel integral is closely related to the fractional Fourier transform (FrFT). Actually, it has been proved that the Fresnel transform can be interpreted as a scaled FrFT with a residual phase curvature [35]. The FrFT is a very useful tool in Fourier optics and information optics. This concept was firstly introduced in 1980 by Namias [36], but had not attracted enough attention

until FrFT was defined physically, based on propagation in quadratic graded-index media (GRIN media). Mendlovic and Ozaktas [37–39] defined the α th FrFT as follows: Let the original function be input from one side of quadratic GRIN medium, at $z = 0$. Then, the light distribution observed at the plane $z = z_0$ corresponds to the α equal to the (z_0/L) -th FrFT of the input fraction, where $L \equiv (\pi/2)(n_1/n_2)^{1/2}$ is a characteristic distance. The FrFT can also be implemented by lenses. Another approach for introducing FrFT was made by Lohmann who pointed out the algorithmic isomorphism among image rotation, rotation of the Wigner distribution function [42], and fractional Fourier transforming [40]. Lohmann proposed the FrFT as the transform performed on a function that leads to a rotation with an angle of the associated Wigner distribution function. In this sense, the FrFT bridges the gap between the classical optics and the optical Wigner distribution theory. Recently, the FrFT has been paid more and more attention within different contexts of both mathematics and physics [36–41]. The FrFT is defined as

$$\begin{aligned}
 \mathcal{F}_\alpha[U_1](x_2, y_2) &= \frac{e^{i(1-\alpha)\frac{\pi}{2}}}{2 \sin(\frac{\pi}{2}\alpha)} \exp\left[-\frac{i(x_2^2 + y_2^2)}{2 \tan(\frac{\pi}{2}\alpha)}\right] \\
 &\times \int \int_{-\infty}^{\infty} \frac{dx_1 dy_1}{\pi} \exp\left[-\frac{i(x_1^2 + y_1^2)}{2 \tan(\frac{\pi}{2}\alpha)}\right] \\
 &\times \exp\left[\frac{i(x_2 x_1 + y_2 y_1)}{\sin(\frac{\pi}{2}\alpha)}\right] U_1(x_1, y_1) \quad (3)
 \end{aligned}$$

We can see that F_0 is the identity operator and $F_{\pi/2}$ is just the Fourier transform. The most important property of FrFT is that F_α obeys the semigroup property, i.e., two successive FrFTs of order α and β makes up the FrFT of order $\alpha + \beta$. A more general form describing the light propagation in an optical system characterized by the $[A, B; C, D]$ ray transfer matrix is the Collins diffraction integral formula [43],

$$\begin{aligned}
 U_2(x_2, y_2) &= \frac{k \exp(ikz)}{2\pi i B} \iint_{-\infty}^{\infty} dx_1 dy_1 U_1(x_1, y_1) \\
 &\times \exp\left\{\frac{ik}{2B}[A(x_1^2 + y_1^2) - 2(x_1 x_2 + y_1 y_2) \right. \\
 &\left. + D(x_2^2 + y_2^2)]\right\} dx_1 dy_1 \quad (4)
 \end{aligned}$$

where $AD - BC = 1$ if the system is lossless. One can easily find the similarity between Collins formula and the FrFT by some scaling transform and relating the $[A, B, C, D]$ matrix to α in the FrFT [44]. Note that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a ray transfer matrix describing optical systems belonging to the unimodular symplectic group. When treating the light propagation in optical elements in near-axis approximation, matrices M repre-

sending linear transformations are a convenient mathematical tool for calculating the fundamental properties of optical systems, which is the origin of the name of matrix optics. In cylindrical coordinates the Collins formula is expressed as [43, 44]

$$U_2(r_2, \varphi) = \frac{i}{\lambda B} \int_0^\infty \int_0^{2\pi} \exp \left\{ -\frac{i\pi}{\lambda B} [Ar_1^2 + Dr_2^2 - 2r_1r_2 \cos(\theta - \varphi)] \right\} U_1(r_1, \theta) r_1 dr_1 d\theta \quad (5)$$

where $x_1 = r_1 \cos \theta$, $y_1 = r_1 \sin \theta$, $x_2 = r_2 \cos \varphi$ and $y_2 = r_2 \sin \varphi$. When $U_1(r_1, \theta)$ has rotational symmetry

$$\begin{aligned} U_1(r_1, \theta) &= u_1(r_1) \exp(im\theta) \\ U_2(r_2, \varphi) &= u_2(r_2) \exp(im\varphi) \end{aligned} \quad (6)$$

then Eq. (5) becomes

$$\begin{aligned} u_2(r_2) &= \frac{2\pi}{\lambda B} \exp \left[i(1+m) \frac{\pi}{2} \right] \\ &\times \int_0^\infty \exp \left[-\frac{i\pi}{\lambda B} (Ar_1^2 + Dr_2^2) \right] \\ &\times J_m \left(\frac{2\pi r_1 r_2}{\lambda B} \right) u_1(r_1) r_1 dr_1 \end{aligned} \quad (7)$$

where we have used the m -order Bessel function

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left[ix \cos \theta + im \left(\theta - \frac{\pi}{2} \right) \right] d\theta \quad (8)$$

When $A = 0$, Eq. (7) reduces to the standard Hankel transform (up to a phase factor)

$$u_2(r_2) \rightarrow \frac{2\pi}{\lambda B} \int_0^\infty J_m \left(\frac{2\pi r_1 r_2}{\lambda B} \right) u_1(r_1) r_1 dr_1 \quad (9)$$

The compact form of 1D Collins formula is

$$g(x_2) = \int_{-\infty}^\infty \mathcal{K}^M(x_2, x_1) f(x_1) dx_1 \quad (10)$$

where the transform kernel is

$$\mathcal{K}^M(x_2, x_1) = \frac{1}{\sqrt{2\pi i B}} \exp \left[\frac{i}{2B} (Ax_1^2 - 2x_2x_1 + Dx_2^2) \right] \quad (11)$$

M is the parameter matrix $[A, B, C, D]$. Eq. (11) is called generalized Fresnel transform [46–50]. In the following sections we will show how we find the quantum optical counterpart for those transformations of classical optics. For this purpose in the next chapter we introduce the IWOP technique to demonstrate how Dirac's symbolic method can be developed and be applied to quantum optics theory. Also, we briefly review some properties of the entangled state [18–27] and reveal the connection between the mutual transform generated by these entangled states and the Hankel transform in classical optics.

3 The IWOP technique and two mutually conjugate entangled states

3.1 The IWOP technique

The history of mathematics tells us that whenever there appears a new important mathematical symbol, there co-exists certain operational rules for it, the quantum mechanical operators in ket-bra projective form (the core of Dirac's symbolic method) also need their own operational rules. The terminology "symbolic method" was first shown in the preface of Dirac's book "The Principle of Quantum Mechanics": "The symbolic method, which deals directly in an abstract way with the quantities of fundamental importance... however, seems to go more deeply into the nature of things. It enables one to express the physical law in a neat and concise way, and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed." [7] Then two questions naturally arise: How to better understand the symbolic method? How to develop Dirac's symbolic method, especially its mathematics? We noticed that Newton–Leibniz integration rule only applies to commuting functions of continuum variables, while operators made of Dirac's symbols (ket versus bra, e.g., $|q/\mu\rangle\langle q|$ of continuous parameter q) in quantum mechanics are usually not commutative. Therefore integrations over the operators of type $|\rangle\langle|$ (where ket- and bra-state vectors need not to be Hermitian-conjugate to each other) can not be directly performed by the Newton–Leibniz rule. Thus we invented an innovative technique of integration within an ordered product (IWOP) of operators that made the integration of non-commutative operators possible. The core of IWOP technique is to arrange non-commutable quantum operators within an ordered product (say, normal ordering) in a way that they become commutable. In this sense the gap between q -numbers and c -numbers is "narrowed". However, the nature of operators which are within $:\ ::$ is not changed. They are still q -numbers, not c -numbers. After the integration over c -numbers within ordered product is performed, we can get rid of the normal ordering symbol after putting the integration result in normal ordering [51]. The IWOP technique thus bridges this mathematical gap between classical mechanics and quantum mechanics, and further reveals the beauty and elegance of Dirac's symbolic method and transformation theory. This technique develops symbolic method significantly, i.e., makes Dirac's representation theory and the transformation theory more plentiful, and consequently to be better understood. The beauty and elegance of Dirac's symbolic method are further revealed. Various applications of the IWOP technique, including constructing the entangled state, developing the nonlinear coherent state

theory, Wigner function theory, etc., are found; many new unitary operators and operator-identities as well as new quantum mechanical representations can be derived too, which are partly summarized in the review articles [22–24].

We begin with listing some properties of normal product of operators which means all the bosonic creation operators a^\dagger are standing on the left of annihilation operators a in a monomial of a^\dagger and a .

(i) The order of Bose operators a and a^\dagger within a normally ordered product can be permuted. That is to say, even though $[a, a^\dagger] = 1$, we can have: $aa^\dagger := a^\dagger a := a^\dagger a$, where $:$ denotes normal ordering.

(ii) c -numbers can be taken out of the symbol $:$ as one wishes.

(iii) The symbol $:$ which is within another symbol $:$ can be deleted.

(iv) The vacuum projection operator $|0\rangle\langle 0|$ has the normal product form

$$|0\rangle\langle 0| := e^{-a^\dagger a} \tag{12}$$

(v) A normally ordered product can be integrated or differentiated with respect to a c -number provided the integration is convergent.

3.2 The IWOP technique for deriving normally ordered Gaussian form of the completeness relations of fundamental quantum mechanical representations

As an application of IWOP, (in the following, unless particularly mentioned, we take $\hbar = \omega = m = 1$ for convenience.), using the Fock representation of the coordinate eigenvector $Q|q\rangle = q|q\rangle$, ($Q = (a + a^\dagger)/\sqrt{2}$)

$$|q\rangle = \pi^{-1/4} e^{-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}} |0\rangle \tag{13}$$

we perform the integration below:

$$\begin{aligned} S_1 &\equiv \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\mu}} \left| \frac{q}{\mu} \right\rangle \langle q| \\ &= \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi\mu}} e^{-\frac{q^2}{2\mu^2} + \sqrt{2}\frac{q}{\mu}a^\dagger - \frac{a^{\dagger 2}}{2}} |0\rangle\langle 0| e^{-\frac{q^2}{2} + \sqrt{2}qa - \frac{a^2}{2}} \end{aligned} \tag{14}$$

Substituting Eq. (12) into Eq. (14), we see

$$\begin{aligned} S_1 &= \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi\mu}} e^{-\frac{q^2}{2\mu^2} + \sqrt{2}\frac{q}{\mu}a^\dagger - \frac{a^{\dagger 2}}{2}} \\ &: e^{-a^\dagger a} : e^{-\frac{q^2}{2} + \sqrt{2}qa - \frac{a^2}{2}} \end{aligned} \tag{15}$$

Note that on the left of $: e^{-a^\dagger a} :$ are all creation operators, while on its right are all annihilation operators, so the whole integral is in normal ordering, thus using property i) we have

$$S_1 = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi\mu}} : e^{-\frac{q^2}{2}(1 + \frac{1}{\mu^2}) + \sqrt{2}q(\frac{a^\dagger}{\mu} + a) - \frac{1}{2}(a + a^\dagger)^2} : \tag{16}$$

As a commutes with a^\dagger within $:$, a^\dagger and a can be considered as if they were parameters while the integration is being performed. Therefore, by setting $\mu = e^\lambda$, $\text{sech } \lambda = \frac{2\mu}{1+\mu^2}$, $\tanh \lambda = \frac{\mu^2-1}{\mu^2+1}$, we are able to perform the integration and obtain

$$S_1 = (\text{sech } \lambda)^{1/2} e^{-\frac{a^{\dagger 2}}{2} \tanh \lambda} : e^{(\text{sech } \lambda - 1)a^\dagger a} : e^{\frac{a^2}{2} \tanh \lambda} \tag{17}$$

which is just the single-mode squeezing operator in normal ordering appearing in many references. It is worth mentioning that we have not used the $SU(1,1)$ Lie algebra method in the derivation. The integration automatically arranges the squeezing operator in normal ordering. With

$$e^{\lambda a^\dagger a} := \exp[(e^\lambda - 1)a^\dagger a] \tag{18}$$

Eq. (17) becomes

$$\int_{-\infty}^{\infty} \frac{dq}{\sqrt{\mu}} \left| \frac{q}{\mu} \right\rangle \langle q| = e^{-\frac{a^{\dagger 2}}{2} \tanh \lambda} e^{(a^\dagger a + \frac{1}{2}) \ln \text{sech } \lambda} e^{\frac{a^2}{2} \tanh \lambda} \tag{19}$$

This shows that the classical dilation $q \rightarrow \frac{q}{\mu}$ is mapped into the normally ordered squeezing operator manifestly. It also exhibits that the fundamental representation theory can be formulated in a less abstract way, as we can now directly perform the integral over ket–bra projection operators. Moreover, the IWOP technique can be employed to perform many complicated integrations for ket–bra projection operators.

There is a deep ditch between quantum mechanical operators (q -numbers) theory and classical numbers (c -numbers) theory. The IWOP technique arranges non-commutable operators within an ordered product symbol in a way that they become commutable. In this sense the “ditch” between q -numbers and c -numbers is “shoaled”. However, the nature of operators are not changed, they are still q -numbers, not c -numbers. After the integration over c -numbers within ordered product is performed, we can finally get rid of the normal ordering symbol by using Eq. (18).

When $\mu = 1$, Eq. (19) becomes

$$\int_{-\infty}^{\infty} dq |q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} : e^{-(q-Q)^2} : = 1 \tag{20}$$

a real simple Gaussian integration! This immediately leads us to put the completeness relation of the momentum representation into the normally ordered Gaussian form

$$\int_{-\infty}^{\infty} dp |p\rangle\langle p| = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{\pi}} : e^{-(p-P)^2} : = 1 \tag{21}$$

where $P = (a - a^\dagger)/(i\sqrt{2})$, and $|p\rangle$ is the momentum eigenvector $P|p\rangle = p|p\rangle$,

$$|p\rangle = \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2}p^2 + i\sqrt{2}pa^\dagger + \frac{1}{2}a^{\dagger 2}\right) |0\rangle \quad (22)$$

In addition, we should notice that $|q\rangle$ and $|p\rangle$ are related by the Fourier transform (FT), i.e., $\langle p|q\rangle = \frac{1}{\sqrt{2\pi}} \exp(-iqp)$. The integral kernel of the Fraunhofer diffraction formula in 1D is such an FT, so FT in classical optics has its correspondence in quantum mechanical representations' transform. This enlightens us that in order to find more general analogy between unitary operators in quantum optics and transformations in classical optics we should construct new representations for quantum optics theory, and these are the bi-partite entangled state and many-particle entangled state. These ideal states can be implemented by optical devices and optical network [52]. In the following we will focus on the bi-partite entangled state.

3.3 Single-mode Wigner operator

When we combine Eqs. (20) and (21) we can obtain

$$\pi^{-1}: e^{-(q-Q)^2 - (p-P)^2} := \Delta(q, p) \quad (23)$$

which is just the normally ordered Wigner operator since its marginal integration gives $|q\rangle\langle q|$ and $|p\rangle\langle p|$ respectively, i.e.,

$$\int_{-\infty}^{\infty} dq \Delta(q, p) = \frac{1}{\sqrt{\pi}} : e^{-(p-P)^2} := |p\rangle\langle p| \quad (24)$$

$$\int_{-\infty}^{\infty} dp \Delta(q, p) = \frac{1}{\sqrt{\pi}} : e^{-(q-Q)^2} := |q\rangle\langle q| \quad (25)$$

Thus the Wigner function of quantum state ρ can be calculated as $W(q, p) = \text{Tr}[\rho \Delta(q, p)]$. On the other hand, the Wigner operator (23) can be recast into the coherent state representation,

$$\Delta(q, p) \rightarrow \Delta(\alpha, \alpha^*) = \int \frac{d^2z}{\pi^2} |\alpha + z\rangle\langle \alpha - z| e^{\alpha z^* - \alpha^* z} \quad (26)$$

where $|z\rangle$ is a coherent state. In fact, using the IWOP technique we can obtain

$$\Delta(\alpha, \alpha^*) = \frac{1}{\pi} : \exp\{-2(a - \alpha)(a^\dagger - \alpha^*)\} : \quad (27)$$

which is the same as Eq. (23).

3.4 Entangled state $|\eta\rangle$ and its Fourier transform in complex form

The concept of quantum entanglement was first employed by Einstein, Rosen and Poldosky (EPR) to challenge that quantum mechanics is incomplete when they observed that two particles' relative position $Q_1 - Q_2$ and the total momentum $P_1 + P_2$ are commutable. Hinted by EPR, the bipartite entangled state $|\eta\rangle$ is introduced as [53–55]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right) |00\rangle \quad (28)$$

$|\eta = \eta_1 + i\eta_2\rangle$ is the common eigenstate of relative coordinate $Q_1 - Q_2$ and the total momentum $P_1 + P_2$,

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \quad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle \quad (29)$$

where $Q_i = (a_j + a_j^\dagger)/\sqrt{2}$, $P_j = (a_j - a_j^\dagger)/(i\sqrt{2})$, $j = 1, 2$. Using the IWOP technique, we can immediately prove that $|\eta\rangle$ possesses the completeness relation

$$\int \frac{d^2\eta}{\pi} |\eta\rangle\langle\eta| = 1, \quad d^2\eta = d\eta_1 d\eta_2 \quad (30)$$

and orthonormal relation

$$\langle\eta|\eta'\rangle = \pi\delta(\eta_1 - \eta'_1)\delta(\eta_2 - \eta'_2) \quad (31)$$

The Schmidt decomposition of $|\eta\rangle$ is

$$|\eta\rangle = e^{-i\eta_2\eta_1} \int_{-\infty}^{\infty} dq |q\rangle_1 \otimes |q - \sqrt{2}\eta_1\rangle_2 e^{i\sqrt{2}\eta_2 q} \quad (32)$$

The $|\eta\rangle$ state can also be Schmidt-decomposed in momentum eigenvector space as

$$|\eta\rangle = e^{i\eta_1\eta_2} \int_{-\infty}^{\infty} dp |p\rangle_1 \otimes |\sqrt{2}\eta_2 - p\rangle_2 e^{-i\sqrt{2}\eta_1 p} \quad (33)$$

The $|\eta\rangle$ is physically appealing in quantum optics theory, because the two-mode squeezing operator has its natural representation on $|\eta\rangle$ basis [55]

$$\int \frac{d^2\eta}{\pi\mu} |\eta/\mu\rangle\langle\eta| = e^{a_1^\dagger a_2^\dagger \tanh\lambda} e^{(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \ln \text{sech}\lambda} \times e^{-a_1 a_2 \tanh\lambda} \equiv S_2, \quad \mu = e^\lambda \quad (34)$$

The proof of Eq. (34) is proceeded by virtue of the IWOP technique. So the necessity of introducing $|\eta\rangle$ into quantum optics is clear. S_2 squeezes $|\eta\rangle$ in the manifest way

$$S_2|\eta\rangle = \frac{1}{\mu} |\eta/\mu\rangle, \quad \mu = e^\lambda \quad (35)$$

and the two-mode squeezed state itself is an entangled state which entangles the idle mode and signal mode as an outcome of a parametric-down conversion process [56–58].

We can also introduce the conjugate state of $|\eta\rangle$ [59],

$$|\xi\rangle = \exp\left(-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right) |00\rangle \quad (36)$$

which obeys the eigen-equations

$$(Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \quad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle \quad (37)$$

where $\xi = \xi_1 + i\xi_2$. Because $[(Q_1 - Q_2), (P_1 - P_2)] = 2i$, we name the conjugacy between $|\xi\rangle$ and $|\eta\rangle$. The completeness and orthonormal relations of $|\xi\rangle$ are

$$\int \frac{d^2\xi}{\pi} |\xi\rangle\langle\xi| = 1, \quad d^2\xi = d\xi_1 d\xi_2 \quad (38)$$

$$\langle\xi|\xi'\rangle = \pi\delta(\xi_1 - \xi'_1)\delta(\xi_2 - \xi'_2) \quad (39)$$

respectively. $|\eta\rangle$ and $|\xi\rangle$ can be related to each other by

$$\langle\eta|\xi\rangle = \frac{1}{2} \exp\left(\frac{\xi\eta^* - \xi^*\eta}{2}\right) \quad (40)$$

Since $\xi^*\eta - \xi\eta^*$ is a pure imaginary number, and Eq. (40) is the Fourier transform kernel in complex form (or named entangled Fourier transform. This concept should also be extended to multipartite entangled states.). It will be shown in later sections that departing from entangled states $|\eta\rangle$ and $|\xi\rangle$, the generalized Fresnel operator would lead to a new entangled Fresnel transform in classical optics.

3.5 Two-mode Wigner operator in the $|\eta\rangle$ representation

Combining Eqs. (30) and (38) we can construct the following operator:

$$\frac{1}{\pi^2} : e^{-[\sigma^* - (a_1^\dagger - a_2)][\sigma - (a_1 - a_2^\dagger)] - [\gamma^* - (a_1^\dagger + a_2)][\gamma - (a_1 + a_2^\dagger)]} : \\ = \Delta(\alpha, \alpha^*) \otimes \Delta(\beta, \beta^*) \equiv \Delta(\sigma, \gamma) \quad (41)$$

where

$$\sigma = \alpha - \beta^*, \quad \gamma = \alpha + \beta^* \quad (42)$$

Eq. (41) is just equal to the direct product of two single-mode Wigner operators. It is convenient to express the Wigner operator in the $|\eta\rangle$ representation as [60]

$$\Delta(\sigma, \gamma) = \int \frac{d^2\eta}{\pi^3} |\sigma - \eta\rangle \langle \sigma + \eta| e^{\eta\gamma^* - \eta^*\gamma} \quad (43)$$

For two-mode correlated system, it prefers to using $\Delta(\sigma, \gamma)$ to calculate quantum states' Wigner function. For example, with $\langle\eta|00\rangle = \exp(-|\eta|^2/2)$, the two-mode squeezed states' Wigner function is

$$\langle 00|S_2^\dagger(\mu)\Delta(\sigma, \gamma)S_2(\mu)|00\rangle \\ = \pi^{-2} \exp(-\mu^2|\sigma|^2 - |\gamma|^2/\mu^2) \quad (44)$$

4 Two deduced entangled state representations and Hankel transform

4.1 Deduced entangled states

Starting from the entangled state $|\eta = re^{i\theta}\rangle$ and introducing an integer m , we can deduce new states [61],

$$|m, r\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta |\eta = re^{i\theta}\rangle e^{-im\theta} \quad (45)$$

which is worthy of attention because when we operate the number-difference operator, $D \equiv a_1^\dagger a_1 - a_2^\dagger a_2$ on $|\eta\rangle$, using Eq. (28) we see

$$D|\eta\rangle = (\eta a_1^\dagger + \eta^* a_2^\dagger)|\eta\rangle = -i \frac{\partial}{\partial \theta} |\eta\rangle, \quad \eta = |\eta| e^{i\theta} \quad (46)$$

so the number-difference operator corresponds to a differential operation $i \frac{\partial}{\partial \theta}$ in the $|\eta\rangle$ representation, which is a remarkable property of $\langle\eta|$. It then follows that

$$D|m, r\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-im\theta} \left(-i \frac{\partial}{\partial \theta} |\eta = re^{i\theta}\rangle \right) = m|m, r\rangle \quad (47)$$

On the other hand, by defining $K \equiv (a_1 - a_2^\dagger)(a_1^\dagger - a_2)$, we see $[D, K] = 0$, and $|m, r\rangle$ is its eigenstate,

$$K|m, r\rangle = r^2|m, r\rangle \quad (48)$$

where K is named correlated-amplitude operator since $K|\eta\rangle = r^2|\eta\rangle$. Thus we name $|m, r\rangle$ correlated-amplitude — number-difference entangled states. It is not difficult to prove the completeness and orthonormal property of $|m, r\rangle$,

$$\sum_{m=-\infty}^{\infty} \int_0^{\infty} d(r^2) |m, r\rangle \langle m, r| = 1 \quad (49)$$

$$\langle m, r|m', r''\rangle = \delta_{m, m'} \frac{1}{2r} \delta(r - r'') \quad (50)$$

On the other hand, from $|\xi\rangle$ we can derive another state

$$|s, r'\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi |\xi = r' e^{i\varphi}\rangle e^{-is\varphi} \quad (51)$$

which satisfies

$$D|\xi\rangle = (a_1^\dagger \xi - a_2^\dagger \xi^*)|\xi\rangle = -i \frac{\partial}{\partial \varphi} |\xi = r' e^{i\varphi}\rangle \quad (52)$$

So D in $\langle\xi = r' e^{i\varphi}|$ representation is equal to $i \frac{\partial}{\partial \varphi}$. Consequently,

$$D|s, r'\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-is\theta} \left(-i \frac{\partial}{\partial \theta} |\xi = r' e^{i\theta}\rangle \right) = s|s, r'\rangle \quad (53)$$

Note $[D, (a_1^\dagger + a_2)(a_1 + a_2^\dagger)] = 0$ and

$$(a_1^\dagger + a_2)(a_1 + a_2^\dagger)|s, r'\rangle = r'^2|s, r'\rangle \quad (54)$$

$|s, r'\rangle$ is qualified to be a new representation since

$$\sum_{s=-\infty}^{\infty} \int_0^{\infty} d(r'^2) |s, r'\rangle \langle s, r'| = 1$$

$$\langle s, r'|s'', r''\rangle = \delta_{s, s'} \frac{1}{2r'} \delta(r' - r'') \quad (55)$$

4.2 Hankel transform between two deduced entangled state representations

Since $|\xi\rangle$ and $|\eta\rangle$ are mutual conjugate, $|s, r'\rangle$ is the conjugate state of $|m, r\rangle$. From the definition of $|m, r\rangle$ and $|s, r'\rangle$ and Eq. (40) we calculate the overlap [33]

$$\langle s, r'|q, r\rangle = \frac{1}{2} \sum_{l=-\infty}^{\infty} \delta_{l, m} \delta_{l, s} J_l(rr') = \frac{1}{2} \delta_{s, m} J_s(rr') \quad (56)$$

where we have identified the generating function of the s -order Bessel function J_l , $e^{ix \sin t} = \sum_{l=-\infty}^{\infty} J_l(x)e^{ilt}$, and

$$J_l(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(l+k)!} \left(\frac{x}{2}\right)^{l+2k} \quad (57)$$

Eq. (56) is remarkable, because $J_s(rr'')$ is just the integral kernel of Hankel transform. In fact, if we define

$$\langle m, r|g \rangle \equiv g(m, r), \quad \langle s, r'|g \rangle \equiv \mathcal{G}(s, r') \quad (58)$$

and use Eq. (49) as well as Eq. (56), we obtain

$$\mathcal{G}(s, r') = \frac{1}{2} \int_0^{\infty} d(r'^2) J_s(rr'') g(s, r) \equiv \mathcal{H}[g(s, r)] \quad (59)$$

which is just the Hankel transform of $g(m, r)$ [or it can be regarded as a simplified form of the Collins formula in cylindrical coordinate, see Eq. (9)]. The inverse transform of Eq. (59) is

$$g(m, r) = \frac{1}{2} \int_0^{\infty} d(r'^2) J_q(rr') \mathcal{G}(m, r') \equiv \mathcal{H}^{-1}[\mathcal{G}(m, r')] \quad (60)$$

Now we know that the quantum optical image of classical Hankel transform just corresponds to the representation transformation between two mutually conjugate entangled states $\langle s, r'|$ and $|m, r\rangle$, which is like the case that the Fourier transform kernel is just the matrix element between the coordinate state and the momentum state, a wonderful result unnoticed before. Therefore the bipartite entangled state representations' transforms, which can lead us to the Hankel transform that was proposed first in classical optics, can find their way back in quantum optics.

4.3 Quantum optical version of classical circular harmonic correlation

From Eq. (45) we can see that its reciprocal relation is the circular harmonic expansion,

$$|\eta = re^{i\theta}\rangle = \sum_{m=-\infty}^{\infty} |m, r\rangle e^{im\theta} \quad (61)$$

or the correlated-amplitude—number-difference entangled state $|m, r\rangle$ can be considered as circular harmonic decomposition of $|\eta = re^{i\theta}\rangle$. Let $g(r, \theta)$, a general 2D function expressed in polar coordinates, be periodic in the variable θ , it can be looked as the wavefunction of the state vector $|g\rangle$ in the $\langle \eta = re^{i\theta}|$ representation

$$g(r, \theta) = \langle \eta = re^{i\theta}|g \rangle \quad (62)$$

Using Eq. (61) we have

$$g(r, \theta) = \sum_{m=-\infty}^{\infty} g_m(r) e^{-im\theta}, \quad g_m(r) = \langle m, r|g \rangle \quad (63)$$

where $g_m(r)$ is the wavefunction of $|g\rangle$ in $\langle m, r|$ repre-

sentation. We see from Eq. (46)

$$\begin{aligned} e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)} |\eta = re^{i\theta}\rangle &= e^{-\alpha \frac{\partial}{\partial \theta}} |\eta = re^{i\theta}\rangle \\ &= |\eta = re^{i(\theta-\alpha)}\rangle \end{aligned} \quad (64)$$

so $e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)}$ behaves a rotation operator in $|\eta\rangle$ representation, the expectation value of $e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)}$ in $|g\rangle$ is

$$\begin{aligned} \pi \langle g|e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)}|g \rangle &= \pi \langle g| \int \frac{d^2\eta}{\pi} e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)} |\eta\rangle \langle \eta|g \rangle \\ &= \int_0^{\infty} r dr \int_0^{2\pi} g^*(r, \theta - \alpha) g(r, \theta) d\theta \equiv R_\alpha \end{aligned} \quad (65)$$

which is just the cross-correlation between $g(r, \theta)$ and an angularly rotated version of the same function, $g^*(r, \theta - \alpha)$. On the other hand, using Eq. (49) we have

$$|g\rangle = \sum_{m=-\infty}^{\infty} \int_0^{\infty} d(r'^2) |m, r\rangle g_m(r) \quad (66)$$

Substituting Eq. (66) into Eq. (65) and using the eigenvector equation (47) as well as Eq. (50), we obtain

$$R_\alpha = 2\pi \sum_{m=-\infty}^{\infty} e^{-im\alpha} \int_0^{\infty} r |g_m(r)|^2 dr \quad (67)$$

from which we see that each of the circular harmonic components of the crosscorrelation undergoes a different phase shift $-m\alpha$, so R_α is not rotation invariant. However, when we consider only one harmonic component

$$R_{\alpha, M} = 2\pi e^{-iM\alpha} \int_0^{\infty} r |g_M(r)|^2 dr \quad (68)$$

is extracted digitally, then from the phase associated with this component it is possible to determine the angular shift that one version of the object has undergone. When an optical filter that is matched to $R_{\alpha, M}$ of a particular object is constructed, if that some object is entered as an input to the system with any angular rotation, a correlation peak of strength proportional to $\int_0^{\infty} r |g_M(r)|^2 dr$ will be produced, independent of rotation. Hence an optical correlator can be constructed that will recognize that object independent of rotation [31].

So far we have studied the circular harmonic correlation in the context of quantum optics, we have endowed the crosscorrelation R_α with a definite quantum mechanical meaning, i.e., the overlap between $\langle g|$ and the rotated state $e^{i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)} |g\rangle$, in the entangled state representation.

Note that Fourier-based correlators is also very sensitive to magnification. However, the magnitude of Mellin transform is independent of scale-size changes in the input [31]. Now we examine when $|g\rangle$ is both rotated and squeezed (by a two-mode squeezing operator $S_2(\lambda) = \exp[\lambda(a_1^\dagger a_2^\dagger - a_1 a_2)]$), then from Eqs. (35) and (62) we

have

$$S_2(\lambda)|g\rangle = \int \frac{d^2\eta}{\pi\mu} |\eta/\mu\rangle \langle \eta|g\rangle = \int \frac{d^2\eta}{\pi\mu} |\eta/\mu\rangle g(r, \theta) \quad (69)$$

it follows the overlap between $\langle g|$ and the state $e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)} S(\lambda)|g\rangle$,

$$\begin{aligned} W_{\alpha,\lambda} &\equiv \pi \langle g| e^{-i\alpha(a_1^\dagger a_1 - a_2^\dagger a_2)} S(\lambda)|g\rangle \\ &= \int_0^\infty \frac{rdr}{\mu} \int_0^{2\pi} g^*(r/\mu, \theta - \alpha) g(r, \theta) d\theta \end{aligned} \quad (70)$$

which corresponds to the cross-correlation arising from combination of squeezing and rotation (joint transform correlator). On the other hand, from Eqs. (45) and (35) we see

$$S_2(\lambda)|m, r\rangle = \frac{1}{2\pi\mu} \int_0^{2\pi} d\theta |\eta\rangle = \frac{r}{\mu} e^{i\theta} e^{-im\theta} = \frac{1}{\mu} |m, \frac{r}{\mu}\rangle \quad (71)$$

and therefore

$$W_{\alpha,\lambda} = \frac{2\pi}{\mu} \sum_{m=-\infty}^\infty e^{-im\alpha} \int_0^\infty r g_m(r) g_{m'}^*(re^{-\lambda}) dr \quad (72)$$

from which one can see that to achieve simultaneous scale and rotation invariance, a two-dimensional object $g(r, \theta)$ should be entered into the optical system in a distorted polar coordinate system, the distortion arising from the fact that the radial coordinate is stretched by a logarithmic transformation ($\lambda = -\ln \mu$), which coincides with Ref. [62]. The quantum optical version is thus established which is a new tie connecting Fourier optics and quantum optics [63].

At the end of this section, using the two-variable Hermite polynomials' definition [64]

$$H_{m,n}(\xi, \xi^*) = \sum_{l=0}^{\min(m,n)} \frac{m!n!(-1)^l \xi^{m-l} \xi^{*n-l}}{l!(m-l)!(n-l)!} \quad (73)$$

which is quite different from the product of two single-variable Hermite polynomials, and its generating function formula is

$$\sum_{m,n=0}^\infty \frac{t^m t'^n}{m!n!} H_{m,n}(\xi, \xi^*) = \exp(-tt' + t\xi + t'\xi^*) \quad (74)$$

and noting $H_{m,n}(\xi, \xi^*) = e^{i(m-n)\varphi} H_{m,n}(r', r')$, we can directly perform the integral in Eq. (51) and derive the explicit form of $|s, r'\rangle$,

$$|s, r'\rangle = e^{-r'^2/2} \sum_{n=0}^\infty \frac{1}{\sqrt{(n+s)!n!}} H_{n+s,n}(r', r') |n+s, n\rangle \quad (75)$$

which is really an entangled state in two-mode Fock space. Eqs. (74) and (75) will be used often in the following discussions.

In the following we concentrate on finding the general-

ized Fresnel operators in both one- and two-mode cases with use of the IWOP technique.

5 Single-mode Fresnel operator as the image of the classical optical Fresnel transform

In this section we shall mainly introduce so-called generalized Fresnel operators (GFO) (in one-mode and two-mode cases both) [65] and some appropriate quantum optical representations (e.g. coherent state representation and entangled state representation) to manifestly link the formalisms in quantum optics to those in classical optics. In so doing, we find that the various transforms in classical optics are just the result of generalized Fresnel operators inducing transforms on appropriate quantum state vectors, i.e., classical optical Fresnel transforms have their counterpart in quantum optics. Besides, we can study the important *ABCD* rule obeyed by Gaussian beam propagation (also the ray propagation in matrix optics) [66] in the domain of quantum optics.

5.1 Single-mode GFO gained via coherent state method

For the coherent state $|z\rangle$ in quantum optics [1–5]

$$|z\rangle = \exp(za^\dagger - z^*a)|0\rangle \equiv \left| \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle \quad (76)$$

which is the eigenstate of annihilation operator a , $a|z\rangle = z|z\rangle$, using the IWOP and (12), we can put the over-completeness relation of $|z\rangle$ into normal ordering $\int \frac{d^2z}{\pi} |z\rangle \langle z| = \int \frac{d^2z}{\pi} : e^{-(z^*-a^\dagger)(z-a)} : = 1$. The canonical form of coherent state $|z\rangle$ is expressed as

$$|z\rangle = |p, q\rangle = \exp[i(pQ - qP)]|0\rangle \equiv \left| \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle \quad (77)$$

where $z = (q + ip)/\sqrt{2}$. It follows that $\langle p, q|Q|p, q\rangle = q$, $\langle p, q|P|p, q\rangle = p$, which indicates that the states $|p, q\rangle$ generate a canonical phase-space representation for a state $|\Psi\rangle$, $\Psi(p, q) = \langle p, q|\Psi\rangle$. Thus the coherent state is a good candidate for providing with classical phase-space description of quantum systems. Considering that the Fresnel transform's parameters (A, B, C, D) are elements of a ray transfer matrix M describing optical systems, M belongs to the unimodular symplectic group; and the coherent state $|p, q\rangle$ is a good candidate for providing with classical phase-space description of quantum systems, we naturally think of that the symplectic transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$ in classical phase space may be mapped onto a generalized Fresnel operator in Hilbert space through the coherent state basis. Thus we construct the following ket–bra projection operator

$$\iint_{-\infty}^{\infty} dx dp \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q \\ p \end{pmatrix} \right| \quad (78)$$

as the GFO. In fact, using notation of $|z\rangle$ (coherent state), and introducing complex numbers s and r ,

$$\begin{aligned} s &= \frac{1}{2}[A + D - i(B - C)] \\ r &= -\frac{1}{2}[A - D + i(B + C)], \quad |s|^2 - |r|^2 = 1 \end{aligned} \quad (79)$$

from Eq. (77) we know

$$\begin{aligned} & \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle \\ &= \left| \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle \equiv |sz - rz^*\rangle \\ &= \exp \left[-\frac{1}{2}|sz - rz^*|^2 + (sz - rz^*)a^\dagger \right] |0\rangle \end{aligned} \quad (80)$$

$\begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix}$ is still a symplectic group element, so Eq. (78) becomes [67–70]

$$F_1(s, r) = \sqrt{s} \int \frac{d^2z}{\pi} |sz - rz^*\rangle \langle z| \quad (81)$$

where the factor \sqrt{s} is attached to anticipate the unitarity of the operator F_1 . Eq. (78) tells us that c -number transform $(q, p) \rightarrow (Aq + Bp, Cq + Dp)$ in coherent state basis is mapped into $F_1(s, r)$. Now we prove $F_1(s, r)$ is really the FO we want. Using the IWOP technique and Eqs. (80) and (12) we can perform the integral

$$\begin{aligned} F_1(s, r) &= \exp \left(-\frac{r}{2s^*} a^{\dagger 2} \right) \exp \left[\left(a^\dagger a + \frac{1}{2} \right) \ln \frac{1}{s^*} \right] \\ &\quad \times \exp \left(\frac{r^*}{2s^*} a^2 \right) \end{aligned} \quad (82)$$

where we have used the mathematical formula [71, 72]

$$\begin{aligned} & \int \frac{d^2z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp \left(\frac{-\zeta\xi\eta + \xi^2g + \eta^2f}{\zeta^2 - 4fg} \right) \end{aligned} \quad (83)$$

with the convergent condition $\text{Re}(\zeta \pm f \pm g) < 0$, $\text{Re}(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}) < 0$. It then follows

$$\begin{aligned} & \langle z|F_1(s, r)|z'\rangle \\ &= \frac{1}{\sqrt{s^*}} \exp \left(-\frac{|z|^2 + |z'|^2}{2} - \frac{rz^*z'}{2s^*} + \frac{r^*z'^2}{2s^*} + \frac{z^*z'}{s^*} \right) \end{aligned} \quad (84)$$

Then using

$$\langle x_i|z\rangle = \pi^{-1/4} \exp \left(-\frac{x_i^2}{2} + \sqrt{2}x_i z - \frac{z^2}{2} - \frac{|z|^2}{2} \right) \quad (85)$$

and the completeness relation of coherent state as well

as Eq. (79), we obtain the matrix element of $F_1(s, r)$ ($\equiv F_1(A, B, C)$) in coordinate representation $\langle x_i|$,

$$\begin{aligned} & \langle x_2|F_1(s, r)|x_1\rangle \\ &= \frac{1}{\sqrt{2\pi i B}} \exp \left[\frac{i}{2B}(Ax_1^2 - 2x_2x_1 + Dx_2^2) \right] \\ &\equiv \mathcal{K}(x_2, x_1) \end{aligned} \quad (86)$$

which is just the kernel of generalized Fresnel transform (GFT) $\mathcal{K}(x_2, x_1)$ in Eq. (11). The above discussions demonstrate how to transit classical Fresnel transform to GFO through the coherent state and the IWOP technique.

Now if we define $g(x_2) = \langle x_2|g\rangle$, $f(x_1) = \langle x_1|f\rangle$ and using Eq. (16), we can rewrite Eq. (10) as

$$\begin{aligned} \langle x_2|g\rangle &= \int_{-\infty}^{\infty} dx_1 \langle x_2|F_1(A, B, C)|x_1\rangle \langle x_1|f\rangle \\ &= \langle x_2|F_1(A, B, C)|f\rangle \end{aligned} \quad (87)$$

which is just the quantum mechanical version of GFO. Therefore, the 1D GFT in classical optics corresponds to the 1-mode GFO $F_1(A, B, C)$ operating on state vector $|f\rangle$ in Hilbert space, i.e., $|g\rangle = F_1(A, B, C)|f\rangle$. One merit of GFO is: using coordinate-momentum representation transform we can immediately obtain GFT in “frequency” domain, i.e.,

$$\langle p_2|F|p_1\rangle = \frac{1}{\sqrt{2\pi i(-C)}} \exp \left[\frac{i}{2(-C)}(Dp_1^2 - 2p_2p_1 + Ap_2^2) \right] \quad (88)$$

Obviously, $F_1(A, B, C)$ induces the following transform

$$F_1^{-1}(A, B, C) \begin{pmatrix} Q \\ P \end{pmatrix} F_1(A, B, C) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \quad (89)$$

5.2 Group multiplication rule for single-mode GFO

Because two successive optical Fresnel transforms is still a Fresnel transform, we wonder if the product of two GFO is still a GFO. On the other hand, we have known that the GFO is the image of the symplectic transform $z \rightarrow sz - rz^*$, we expect that the product of two symplectic transforms are mapped into the GFO which is just the product of two GFOs. If so, then the correspondence between GFT and GFO is perfect. Using Eq. (81), $\langle z|z'\rangle = \exp[-\frac{1}{2}(|z|^2 + |z'|^2) + z^*z']$ and the IWOP technique we can directly perform the following integrals

$$\begin{aligned} & F_1(s, r)F_1(s', r') \\ &= \sqrt{ss'} \int \frac{d^2z d^2z'}{\pi^2} |sz - rz^*\rangle \langle z|s'z' - r'z'^*\rangle \langle z'| \\ &= \sqrt{s''} \int \frac{d^2z}{\pi} |s''z - r''z^*\rangle \langle z| = F_1(s'', r'') \end{aligned} \quad (90)$$

where we have set

$$s'' = ss' + rr'^*, \quad r'' = r's + rs'^* \tag{91}$$

or

$$M'' \equiv \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \begin{pmatrix} s' & -r' \\ -r'^* & s'^* \end{pmatrix} = MM' \tag{92}$$

$$|s''|^2 - |r''|^2 = 1$$

From which we see that it is just the mapping of the above (A, B, C, D) matrices multiplication. Hence $F_1(s, r)F_1(s', r')$ is the loyal representation of the product of two symplectic group elements shown in Eq. (92). The above discussion actually reveals an important property of coherent states, though two coherent state vectors are not orthogonal, but the equation

$$\begin{aligned} & \sqrt{ss'} \int \frac{d^2z d^2z'}{\pi^2} |sz - rz^*\rangle \langle z|s'z' - r'z'^*\rangle \langle z'| \\ & = \sqrt{s''} \int \frac{d^2z}{\pi} |s''z - r''z^*\rangle \langle z| \end{aligned} \tag{93}$$

seems as if their overlap $\langle z|s'z' - r'z'^*\rangle$ was a δ -function. The coherent state representation for GFOs' product may be visualized very easily, but it achieves striking importance, because it does not change its form when treating symplectic transform according to $z \rightarrow sz - rz^*$.

As a result of this group multiplication rule of GFO, we immediately obtain

$$\begin{aligned} \mathcal{K}^{M''}(x_2, x_1) &= \langle x_2 | F_1(s'', r'') | x_1 \rangle \\ &= \int_{-\infty}^{\infty} dx_3 \mathcal{K}^M(x_2, x_3) \mathcal{K}^{M'}(x_3, x_1) \end{aligned} \tag{94}$$

provided that the parameter matrices (s'', r'') satisfy Eq. (91). Thus by virtue of the group multiplication property of GFO we immediately find the successive transform property of GFTs.

6 Quantum optical $ABCD$ law for optical propagation — single-mode case

In classical optics, ray-transfer matrices, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $AD - BC = 1$ have been used to describe the geometrical formation of images by a centered lens system. For an optical ray (a centered spherical wavefront) passing through optical instruments there is a famous law, named $ABCD$ law, governing the relation between input ray (r_1, α_1) and output ray (r_2, α_2) , i.e.,

$$\begin{pmatrix} r_2 \\ \alpha_2 \end{pmatrix} = N \begin{pmatrix} r_1 \\ \alpha_1 \end{pmatrix} \tag{95}$$

where r_1 is the ray height from the optical axis, and α_1 is named the optical direction-cosine; $r_1/\alpha_1 \equiv R_1$ specifies the ray's wavefront shape. Eq. (95) implies

$$R_2 \equiv \frac{r_2}{\alpha_2} = \frac{AR_1 + B}{CR_1 + D} \tag{96}$$

This law is the core of matrix optics, since it tells us how the curvature of a centered spherical wavefront changes from one reference plane to the next. Besides, the multiplication rule of matrix optics implies that if the ray-transfer matrices of the n optical components are $M_1, M_2, M_3, \dots, M_n$, respectively, then the whole system is determined by a matrix $M = M_1 M_2 M_3 \dots M_n$.

One of the remarkable things of modern optics is the case in which geometrical ray-transfer methods, constituting the matrix optics, can be adapted to describe the generation and propagation of Laser beams. In 1965, Kogelnik [73] pointed out that propagation of Gaussian beam also obeys $ABCD$ law via optical diffraction integration, i.e., the input light field $f(x_1)$ and output light field $g(x_2)$ are related to each other by so-called Fresnel integration [31] $g(x_2) = \int_{-\infty}^{\infty} \mathcal{K}(A, B, C; x_2, x_1) f(x_1) dx_1$, where

$$\begin{aligned} \mathcal{K}(A, B, C; x_2, x_1) &= \frac{1}{\sqrt{2\pi i B}} \exp \left[\frac{i}{2B} (Ax_1^2 - 2x_2x_1 + Dx_2^2) \right] \end{aligned}$$

The $ABCD$ law for Gaussian beam passing through an optical system is [74, 75]

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{97}$$

where $q_1(q_2)$ represents the complex curvature of the input (output) Gaussian beam, Eq. (97) has the similar form as Eq. (96). An interesting and important question naturally arises [76]: Does $ABCD$ law also exhibit in quantum optics? Since classical Fresnel transform should have its quantum optical counterpart?

To see the $ABCD$ law more explicitly, using Eq. (79) we can re-express Eq. (82) as

$$\begin{aligned} F_1(A, B, C) &= \sqrt{\frac{2}{A + D + i(B - C)}} \\ &: \exp \left\{ \frac{A - D + i(B + C)}{2[A + D + i(B - C)]} a^{\dagger 2} \right. \\ &+ \left[\frac{2}{A + D + i(B - C)} - 1 \right] a^{\dagger} a \\ &\left. - \frac{A - D - i(B + C)}{2[A + D + i(B - C)]} a^2 \right\} : \end{aligned} \tag{98}$$

and the multiplication rule for F_1 is $F(A', B', C', D') F(A, B, C, D) = F(A'', B'', C'', D'')$, where

$$\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{99}$$

Next, we directly use the GFO to derive $ABCD$ law in quantum optics. From Eq. (99) we see that the GFO generates

$$F_1(A, B, C)|0\rangle = \sqrt{\frac{2}{A + iB - i(C + iD)}}$$

$$\times \exp \left\{ \frac{A - D + i(B + C)}{2[A + D + i(B - C)]} a^{\dagger 2} \right\} |0\rangle \tag{100}$$

If we identify

$$\frac{A - D + i(B + C)}{A + D + i(B - C)} = \frac{q_1 - i}{q_1 + i} \tag{101}$$

then

$$F_1(A, B, C)|0\rangle = \sqrt{-\frac{2/(C + iD)}{q_1 + i}} \exp \left[\frac{q_1 - i}{2(q_1 + i)} a^{\dagger 2} \right] |0\rangle \tag{102}$$

The solution of Eq. (101) is

$$q_1 \equiv -\frac{A + iB}{C + iD} \tag{103}$$

Let $F_1(A, B, C)|0\rangle$ expressed by Eq. (102) be an input state for an optical system which is characteristic by parameters A', B', C', D' , then the *quantum optical ABCD law* states that the output state is

$$F_1(A', B', C')F_1(A, B, C)|0\rangle = \sqrt{\frac{-2/(C'' + iD'')}{q_2 + i}} \exp \left[\frac{q_2 - i}{2(q_2 + i)} a^{\dagger 2} \right] |0\rangle \tag{104}$$

which has the similar form as Eq. (102), where (C'', D'') is determined by Eq. (99), and

$$\bar{q}_2 = \frac{A'\bar{q}_1 + B'}{C'\bar{q}_1 + D'}, \quad \bar{q}_j \equiv -q_j, \quad j = 1, 2 \tag{105}$$

which resembles Eq. (97).

Proof: According to the multiplication rule of two GFOs and Eqs. (98) and (99), we have

$$F_1(A', B', C')F_1(A, B, C)|0\rangle = \sqrt{\frac{-2/(C + iD)}{A'q_1 - B' - i(C'q_1 - D')}} \times \exp \left\{ \frac{A'q_1 - B' + i(C'q_1 - D')}{2[A'q_1 - B' - i(C'q_1 - D')]} a^{\dagger 2} \right\} |0\rangle \tag{106}$$

Using Eq. (103) we see $\frac{2/(C+iD)}{C'q_1 - D'} = -2/(C'' + iD'')$, along with Eq. (105) we can reach Eq. (104), thus the law is proved. Using Eq. (102) we can re-express Eq. (105) as

$$q_2 = -\frac{A'(A + iB) + B'(C + iD)}{C'(A + iB) + D'(C + iD)} = -\frac{A'' + iB''}{C'' + iD''} \tag{107}$$

which is in consistent to Eq. (103). Eqs. (102)–(107) are therefore self-consistent.

As an application of quantum optical ABCD law, we apply it to tackle the time-evolution of a time-dependent harmonic oscillator whose Hamiltonian is

$$H = \frac{1}{2}e^{-2\gamma t}P^2 + \frac{1}{2}\omega_0^2e^{2\gamma t}Q^2, \quad \hbar = 1 \tag{108}$$

where we have set the initial mass $m_0 = 1$, γ denotes

damping. Using $u(t) = e^{\frac{i\gamma}{2}Q^2}e^{-\frac{i\gamma t}{2}(QP+PQ)}$ to perform the transformation

$$u(t)Qu^{-1}(t) = e^{-\gamma t}Q \tag{109}$$

$$u(t)Pu^{-1}(t) = e^{\gamma t}P - \gamma e^{\gamma t}Q$$

then $i\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle$ leads to $i\frac{\partial|\phi\rangle}{\partial t} = \mathcal{H}|\phi\rangle$, $|\phi\rangle = u(t)|\psi(t)\rangle$,

$$H \rightarrow \mathcal{H} = u(t)Hu^{-1}(t) - iu(t)\frac{\partial u^{-1}(t)}{\partial t} = \frac{1}{2}P^2 + \frac{1}{2}\omega^2Q^2 \tag{110}$$

where $\omega^2 = \omega_0^2 - \gamma^2$. \mathcal{H} does not contain t explicitly. The dynamic evolution of a mass-varying harmonic oscillator from the Fock state $|0\rangle$ at initial time to a squeezed state at time t is

$$|\psi(t)\rangle_0 = u^{-1}(t)|0\rangle = e^{\frac{i\gamma t}{2}(QP+PQ)}e^{-\frac{i\gamma}{2}Q^2}|0\rangle \tag{111}$$

if we let $A = D = 1, B = 0, C = -\gamma$; and $A' = e^{-\gamma t}, D' = e^{\gamma t}, B' = C' = 0$, then $q_1 = \frac{1}{\gamma - i}, q_2 = \frac{e^{-2\gamma t}}{\gamma - i}$, according to Eq. (104) we directly obtain

$$u^{-1}(t)|0\rangle = \sqrt{\frac{2e^{-\gamma t}}{e^{-2\gamma t} + i\gamma + 1}} \times \exp \left[\frac{e^{-2\gamma t} - 1 - i\gamma}{2(e^{-2\gamma t} + 1 + i\gamma)} a^{\dagger 2} \right] |0\rangle \tag{112}$$

so the time evolution of the damping oscillator embodies the quantum optical ABCD law.

7 Optical operator method studied via GFO's decomposition

Fresnel diffraction is the core of Fourier optics [31, 43, 48–50]. Fresnel transform is frequently used in optical imaging, optical propagation and optical instrument design. The GFT represents a class of optical transforms which are of great importance for their applications to describe various optical systems. It is easily seen that when we let the transform kernel $\mathcal{K}(x_2, x_1) = \exp(ix_2x_1)$, the GFT changes into the well-known Fourier transform, which is adapted to express mathematically the Fraunhofer diffraction. And if $\mathcal{K}(x_2, x_1) = \exp[i(x_2 - x_1)^2]$, the GFT then describes a Fresnel diffraction. In studying various optical transformations someone also proposed so-called optical operator method [77–79] which used quantum mechanical operators' ordered product to express the mechanism of optical systems, such that the ray transfer through optical instruments and the diffraction can be discussed by virtue of the commutative relations of operators and the matrix algebra. Two important questions thus naturally arises: how to directly map the classical optical transformations to the optical operator method? How to combine the usual opti-

cal transformation operators, such as the square phase operators, scaling operator, Fourier transform operator and the propagation operator in free space, into a concise and unified form? In this section we shall solve these two problems and develop the optical operator method onto a new stage.

7.1 Four fundamental optical operators derived by decomposing GFO

The GFO $F_1(A, B, C)$ can also be expressed in the form of quadratic combination of canonical operators Q and P [80], i.e.,

$$F_1(A, B, C) = \exp\left(\frac{iC}{2A}Q^2\right) \exp\left[-\frac{i}{2}(QP + PQ) \ln A\right] \times \exp\left(-\frac{iB}{2A}P^2\right) \tag{113}$$

where we have set $\hbar = 1, A \neq 0$. To confirm this, we first calculate matrix element

$$\langle x|F_1(A, B, C)|p\rangle = \frac{1}{\sqrt{2\pi A}} \exp\left(\frac{iC}{2A}x^2 - \frac{iB}{2A}p^2 + \frac{ipx}{A}\right) \tag{114}$$

where we have used the squeezing property

$$\exp\left[-\frac{i}{2}(QP + PQ) \ln A\right] |p\rangle = \frac{1}{\sqrt{A}} |p/A\rangle \tag{115}$$

It then follows from Eq. (114) and $AD - BC = 1$, we have

$$\begin{aligned} \langle x_2|F_1(A, B, C)|x_1\rangle &= \int_{-\infty}^{\infty} dp \langle x_2|F_1(A, B, C)|p\rangle \langle p|x_1\rangle \\ &= \mathcal{K}^M(x_2, x_1) \end{aligned} \tag{116}$$

Thus $F_1(A, B, C)$ in Eq. (113) is really the expected GFO. Next we directly use Eq. (78) and the canonical operator (Q, P) representation (113) to develop the optical operator method.

Noticing the matrix decompositions [44]

$$F_1^{-1}(A, B, C) = \exp\left(-\frac{iC}{2D}Q^2\right) \exp\left[-\frac{i}{2}(QP + PQ) \ln D\right] \exp\left(\frac{iB}{2D}P^2\right) \tag{125}$$

It then follows that

$$F_1(A, B, C) = \exp\left(-\frac{iB}{2D}P^2\right) \exp\left[\frac{i}{2}(QP + PQ) \ln D\right] \exp\left(\frac{iC}{2D}Q^2\right), \quad D \neq 0 \tag{126}$$

Besides, when we notice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/B & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1/B \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A/B & 1 \end{pmatrix} \tag{127}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & A/C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/C & 0 \\ 0 & -C \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & D/C \\ 0 & 1 \end{pmatrix} \tag{128}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C/A & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & B/A \\ 0 & 1 \end{pmatrix} \tag{117}$$

and comparing Eqs. (78) and (113) as well as using (99) we know

$$F_1(A, B, C) = F_1(1, 0, C/A)F_1(A, 0, 0)F_1(1, B/A, 0) \tag{118}$$

where

$$F_1(1, 0, C/A) = \exp\left(\frac{iC}{2A}Q^2\right) \tag{119}$$

$$F_1(1, B/A, 0) = \exp\left(-\frac{iB}{2A}P^2\right) \tag{120}$$

$$F_1(A, 0, 0) = \exp\left[-\frac{i}{2}(QP + PQ) \ln A\right] \tag{121}$$

which are named quadrature phase operator, Fresnel propagator in free space, and scaling operator (squeezed operator [14–17]), respectively. When $A = D = 0, B = 1, C = -1$, from Eq. (98) we see

$$F_1(0, 1, -1) = \exp\left[-i\frac{\pi}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \tag{122}$$

which is named the Fourier operator, since it quantum mechanically transforms [71, 72]

$$\begin{aligned} F_1^\dagger(0, 1, -1)QF_1(0, 1, -1) &= P \\ F_1^\dagger(0, 1, -1)PF_1(0, 1, -1) &= -Q \end{aligned} \tag{123}$$

7.2 Alternate decompositions of GFO

Note that when $A = 0$, the decomposition (113) is not available, instead, from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \tag{124}$$

and Eqs. (98), (99) and (113) we have

we have another decomposition for $B \neq 0$,

$$F_1(A, B, C) = \exp\left(\frac{iD}{2B}Q^2\right) \exp\left[-\frac{i}{2}(QP + PQ) \ln B\right] \\ \times \exp\left[-\frac{i\pi}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \exp\left(\frac{iA}{2B}Q^2\right) \tag{129}$$

and for $C \neq 0$,

$$F_1(A, B, C) = \exp\left(-\frac{iA}{2C}P^2\right) \\ \times \exp\left[-\frac{i}{2}(QP + PQ) \ln\left(\frac{-1}{C}\right)\right] \\ \times \exp\left[-\frac{i\pi}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \exp\left(-\frac{iD}{2C}P^2\right) \tag{130}$$

7.3 Some optical operator identities

For a special optical systems with the parameter $A = 0$, $C = -B^{-1}$,

$$\begin{pmatrix} 0 & B \\ -B^{-1} & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/B & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{131}$$

we have

$$\exp\left(-\frac{iB}{2D}P^2\right) \exp\left[\frac{i}{2}(QP + PQ) \ln D\right] \\ \times \exp\left(\frac{-i}{2DB}Q^2\right) \\ = \exp\left(\frac{-iD}{2B}Q^2\right) \exp\left[\frac{i}{2}(QP + PQ) \ln B\right] \\ \times \exp\left[-\frac{i\pi}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \tag{132}$$

In particular, when $A = D = 0$, $C = -B^{-1}$, from

$$\begin{pmatrix} 0 & B \\ -\frac{1}{B} & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & \frac{1}{B} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{133}$$

we have

$$\exp\left[-\frac{B^2 - 1}{2(B^2 + 1)}a^{\dagger 2}\right] \exp\left[\left(a^\dagger a + \frac{1}{2}\right) \ln\left(\frac{-2Bi}{B^2 + 1}\right)\right] \\ \times \exp\left[-\frac{B^2 - 1}{2(B^2 + 1)}a^2\right] \\ = \exp\left[-\frac{i}{2}(QP + PQ) \ln B\right] \exp\left[-i\frac{\pi}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \tag{134}$$

Using the following relations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & (A-1)/C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & (D-1)/C \\ 0 & 1 \end{pmatrix} \tag{135}$$

it then follows that

$$F_1(A, B, C) = \exp\left[-\frac{i(A-1)}{2C}P^2\right] \exp\left(\frac{iC}{2}Q^2\right) \\ \times \exp\left[-\frac{i(D-1)}{2C}P^2\right] \tag{136}$$

While from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (D-1)/B & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \\ \times \begin{pmatrix} 1 & 0 \\ (A-1)/B & 1 \end{pmatrix} \tag{137}$$

we obtain

$$F_1(A, B, C) = \exp\left[\frac{i(D-1)}{2B}Q^2\right] \exp\left(-\frac{iB}{2}P^2\right) \\ \times \exp\left[\frac{i(A-1)}{2B}Q^2\right] \tag{138}$$

so we have

$$\exp\left[\frac{i(D-1)}{2B}Q^2\right] \exp\left(-\frac{iB}{2}P^2\right) \exp\left[\frac{i(A-1)}{2B}Q^2\right] \\ = \exp\left[-\frac{i(A-1)}{2C}P^2\right] \exp\left(\frac{iC}{2}Q^2\right) \exp\left[-\frac{i(D-1)}{2C}P^2\right] \tag{139}$$

In this section, based on a one-to-one correspondence between classical Fresnel transform in phase space and quantum unitary transform in state-vector space and the IWOP technique as well as the coherent state representation, we have found a way to directly map the classical optical transformations to the optical operator method. We have combined the usual optical transformation operators, such as the square phase operators, scaling operator, Fourier transform operator and the propagation operator in free space, into a concise and unified form. The various decompositions of Fresnel operator into the exponential canonical operators are also obtained.

8 Quantum tomography and probability distribution for the Fresnel quadrature phase

In quantum optics theory all possible linear combinations of quadratures Q and P of the oscillator field mode a and a^\dagger can be measured with the homodyne measurement just by varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of Wigner function (WF). Thus the homodyne measurement of light field permits the reconstruction of the WF of a quantum

system by varying the phase shift between two oscillators. In Ref. [81] Vogel and Risken pointed out that the probability distribution for the rotated quadrature phase $Q_\theta \equiv [a^\dagger \exp(i\theta) + a \exp(-i\theta)]/\sqrt{2}$, $[a, a^\dagger] = 1$, which depends on only one θ angle, can be expressed in terms of WF, and that the reverse is also true (named as Vogel–Risken relation). That is, one can obtain the Wigner distribution by tomographic inversion of a set of measured probability distributions, $P_\theta(q_\theta)$, of the quadrature amplitude. Once the distribution $P_\theta(q_\theta)$ are obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the Wigner distribution and density matrix. The Radon transform of the WF is closely related to the expectation values or densities formed with the eigenstates to the rotated canonical observables. The field of problems of the reconstruction of the density operator from such data is called quantum tomography. (Optical tomographic imaging techniques derive 2D data from a three-dimensional object to obtain a slice image of the internal structure and thus have the ability to peer inside the object noninvasively. The slice image is equated with tomogram.) The theoretical development in quantum tomography in the last decade has progressed in the direction of determining more physical relevant parameters of the density from tomographic data [81–86].

8.1 Relation between Fresnel transform and Radon transform of WF

In Refs. [87, 88], the Radon transform of WF which depends on two continuous parameters is introduced. This has the advantage in conveniently associating quantum tomography theory with squeezed coherent state theory. In this subsection we want to derive relations between the Fresnel transform and the Radon transform of WF in quantum optics in tomography theory.

By extending the rotated quadrature phase Q_θ to the Fresnel quadrature phase

$$Q_F \equiv (s^*a + ra^\dagger + sa^\dagger + r^*a)/\sqrt{2} = F_1 Q F_1^\dagger \quad (140)$$

where s and r are related to ABCD through Eq. (79), we shall prove that the (D, B) related Radon transform of Wigner operator $\Delta(q, p)$ is just the pure state density operator $|q\rangle_{s,r,s,r}\langle q|$ (named as the tomographic density operator) formed with the eigenstates belonging to the quadrature Q_F ($|q\rangle_{s,r} = F_1|q\rangle$; Q is the coordinate operator),

$$\begin{aligned} F_1|q\rangle\langle q|F_1^\dagger &= |q\rangle_{s,r,s,r}\langle q| \\ &= \int_{-\infty}^{\infty} dq' dp' \delta[q - (Dq' - Bp')] \Delta(q', p') \end{aligned} \quad (141)$$

Since F corresponds to classical Fresnel transform in optical diffraction theory, so Eq. (141) indicates that the

probability distribution for the Fresnel quadrature phase is the Radon transform of WF [89].

Proof: First, from Eq. (82) we see

$$F_1(s, r) a F_1^\dagger(s, r) = s^* a + r a^\dagger \quad (142)$$

so from $Q = \frac{a+a^\dagger}{\sqrt{2}}$, $P = i\frac{a^\dagger-a}{\sqrt{2}}$, we have indeed

$$\begin{aligned} F_1 Q F_1^\dagger &= F_1 \frac{a + a^\dagger}{\sqrt{2}} F_1^\dagger \\ &= (s^*a + r a^\dagger + s a^\dagger + r^*a)/\sqrt{2} = Q_F \end{aligned} \quad (143)$$

Second, we can derive the explicit form of $|q\rangle_{s,r}$. Starting from $s^* + r^* = D + iB$, $s^* - r^* = A - iC$, we set up the eigenvector equation

$$Q_F |q\rangle_{s,r} = (DQ - BP) |q\rangle_{s,r} = q |q\rangle_{s,r} \quad (144)$$

It follows that

$$|q\rangle_{s,r} = F_1(s, r) |q\rangle \quad (145)$$

In the coordinate and momentum representations we have

$$\langle q' | Q_F | q \rangle_{s,r} = \left(Dq' + iB \frac{d}{dq'} \right) \langle q' | q \rangle_{s,r} = q \langle q' | q \rangle_{s,r} \quad (146)$$

$$\langle p | Q_F | q \rangle_{s,r} = \left(iD \frac{d}{dp} - Bp \right) \langle p | q \rangle_{s,r} = q \langle p | q \rangle_{s,r} \quad (147)$$

The normalizable solutions to Eqs. (146) and (147) are

$$\langle q' | q \rangle_{s,r} = c(q) \exp \left[\frac{iq'(Dq' - 2q)}{2B} \right] \quad (148)$$

$$\langle p | q \rangle_{s,r} = d(q) \exp \left[\frac{ip(-Bp - 2q)}{2D} \right] \quad (149)$$

Using the Fock representation of $|q\rangle$ and $|p\rangle$ in Eqs. (13) and (22), we obtain

$$\begin{aligned} |q\rangle_{s,r} &= \pi^{-1/4} c(q) \sqrt{\frac{2B\pi}{B - iD}} \exp \left[-\frac{q^2}{2B(B - iD)} \right. \\ &\quad \left. + \frac{\sqrt{2}a^\dagger q}{D + iB} - \frac{D - iB}{D + iB} \frac{a^{\dagger 2}}{2} \right] |0\rangle \end{aligned} \quad (150)$$

and

$$\begin{aligned} |q\rangle_{s,r} &= d(q) \pi^{-1/4} \sqrt{\frac{2\pi D}{D + iB}} \exp \left[-\frac{q^2}{2D(D + iB)} \right. \\ &\quad \left. + \frac{\sqrt{2}a^\dagger q}{D + iB} - \frac{D - iB}{D + iB} \frac{a^{\dagger 2}}{2} \right] |0\rangle \end{aligned} \quad (151)$$

Comparing Eq. (150) with Eq. (151) we see

$$\frac{c(q)}{d(q)} = \sqrt{\frac{D}{iB}} \exp \left(\frac{iA}{2B} q^2 - \frac{iCq^2}{2D} \right) \quad (152)$$

On the other hand, according to the orthogonalization of $|q\rangle_{s,r}$, ${}_{s,r}\langle q' | q'' \rangle_{s,r} = \delta(q' - q'')$, we have

$$|c(q)|^2 = \frac{1}{2\pi B}, \quad |d(q)|^2 = \frac{1}{2\pi D} \quad (153)$$

Thus, combining Eqs. (152) and (153) we deduce

$$c(q) = \frac{1}{\sqrt{2\pi iB}} \exp\left(\frac{iA}{2B}q^2\right)$$

$$d(q) = \frac{1}{\sqrt{2\pi D}} \exp\left(\frac{iCq^2}{2D}\right) \quad (154)$$

and

$$|q\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{D+iB}} \exp\left(-\frac{A-iC}{D+iB}\frac{q^2}{2} + \frac{\sqrt{2}q}{D+iB}a^\dagger - \frac{D-iB}{D+iB}\frac{a^{\dagger 2}}{2}\right)|0\rangle \quad (155)$$

or

$$|q\rangle_{s,r} \equiv \frac{\pi^{-1/4}}{\sqrt{s^*+r^*}} \exp\left(-\frac{s^*-r^*}{s^*+r^*}\frac{q^2}{2} + \frac{\sqrt{2}q}{s^*+r^*}a^\dagger - \frac{s+r}{s^*+r^*}\frac{a^{\dagger 2}}{2}\right)|0\rangle \quad (156)$$

It is easily seen that that $|q\rangle_{s,r}$ make up a complete set $\int_{-\infty}^{\infty} dq|q\rangle_{s,r,s,r}\langle q| = 1$ (so $|q\rangle_{s,r}$ can be named as the tomography representation). Then according to the Weyl quantization scheme [51]

$$H(Q, P) = \int_{-\infty}^{\infty} dpdq\Delta(q, p)h(q, p) \quad (157)$$

where $h(q, p)$ is the Weyl correspondence of $H(Q, P)$,

$$h(q, p) = 2\pi T_r[H(Q, P)\Delta(q, p)] \quad (158)$$

$\Delta(q, p)$ is the Wigner operator [42, 90–94],

$$\Delta(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{ipu} |q + \frac{u}{2}\rangle\langle q - \frac{u}{2}| \quad (159)$$

and using Eqs. (158), (159) and (150) we know that the classical Weyl correspondence (Weyl image) of the projection operator $|q\rangle_{s,r,s,r}\langle q|$ is

$$2\pi \text{Tr}[\Delta(q', p')|q\rangle_{s,r,s,r}\langle q|] = \delta[q - (Dq' - Bp')] \quad (160)$$

which means

$$|q\rangle_{s,r,s,r}\langle q| = \int_{-\infty}^{\infty} dq' dp' \delta[q - (Dq' - Bp')] \Delta(q', p') \quad (161)$$

Combining Eqs. (142)–(145) together we complete the proof. Therefore, the probability distribution for the Fresnel quadrature phase is the Radon transform of WF

$$|\langle q|F_1^\dagger|\psi\rangle|^2 = |{}_{s,r}\langle q|\psi\rangle|^2$$

$$= \int_{-\infty}^{\infty} dq' dp' \delta[q - (Dq' - Bp')] \langle\psi|\Delta(q', p')|\psi\rangle \quad (162)$$

So we name $|q\rangle_{s,r,s,r}\langle q|$ the tomographic density. Moreover, the tomogram of quantum state $|\psi\rangle$ is just the squared modulus of the wave function ${}_{s,r}\langle q|\psi\rangle$. This new relation between quantum tomography and optical Fresnel transform may provide experimentalists to figure out

new approach for generating tomography.

The introduction of $|q\rangle_{s,r}$ also brings convenience to obtaining the inverse of Radon transformation. Using the complete relation of $|q\rangle_{s,r}$ and Eq. (144) we have

$$e^{-igQ_F} = \int_{-\infty}^{\infty} dq|q\rangle_{s,r,s,r}\langle q|e^{-igq}$$

$$= \int_{-\infty}^{\infty} dqdp\Delta(q, p)e^{-ig(Dq-Bp)} \quad (163)$$

Considering its right hand-side as a Fourier transformation, its reciprocal transform is

$$\Delta(q, p) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dg'|g'| \int_0^\pi d\varphi|q'\rangle_{s,r,s,r}\langle q'|$$

$$\times \exp\left[-ig'\left(\frac{q'}{\sqrt{D^2+B^2}} - q\cos\varphi - p\sin\varphi\right)\right] \quad (164)$$

where $g' = g\sqrt{D^2+B^2}$, $\cos\varphi = \frac{D}{\sqrt{D^2+B^2}}$, $\sin\varphi = \frac{-B}{\sqrt{D^2+B^2}}$. So once the distribution $|{}_{s,r}\langle q|\psi\rangle|^2$ are obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the Wigner distribution.

By analogy, we can conclude that the (A, C) related Radon transform of $\Delta(q, p)$ is just the pure state density operator $|p\rangle_{s,r,s,r}\langle p|$ formed with the eigenstates belonging to the conjugate quadrature of Q_F ,

$$F_1|p\rangle\langle p|F_1^\dagger = |p\rangle_{s,r,s,r}\langle p|$$

$$= \int_{-\infty}^{\infty} dq' dp' \delta[p - (Aq' - Cp')] \Delta(q', p') \quad (165)$$

Similarly, we find that for the momentum density,

$$F_1|p\rangle\langle p|F_1^\dagger = |p\rangle_{s,r,s,r}\langle p|$$

$$= \int_{-\infty}^{\infty} dq' dp' \delta[p - (Ap - Cq')] \Delta(q', p') \quad (166)$$

where

$$F_1|p\rangle = |p\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{A-iC}} \exp\left(-\frac{D+iB}{A-iC}\frac{p^2}{2} + \frac{\sqrt{2}ip}{A-iC}a^\dagger + \frac{A+iC}{A-iC}\frac{a^{\dagger 2}}{2}\right)|0\rangle \quad (167)$$

As an application of the relation (141), recalling that the $F_1(r, s)$ makes up a faithful representation of the symplectic group, it then follows from Eq. (141) that

$$F_1'(r', s')F_1(r, s)|q\rangle\langle q|F_1^\dagger(r, s)F_1'^{\dagger}(r', s')$$

$$= |q\rangle_{s'', r'' s'', r''}\langle q|$$

$$= \int \int_{-\infty}^{\infty} dq' dp' \delta[q - ((B'C + DD')q' - (AB' + BD')p')] \Delta(q', p') \quad (168)$$

In this way a complicated Radon transform of tomography can be viewed as the sequential operation of

two Fresnel transforms. This confirms that the continuous Radon transformation corresponds to the symplectic group transformation [87, 88]. This is an advantage of introducing the Fresnel operator. The group property of Fresnel operators help us to analyze complicated Radon transforms in terms of some sequential Fresnel transformations. The new relation may provide experimentalists to figure out new approach for realizing tomography.

8.2 A theorem for the calculation of the tomogram

In this subsection, we introduce a new theorem, which states that the tomogram of a density operator ρ is equal to the marginal integration of the classical Weyl correspondence function of $F^\dagger \rho F_1$, where F_1 is the Fresnel operator.

Multiply the both sides of Eq. (161) by a density matrix ρ . Then perform the trace. Note the Wigner function $W(p, q) = \text{Tr}[\rho \Delta(p, q)]$. One can see

$$\begin{aligned} \text{Tr} \left[\iint_{-\infty}^{\infty} dq' dp' \delta[q - (Dq' - Bp')] \Delta(q', p') \rho \right] \\ = \text{Tr}(|q\rangle_{s,r,s,r} \langle q| \rho) =_{s,r} \langle q | \rho | q \rangle_{s,r} = \langle q | F^\dagger \rho F_1 | q \rangle \\ = \iint_{-\infty}^{\infty} dq' dp' \delta[q - (Dq' - Bp)] W(p, q) \end{aligned} \quad (169)$$

The right hand side of Eq. (169) is commonly defined as the tomogram of quantum states in (B, D) direction, so in our view the calculation of tomogram in (B, D) direction is ascribed to the calculation of

$$\langle q | F^\dagger \rho F_1 | q \rangle \equiv \Xi \quad (170)$$

This is a concise and neat formula. Similarly, the tomogram in (A, C) direction is ascribed to $\langle p | F^\dagger \rho F_1 | p \rangle$.

According to the Weyl correspondence rule

$$H(X, P) = \iint_{-\infty}^{\infty} dp dx \mathfrak{h}(p, x) \Delta(p, x) \quad (171)$$

and the Weyl ordering form of $\Delta(p, q)$

$$\Delta(p, q) = \dot{\dot{\delta}}(q - Q) \delta(p - P) \dot{\dot{\delta}} \quad (172)$$

where the symbol $\dot{\dot{\delta}}$ denotes Weyl ordering, the classical correspondence of a Weyl ordered operator $\dot{\dot{\mathfrak{h}}}(Q, P) \dot{\dot{\delta}}$ is obtained just by replacing $Q \rightarrow q, P \rightarrow p$ in h , i.e.,

$$\dot{\dot{\mathfrak{h}}}(Q, P) \dot{\dot{\delta}} = \iint_{-\infty}^{\infty} dp dq \mathfrak{h}(p, q) \Delta(p, q) \quad (173)$$

Let the classical Weyl correspondence of $F^\dagger \rho F_1$ be $h(p, q)$

$$F^\dagger \rho F = \iint_{-\infty}^{\infty} dp dq h(p, q) \Delta(p, q)$$

then using Eqs. (170) and (159) we have

$$\begin{aligned} \Xi &= \langle q | \iint dp dq' h(p, q') \Delta(p, q') | q \rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} h(p, q) \end{aligned} \quad (174)$$

Thus we reach a theorem as below.

The tomogram of a density operator ρ is equal to the marginal integration of the classical Weyl correspondence $h(p, q)$ of $F^\dagger \rho F_1$, where F is the Fresnel operator, expressed by

$$\text{Tr} [\rho | q \rangle_{s,r,s,r} \langle q |] = \int_{-\infty}^{\infty} \frac{dp}{2\pi} h(p, q) \quad (175)$$

or

$$\text{Tr} [\rho | p \rangle_{s,r,s,r} \langle p |] = \int_{-\infty}^{\infty} \frac{dx}{2\pi} h(p, q) \quad (176)$$

In this way the relationship between tomogram of a density operator ρ and the Fresnel transformed ρ 's classical Weyl function is established.

9 Two-mode GFO and its application

For two-dimensional optical Fresnel transforms [see Eq. (4)] in the x - y plane one may naturally think that the 2-mode GFO is just the direct product of two independent 1-mode GFOs, but with the same (A, B, C, D) matrix. However, here we present another 2-mode Fresnel operator which cannot only lead to the usual 2-dimensional optical Fresnel transforms in some appropriate quantum mechanical representations, but also provide us with some new classical transformations (we name them entangled Fresnel transformations.).

9.1 Two-mode GFO gained via coherent state representation

Similar to the single-mode case, we introduce the two-mode GFO $F_2(r, s)$ through the following 2-mode coherent state representation [67–70]

$$F_2(r, s) = s \int \frac{d^2 z_1 d^2 z_2}{\pi^2} |sz_1 + rz_2^*, rz_1^* + sz_2\rangle \langle z_1, z_2| \quad (177)$$

which indicates that $F_2(r, s)$ is a mapping of classical symplectic transform $(z_1, z_2) \rightarrow (sz_1 + rz_2^*, rz_1^* + sz_2)$ in phase space. Concretely, the ket in Eq. (177) is

$$|sz_1 + rz_2^*, rz_1^* + sz_2\rangle \equiv |sz_1 + rz_2^*\rangle_1 \otimes |rz_1^* + sz_2\rangle_2 \quad (178)$$

s and r are complex and satisfy the unimodularity condition. Using the IWOP technique we perform the integral in Eq. (177) and obtain

$$F_2(r, s) = \exp\left(\frac{r}{s^*} a_1^\dagger a_2^\dagger\right) \exp[(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \ln(s^*)^{-1}]$$

$$\times \exp\left(-\frac{r^*}{s^*}a_1a_2\right) \quad (179)$$

Thus $F_2(r, s)$ induces the transform

$$\begin{aligned} F_2(r, s)a_1F_2^{-1}(r, s) &= s^*a_1 - ra_2^\dagger \\ F_2(r, s)a_2F_2^{-1}(r, s) &= s^*a_2 - ra_1^\dagger \end{aligned} \quad (180)$$

and F_2 is actually a general 2-mode squeezing operator. Recall that Eq. (35) implies the intrinsic relation between the EPR entangled state and the two-mode squeezed state, which has physical implementation. That is, in the output of a parametric down-conversion the idler-mode and the signal-mode constitute a two-mode squeezed state. Meanwhile, they are entangled with each other in frequency domain. We naturally select the entangled state representation to relate $F_2(r, s)$ to two-dimensional GFT. Letting $|g\rangle = F_2(r, s)|f\rangle$, and then project $|\psi\rangle$ onto the entangled state $\langle\eta'|$ defined by Eq. (28), and using the completeness relation (30) of $|\eta\rangle$, we obtain

$$\begin{aligned} g(\eta') &\equiv \langle\eta'|g\rangle = \langle\eta'|F_2(r, s)|f\rangle \\ &= \int \frac{d^2\eta}{\pi} \langle\eta'|F_2(r, s)|\eta\rangle \langle\eta|f\rangle \\ &\equiv \int d^2\eta \mathcal{K}_2^{(r,s)}(\eta', \eta) f(\eta) \end{aligned} \quad (181)$$

Then using the overcompleteness relation of the coherent state, we can calculate the integral kernel

$$\begin{aligned} \mathcal{K}_2^{(r,s)}(\eta', \eta) &= \frac{1}{2iB\pi} \\ &\times \exp\left[\frac{i}{2B}(A|\eta|^2 - (\eta\eta'^* + \eta'^*\eta) + D|\eta'|^2)\right] \\ &\equiv \mathcal{K}_2^M(\eta', \eta) \end{aligned} \quad (182)$$

where the relation between s, r and (A, B, C, D) in Eq. (79) is used and the superscript M only means that the parameters of \mathcal{K}_2^M are $[A, B; C, D]$, and the subscript 2 means the 2D kernel. Eq. (182) has the similar form as Eq. (11) except for its complex form. Taking $\eta_1 = x_1, \eta_2 = x_2$ and $\eta'_1 = x'_1, \eta'_2 = x'_2$, we have

$$\begin{aligned} \mathcal{K}_2^M(\eta', \eta) &= \mathcal{K}_2^M(x'_1, x'_2; x_1, x_2) \\ &= \mathcal{K}_1^M(x_1, x'_1) \otimes \mathcal{K}_1^M(x_2, x'_2) \end{aligned} \quad (183)$$

This shows that $F_2(r, s)$ is really the counterpart of the 2D GFT.

If taking the matrix element of $F_2(r, s)$ in the $|\xi\rangle$ representation which is conjugate to $|\eta\rangle$, we obtain the 2-dimensional GFT in its “frequency domain”, i.e.,

$$\begin{aligned} \langle\xi'|F_2(r, s)|\xi\rangle &= \frac{1}{2i(-C)\pi} \exp\left[\frac{i}{2(-C)}(D|\xi|^2 + A|\xi'|^2 \right. \\ &\left. - \xi'^*\xi - \xi'\xi^*)\right] \equiv \mathcal{K}_2^N(\xi', \xi) \end{aligned} \quad (184)$$

where the superscript N means that this transform kernel corresponds to the parameter matrix $M =$

$[D, -C, -B, A]$.

The two-mode GFO also abides by the group multiplication rule. Using the IWOP technique and Eq. (177) we can also obtain

$$F_2(r, s)F_2(r', s') = F_2(r'', s'') \quad (185)$$

where (r'', s'') are given by Eq. (91) or Eq. (92). Therefore, Eq. (185) is a loyal representation of the multiplication rule for ray transfer matrices in the sense of *Matrix Optics*.

9.2 Quantum optical ABCD law for two-mode GFO

Next we extended quantum optical ABCD law to the two-mode case. Operating with $F_2(r, s)$ on two-mode number state $|m, n\rangle$ and using the overlap between coherent state and number state, i.e., $\langle z_1, z_2 | m, n \rangle = z_1^{*m} z_2^{*n} / \sqrt{m!n!} \exp[-\frac{1}{2}(|z_1|^2 + |z_2|^2)]$, and the integral formula [71, 72]

$$H_{m,n}(\xi, \eta) = (-1)^n e^{\xi\eta} \int \frac{d^2z}{\pi} z^n z^{*m} e^{-|z|^2 + \xi z - \eta z^*} \quad (186)$$

we can calculate

$$\begin{aligned} F_2(r, s)|m, n\rangle &= \frac{r^{*n}}{s^{*n+1}\sqrt{m!n!}} H_{m,n} \left[\frac{a_1^\dagger}{s^*}, \frac{a_2^\dagger}{r^*} \right] \\ &\times \exp\left(\frac{ra_1^\dagger a_2^\dagger}{s^*}\right) |00\rangle \end{aligned} \quad (187)$$

where $H_{m,n}(\epsilon, \varepsilon)$ is the two variables Hermite polynomial [87, 88], shown in Eqs. (73) and (74).

Using Eqs. (79) and (103), we recast Eq. (187) into

$$\begin{aligned} F_2(r, s)|m, n\rangle &= \frac{-2/(C+iD)}{(q_1+i)\sqrt{m!n!}} \left(-\frac{q_1^*+iC-iD}{q_1+iC+iD}\right)^n \\ &\times H_{m,n} \left[-\frac{2a_1^\dagger/(C+iD)}{q_1+i}, \frac{2a_2^\dagger/(C-iD)}{q_1^*+i} \right] \\ &\times \exp\left(-\frac{q_1-i}{q_1+i} a_1^\dagger a_2^\dagger\right) |00\rangle \end{aligned} \quad (188)$$

Taking notice of the multiplication rule of $F_2(r, s)$ in Eq. (185), which is equivalent to

$$F_2(A', B', C')F_2(A, B, C) = F_2(A'', B'', C'') \quad (189)$$

where $(A', B', C'), (A, B, C)$ and (A'', B'', C'') are related to each other by Eq. (99).

Next we directly use the GFO to derive ABCD rule in quantum optics for Gaussian beam in the two-mode case. According to Eq. (187) and Eq. (189) we obtain

$$\begin{aligned} F_2(A', B', C')F_2(A, B, C)|m, n\rangle &= \frac{r''^{*n}}{s''^{*n+1}\sqrt{m!n!}} H_{m,n} \left[\frac{a_1^\dagger}{s''^*}, \frac{a_2^\dagger}{r''^*} \right] \exp\left(\frac{r''a_1^\dagger a_2^\dagger}{s''^*}\right) |00\rangle \end{aligned}$$

(190)

Similar to the way of deriving Eq. (188), we can simplify Eq. (190) as

$$\begin{aligned}
 & F_2(A', B', C')F_2(A, B, C)|00\rangle \\
 &= \frac{-2/(C'' + iD'')}{(q_2 + i)\sqrt{m!n!}} \left(-\frac{q_2^* + iC'' - iD''}{q_2 + iC'' + iD''} \right)^n \\
 &\times H_{m,n} \left[-\frac{2a_1^\dagger/(C'' + iD'')}{q_2 + i}, \frac{2a_2^\dagger/(C'' - iD'')}{q_2^* + i} \right] \\
 &\times \exp \left[-\frac{q_2 - i}{q_2 + i} a_1^\dagger a_2^\dagger \right] |00\rangle \tag{191}
 \end{aligned}$$

where the relation between q_2 and q_1 are determined by Eq. (105) which resembles Eq. (97). This is just the new *ABCD* law for two-mode case in quantum optics.

9.3 Optical operators derived by decomposing GFO

9.3.1 GFO as quadratic combinations of canonical operators

In order to obtain the quadratic combinations of canonical operators, first derive an operator identity. Note $Q_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/(\sqrt{2}i)$, and Eqs. (29), (30), and we can prove the operator identity

$$e^{\frac{\lambda}{2}[(Q_1 - Q_2)^2 + (P_1 + P_2)^2]} = \frac{1}{1 - \lambda} : \exp \left(\frac{2\lambda}{1 - \lambda} K_+ \right) : \tag{192}$$

where we have set

$$K_+ \equiv \frac{1}{4} [(Q_1 - Q_2)^2 + (P_1 + P_2)^2] \tag{193}$$

When $B = 0, A = 1, C \rightarrow C/A, D = 1$, and using Eq. (179) we see that

$$F_2(1, 0, C/A) = \exp \left(\frac{iC}{A} K_+ \right) \tag{194}$$

which is corresponding to the square phase operator in single-mode case. In a similar way, using Eqs. (37) and (39) we can derive another operator identity

$$e^{\frac{\lambda}{2}[(Q_1 + Q_2)^2 + (P_1 - P_2)^2]} = \frac{1}{1 - \lambda} : \exp \left[\frac{2\lambda}{1 - \lambda} K_- \right] : \tag{195}$$

where

$$K_- = \frac{1}{4} [(Q_1 + Q_2)^2 + (P_1 - P_2)^2] \tag{196}$$

It then follows from Eqs. (179) and (195) that

$$F_2(1, B/A, 0) = \exp \left(-\frac{iB}{A} K_- \right) \tag{197}$$

which is corresponding to Fresnel propagator in free space (single-mode case). In particular, when $B = C = 0$, and $D = A^{-1}$, Eq. (179) becomes

$$\begin{aligned}
 F_2(A, 0, 0) &= \text{sech}\lambda : \exp \left[-a_1^\dagger a_2^\dagger \tanh \lambda + (\text{sech}\lambda - 1) \right. \\
 &\quad \left. \times (a_1^\dagger a_1 + a_2^\dagger a_2) + a_1 a_2 \tanh \lambda \right] : \tag{198}
 \end{aligned}$$

where $\frac{A-A^{-1}}{A+A^{-1}} = \tanh \lambda, A = e^\lambda$. Eq. (198) is just the two-mode squeezing operator,

$$F_2(A, 0, 0) = \exp(-2K_0 \ln A) \tag{199}$$

$$K_0 \equiv -\frac{i}{2}(Q_1 P_2 + Q_2 P_1) \tag{200}$$

which actually squeezes the entangled state $|\xi\rangle$ (its conjugate state is $|\eta\rangle$),

$$F_2(A, 0, 0)|\xi\rangle = \int \frac{d^2\xi'}{\pi A} |\xi'/A\rangle \langle \xi'|\xi\rangle = \frac{1}{A} |\xi/A\rangle \tag{201}$$

Using the decomposition (117) of the matrix and combining equations (194), (197) and (199) together, we see that

$$\begin{aligned}
 F_2(A, B, C) &= \exp \left(\frac{iC}{A} K_+ \right) \exp(-2K_0 \ln A) \\
 &\quad \times \exp \left(-\frac{iB}{A} K_- \right) \tag{202}
 \end{aligned}$$

This is the two-mode quadratic canonical operator representation of $F_2(A, B, C)$.

To prove Eq. (202), using Eqs. (201) and (40) we have

$$\begin{aligned}
 \langle \eta | F_2(A, B, C) | \xi \rangle &= \frac{1}{2A} \exp \left(\frac{iC}{2A} |\eta|^2 - \frac{iB}{2A} |\xi|^2 \right) \\
 &\quad \times \exp \left[\frac{1}{2A} (\eta^* \xi - \eta \xi^*) \right] \tag{203}
 \end{aligned}$$

It then follows that

$$\begin{aligned}
 & \langle \eta' | F_2(A, B, C) | \eta \rangle \\
 &= \frac{1}{2iB} \exp \left[\frac{i}{2B} (A|\eta|^2 - i(\eta\eta'^* + \eta^*\eta') + D|\eta'|^2) \right] \\
 &\equiv \mathcal{K}_2^M(\eta', \eta) \tag{204}
 \end{aligned}$$

which is just the transform kernel of a 2-dimensional GFT and the definition given in Eq. (202) is true.

Note that the quadratic combinations in Eqs. (193), (196) and (200) of the four canonical operators $(Q_1, Q_2; P_1, P_2)$ obey the commutative relations $[K_+, K_-] = 2K_0, [K_0, K_\pm] = \pm K_\pm$, so $F_2(A, B, C)$ involves a $SU(2)$ Lie algebra structure. (This structure is also compiled by $Q^2/2, P^2/2$ and $-i(QP + PQ)/2$ that have been used in constructing $F_1(A, B, C)$).

9.3.2 Alternate decompositions of GFO and new optical operator identities

When $A = D = 0, B = 1, C = -1$, from Eq. (179) we see

$$F_2(0, 1, -1) = \exp \left[-i\frac{\pi}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \right] \equiv \mathcal{F} \tag{205}$$

which can also be named the Fourier operator, since it induces the quantum mechanically transforms [71, 72]

$$\mathcal{F}^\dagger Q_i \mathcal{F} = P_i, \mathcal{F}^\dagger P_i \mathcal{F}^\dagger = -Q_i \quad (206)$$

It then follows that

$$\mathcal{F}^\dagger K_+ \mathcal{F} = K_- \quad (207)$$

On the other hand, in order to obtain the decomposition of $F_2(A, B, C)$ for $A = 0$, similar to deriving Eq. (126), we have

$$F_2(A, B, C) = \exp\left(-\frac{iB}{D}K_-\right) \exp(2K_0 \ln D) \times \exp\left(\frac{iC}{D}K_+\right), D \neq 0 \quad (208)$$

While for $B \neq 0$ or $C \neq 0$, using Eqs. (127) and (128) we have another decomposition of $F_2(A, B, C)$, i.e.,

$$F_2(A, B, C) = \exp\left(\frac{iD}{B}K_+\right) \exp(-2K_0 \ln B) \mathcal{F} \times \exp\left(\frac{iA}{B}K_+\right), B \neq 0 \quad (209)$$

and

$$F_2(A, B, C) = \exp\left(-\frac{iA}{C}K_-\right) \exp\left(-2K_0 \ln \frac{-1}{C}\right) \mathcal{F} \times \exp\left(-\frac{iD}{C}K_-\right), C \neq 0 \quad (210)$$

In addition, noticing Eqs. (137) and (135), we can rewrite Eqs. (209) and (210) as follows:

$$F_2(A, B, C) = \exp\left[\frac{i}{B}(D-1)K_+\right] \exp(-iBK_-) \times \exp\left[\frac{i}{B}(A-1)K_+\right], B \neq 0 \quad (211)$$

and

$$F_2(A, B, C) = \exp\left[\frac{-i}{C}(A-1)K_-\right] \exp(iCK_+) \times \exp\left[\frac{-i}{C}(D-1)K_-\right], C \neq 0 \quad (212)$$

respectively.

Next, according to some optical systems used frequently in physical optics, we derive some new entangled optical operator identities. For a special optical system Eq. (131) with the parameter $A = 0, C = -B^{-1}$, which corresponds to the Fourier transform system, we have

$$\exp\left(-\frac{iB}{D}K_-\right) \exp(2K_0 \ln D) \exp\left(-\frac{i}{BD}K_+\right) = \exp\left(\frac{iD}{B}K_+\right) \exp(-2K_0 \ln B) \mathcal{F} \quad (213)$$

In particular, when $A = D = 0, C = -B^{-1}$, and with Eq. (133) corresponding to the ideal spectrum analyzer,

we have

$$\exp(-iBK_-) \exp\left(-\frac{i}{B}K_+\right) \exp(-iBK_-) = \exp(-2K_0 \ln B) \mathcal{F} \quad (214)$$

When $B = 0, D = A^{-1}$,

$$\begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C/A & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

which corresponds to the form of the image system, another operator identity is given by

$$\exp(-2K_0 \ln A) \exp(iACK_+) = \exp\left(\frac{iC}{A}K_+\right) \exp(-2K_0 \ln A) \quad (215)$$

When $C = 0, A = D^{-1}, \begin{pmatrix} D^{-1} & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 1 & B/D \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}$, which corresponds to the far foci system,

$$\exp\left(\frac{iD}{B}K_+\right) \exp(-2K_0 \ln B) \mathcal{F} \exp\left(\frac{i}{BD}K_+\right) = \exp\left(-\frac{iB}{D}K_-\right) \exp(2K_0 \ln D) \quad (216)$$

When $D = 0, C = -B^{-1}$, corresponding to the Fresnel transform system,

$$\begin{pmatrix} A & B \\ -B^{-1} & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A/B & 1 \end{pmatrix}$$

we have

$$\exp\left(-\frac{i}{AB}K_+\right) \exp(-2K_0 \ln A) \exp\left(-\frac{iB}{A}K_-\right) = \exp(-2K_0 \ln B) \mathcal{F} \exp\left(\frac{iA}{B}K_+\right) \quad (217)$$

The GFO can unify those optical operators in the two-mode case. Various decompositions of the GFO into the exponential canonical operators, corresponding to the decomposition of ray transfer matrix $[A, B, C, D]$, are also derived. In our derivation, the entangled state representation has been fully used.

9.4 Quantum tomography and probability distribution for the Fresnel quadrature phase — two-mode entangled case

In Section 8 we have found that under the Fresnel transformation the pure position density $|q\rangle\langle q|$ becomes the tomographic density $|q\rangle_{s,r,s,r}\langle q|$, which is just the Radon transform of the Wigner operator $\Delta(q, p)$. In this section we want to generalize the above conclusion to the two-mode entangled case. Here we shall prove

$$F_2|\eta\rangle\langle\eta|F_2^\dagger = |\eta\rangle_{s,r,s,r}\langle\eta| = \pi \int d^2\gamma d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma_1)\delta(\eta_1 - D\sigma_1 - B\gamma_2)\Delta(\sigma, \gamma) \quad (218)$$

That is, we show that $|\eta\rangle_{s,r,s,r}\langle\eta|$ is just the Radon transform of the entangled Wigner operator $\Delta(\sigma, \gamma)$.

Similar to the single-mode case, operating $F_2(r, s)$ on entangled state representation $|\eta\rangle$, we obtain $(F_2(r, s)|\eta) \equiv |\eta\rangle_{s,r}$

$$|\eta\rangle_{s,r} = \frac{1}{s^* + r^*} \exp\left[-\frac{s^* - r^*}{2(s^* + r^*)}|\eta|^2 + \frac{\eta a_1^\dagger}{s^* + r^*} - \frac{\eta^* a_2^\dagger}{s^* + r^*} + \frac{s+r}{s^* + r^*} a_1^\dagger a_2^\dagger\right]|00\rangle \quad (219)$$

or

$$|\eta\rangle_{s,r} = \frac{1}{D + iB} \exp\left[-\frac{A - iC}{2(D + iB)}|\eta|^2 + \frac{\eta a_1^\dagger}{D + iB} - \frac{\eta^* a_2^\dagger}{D + iB} + \frac{D - iB}{D + iB} a_1^\dagger a_2^\dagger\right]|00\rangle \quad (220)$$

where we have used the integration formula

$$\int \frac{d^2z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^*) = -\frac{1}{\zeta} e^{-\frac{\xi\eta}{\zeta}}, \quad \text{Re}(\zeta) < 0 \quad (221)$$

Noticing the completeness relation and the orthogonality of $|\eta\rangle$, we immediately derive

$$\int \frac{d^2\eta}{\pi} |\eta\rangle_{s,r,s,r}\langle\eta| = 1, \quad {}_{s,r}\langle\eta|\eta'\rangle_s = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*) \quad (222)$$

a generalized entangled state representation $|\eta\rangle_{s,r}$ with the completeness relation (222). From Eq. (220) we can see that

$$a_1|\eta\rangle_{s,r} = \left(\frac{\eta}{D + iB} + \frac{D - iB}{D + iB} a_2^\dagger\right)|\eta\rangle_{s,r} \quad (223)$$

$$a_2|\eta\rangle_{s,r} = \left(-\frac{\eta^*}{D + iB} + \frac{D - iB}{D + iB} a_1^\dagger\right)|\eta\rangle_{s,r} \quad (224)$$

so we have the eigen-equations for $|\eta\rangle_{s,r}$ as follows:

$$[D(Q_1 - Q_2) - B(P_1 - P_2)]|\eta\rangle_{s,r} = \sqrt{2}\eta_1|\eta\rangle_{s,r} \quad (225)$$

$$[B(Q_1 + Q_2) + D(P_1 + P_2)]|\eta\rangle_{s,r} = \sqrt{2}\eta_2|\eta\rangle_{s,r} \quad (226)$$

We can also check Eqs. (223)–(226) by another way.

9.4.1 $|\eta\rangle_{s,r}$ ${}_{s,r}\langle\eta|$ as Radon transform of the entangled Wigner operator

For a two-mode correlated system, we have introduced the Wigner operator in Eq. (43). According to the Weyl correspondence rule [51]

$$H(a_1^\dagger, a_2^\dagger; a_1, a_2) = \int d^2\gamma d^2\sigma h(\sigma, \gamma)\Delta(\sigma, \gamma) \quad (227)$$

where $h(\sigma, \gamma)$ is the Weyl correspondence of $H(a_1^\dagger, a_2^\dagger;$

$a_1, a_2)$, and

$$h(\sigma, \gamma) = 4\pi^2 \text{Tr} \left[H(a_1^\dagger, a_2^\dagger; a_1, a_2)\Delta(\sigma, \gamma) \right] \quad (228)$$

The classical Weyl correspondence of the projection operator $|\eta\rangle_{r,s,r,s}\langle\eta|$ can be calculated as

$$4\pi^2 \text{Tr} [|\eta\rangle_{r,s,r,s}\langle\eta|\Delta(\sigma, \gamma)] = 4\pi^2 \int \frac{d^2\eta'}{\pi^3} \langle\eta|F_2^\dagger|\sigma - \eta'\rangle\langle\sigma + \eta'|F_2|\eta\rangle \times \exp(\eta'\gamma^* - \eta'^*\gamma) \quad (229)$$

Then using Eq. (182) we have

$$4\pi^2 \text{Tr} [|\eta\rangle_{s,r,s,r}\langle\eta|\Delta(\sigma, \gamma)] = \pi\delta(\eta_2 - D\sigma_2 + B\gamma_1)\delta(\eta_1 - D\sigma_1 - B\gamma_2) \quad (230)$$

which means the following Weyl correspondence

$$|\eta\rangle_{s,r,s,r}\langle\eta| = \pi \int d^2\gamma d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma_1) \times \delta(\eta_1 - D\sigma_1 - B\gamma_2)\Delta(\sigma, \gamma) \quad (231)$$

so the projector operator $|\eta\rangle_{s,r,s,r}\langle\eta|$ is just the Radon transformation of $\Delta(\sigma, \gamma)$, and D and B are the Radon transformation parameter. Combining Eqs. (219)–(231) together we complete the proof (218). Therefore, the quantum tomography in the two-mode entangled case is expressed as

$${}_{s,r}\langle\eta|\psi\rangle|^2 = |\langle\eta|F^\dagger|\psi\rangle|^2 = \pi \int d^2\gamma d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma_1)\delta(\eta_1 - D\sigma_1 - B\gamma_2)\langle\psi|\Delta(\sigma, \gamma)|\psi\rangle \quad (232)$$

where $\langle\psi|\Delta(\sigma, \gamma)|\psi\rangle$ is the Wigner function. So the probability distribution for the Fresnel quadrature phase is the tomography (Radon transform of the two-mode Wigner function). This new relation between quantum tomography and optical Fresnel transform may provide experimentalists to figure out new approach for generating tomography.

Next we turn to the “frequency” domain, that is to say, we shall prove that the (A, C) related Radon transform of entangled Wigner operator $\Delta(\sigma, \gamma)$ is just the pure state density operator $|\xi\rangle_{s,r,s,r}\langle\xi|$, i.e.,

$$F_2|\xi\rangle\langle\xi|F_2^\dagger = |\xi\rangle_{s,r,s,r}\langle\xi| = \pi \int \delta(\xi_1 - A\sigma_1 - C\gamma_2) \times \delta(\xi_2 - A\sigma_2 + C\gamma_1)\Delta(\sigma, \gamma)d^2\sigma d^2\gamma \quad (233)$$

where $|\xi\rangle$ is the conjugated entangled state to $|\eta\rangle$.

By analogy with the above procedure, we obtain the 2-dimensional Fresnel transformation in its “frequency domain”, i.e.,

$$\mathcal{K}_2^N(\xi', \xi) \equiv \frac{1}{\pi} \langle\xi'|F_2(r, s)|\xi\rangle = \frac{1}{2i(-C)\pi} \times \exp\left[\frac{i}{2(-C)}(D|\xi|^2 + A|\xi'|^2 - \xi'^*\xi - \xi'\xi^*)\right] \quad (234)$$

where the superscript N means that this transform kernel corresponds to the parameter matrix $N = [D, -C, -B, A]$. Thus the 2D Fresnel transformation in its “frequency domain” is given by

$$\Psi(\xi') = \int \mathcal{K}_2^N(\xi', \xi) \Phi(\xi) d^2\xi \quad (235)$$

Operating $F_2(r, s)$ on $|\xi\rangle$ we also have

$$\begin{aligned} |\xi\rangle_{s,r} &= \frac{1}{A - iC} \exp \left[-\frac{D + iB}{2(A - iC)} |\eta|^2 + \frac{\xi a_1^\dagger}{A - iC} \right. \\ &\quad \left. + \frac{\xi^* a_2^\dagger}{A - iC} - \frac{A + iC}{A - iC} a_1^\dagger a_2^\dagger \right] |00\rangle \end{aligned} \quad (236)$$

or

$$\begin{aligned} |\xi\rangle_{s,r} &= \frac{1}{s^* - r^*} \exp \left[-\frac{s^* + r^*}{2(s^* - r^*)} |\xi|^2 + \frac{\xi a_1^\dagger}{s^* - r^*} \right. \\ &\quad \left. + \frac{\xi^* a_2^\dagger}{s^* - r^*} - \frac{s - r}{s^* - r^*} a_1^\dagger a_2^\dagger \right] |00\rangle \end{aligned} \quad (237)$$

Noticing that the entangled Wigner operator in $|\xi\rangle$ representation is expressed as

$$\Delta(\sigma, \gamma) = \int \frac{d^2\xi}{\pi^3} |\gamma + \xi\rangle \langle \gamma - \xi| \exp(\xi^* \sigma - \sigma^* \xi) \quad (238)$$

and using the classical correspondence of $|\xi\rangle_{s,r,s,r}\langle\xi|$ which is calculated by

$$\begin{aligned} h(\sigma, \gamma) &= 4\pi^2 \text{Tr} [|\xi\rangle_{s,r,s,r}\langle\xi| \Delta(\sigma, \gamma)] \\ &= \pi \delta(\xi_1 - A\sigma_1 - C\gamma_2) \delta(\xi_2 - A\sigma_2 + C\gamma_1) \end{aligned} \quad (239)$$

we obtain

$$\begin{aligned} |\xi\rangle_{s,r,s,r}\langle\xi| &= \pi \int \delta(\xi_1 - A\sigma_1 - C\gamma_2) \\ &\quad \times \delta(\xi_2 - A\sigma_2 + C\gamma_1) \Delta(\sigma, \gamma) d^2\sigma d^2\gamma \end{aligned} \quad (240)$$

so the projector operator $|\xi\rangle_{s,r,s,r}\langle\xi|$ is another Radon transformation of the two-mode Wigner operator, with A and C being the Radon transformation parameter (“frequency domain”). Therefore, the quantum tomography in ${}_{s,r}\langle\xi|$ representation is expressed as the Radon transformation of the Wigner function

$$\begin{aligned} |\langle\xi|F_2^\dagger|\psi\rangle|^2 &= |{}_{s,r}\langle\xi|\psi\rangle|^2 = \pi \int d^2\gamma d^2\sigma \delta(\xi_1 - A\sigma_1 \\ &\quad - C\gamma_2) \times \delta(\xi_2 - A\sigma_2 + C\gamma_1) \langle\psi|\Delta(\sigma, \gamma)|\psi\rangle \end{aligned} \quad (241)$$

and ${}_{s,r}\langle\xi| = \langle\xi|F_2^\dagger$.

9.4.2 Inverse Radon transformation

Now we consider the inverse Radon transformation. For instance, using Eq. (231) we see that the Fourier transformation of $|\eta\rangle_{s,r,s,r}\langle\eta|$ is

$$\int d^2\eta |\eta\rangle_{s,r,s,r}\langle\eta| \exp(-i\zeta_1\eta_1 - i\zeta_2\eta_2)$$

$$\begin{aligned} &= \pi \int d^2\gamma d^2\sigma \Delta(\sigma, \gamma) \\ &\quad \times \exp[-i\zeta_1(D\sigma_1 + B\gamma_2) - i\zeta_2(D\sigma_2 - B\gamma_1)] \end{aligned} \quad (242)$$

The right-hand side of Eq. (242) can be regarded as a special Fourier transformation of $\Delta(\sigma, \gamma)$, so by making its inverse Fourier transformation, we get

$$\begin{aligned} \Delta(\sigma, \gamma) &= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dr_1 |r_1| \int_{-\infty}^{\infty} dr_2 |r_2| \int_0^\pi d\theta_1 d\theta_2 \\ &\quad \times \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} |\eta\rangle_{s,r,s,r}\langle\eta| K(r_1, r_2, \theta_1, \theta_2) \end{aligned} \quad (243)$$

where $\cos\theta_1 = \cos\theta_2 = \frac{D}{\sqrt{B^2 + D^2}}$, $r_1 = \zeta_1 \sqrt{B^2 + D^2}$, $r_2 = \zeta_2 \sqrt{B^2 + D^2}$ and

$$\begin{aligned} K(r_1, r_2, \theta_1, \theta_2) &\equiv \exp \left[-ir_1 \left(\frac{\eta_1}{\sqrt{B^2 + D^2}} - \sigma_1 \cos\theta_1 - \gamma_2 \sin\theta_1 \right) \right] \\ &\quad \times \exp \left[-ir_2 \left(\frac{\eta_2}{\sqrt{B^2 + D^2}} - \sigma_2 \cos\theta_2 + \gamma_1 \sin\theta_2 \right) \right] \end{aligned} \quad (244)$$

Eq. (243) is just the inverse Radon transformation of entangled Wigner operator in the entangled state representation. This is different from the two independent Radon transformations' direct product of the two independent single-mode Wigner operators, because in Eq. (220) the $|\eta\rangle_{s,r}$ is an entangled state. Therefore the Wigner function of quantum state $|\psi\rangle$ can be reconstructed from the tomographic inversion of a set of measured probability distributions ${}_{s,r}\langle\eta|\psi\rangle|^2$, i.e.,

$$\begin{aligned} W_\psi &= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dr_1 |r_1| \int_{-\infty}^{\infty} dr_2 |r_2| \int_0^\pi d\theta_1 d\theta_2 \\ &\quad \times \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} |{}_{s,r}\langle\eta|\psi\rangle|^2 K(r_1, r_2, \theta_1, \theta_2) \end{aligned} \quad (245)$$

Thus, based on the previous section, we have further extended the relation connecting optical Fresnel transformation with quantum tomography to the entangled case. The tomography representation ${}_{s,r}\langle\eta| = \langle\eta|F_2^\dagger$ is set up, based on which the tomogram of quantum state $|\psi\rangle$ is just the squared modulus of the wave function ${}_{s,r}\langle\eta|\psi\rangle$. In other words, the probability distribution for the Fresnel quadrature phase is the tomogram (Radon transform of the Wigner function).

10 Fractional Fourier transformation (FrFT) for a 1D case

The fractional Fourier transform (FrFT) has been shown to be a very useful tool in Fourier optics and information optics. The concept of FrFT was first introduced mathematically in 1980 by Namias [36] as a mathematical tool to solve theoretical physical problems [95], but did

not bring enough attention until Mendlovic and Ozaktas [37–39] defined the α -th FrFT physically, based on propagation in quadratic graded-index media (GRIN media with medium parameters $n(r) = n_1 - n_2 r^2/2$). Since then a lot of work have been done on its properties, optical implementations and applications [96–100].

10.1 Quantum version of FrFT

The FrFT of θ -order is defined in such a manner as

$$\mathcal{F}_\theta [f(x)] = \sqrt{\frac{e^{i(\frac{\pi}{2}-\theta)}}{2\pi \sin \theta}} \int_{-\infty}^{\infty} \exp \left[-i \frac{x^2 + y^2}{2 \tan \theta} + \frac{ixx'}{\sin \theta} \right] f(x) dx \quad (246)$$

where the exponential function is an integral kernel. In order to find the quantum correspondence of FrFT, multiply the function $\exp \left(-i \frac{x^2 + y^2}{2 \tan \theta} + \frac{ixx'}{\sin \theta} \right)$ by the ket $\int dy |y\rangle$ and bra $\int dx \langle x|$ from left and right, respectively, where $|y\rangle$ and $|x\rangle$ are coordinate eigenvectors, $X|x\rangle = x|x\rangle$, and then use Eq. (13) and the IWOP technique to perform the integration. We obtain

$$\int_{-\infty}^{\infty} dx dy |y\rangle \exp \left(-i \frac{x^2 + y^2}{2 \tan \theta} + \frac{ixy}{\sin \theta} \right) \langle x| = \sqrt{-2\pi i \sin \theta} e^{i\theta} \exp(i\theta a^\dagger a) \quad (247)$$

where we have used the operator identity $\exp(fa^\dagger a) = : \exp[(e^f - 1)a^\dagger a] :$ in the last step of Eq. (247).

From the orthogonal relation $\langle x'|x\rangle = \delta(x - x')$, we know that Eq. (247) indicates

$$\sqrt{\frac{e^{i(\frac{\pi}{2}-\theta)}}{2\pi \sin \theta}} \exp \left(-i \frac{x^2 + y^2}{2 \tan \theta} + \frac{ixy}{\sin \theta} \right) = \langle y | e^{i\theta a^\dagger a} | x \rangle \quad (248)$$

which implies that the integral kernel in Eq. (246) is just the matrix element of operator $\exp(i\theta a^\dagger a)$ in coordinate state $\exp(i\theta a^\dagger a)$ called as Fractional Fourier Operator [70]. Therefore, if we consider $f(x)$ as $\langle x|f\rangle$, the wavefunction of quantum state $|f\rangle$ in the coordinate representation, from Eqs. (246) and (248) it then follows that

$$\mathcal{F}_\theta [f(x)] = \int_{-\infty}^{\infty} dx \langle y | e^{i\theta a^\dagger a} | x \rangle f(x) = \langle y | e^{i\theta a^\dagger a} | f \rangle \equiv g(y) \quad (249)$$

which suggests

$$|g\rangle = e^{i\theta a^\dagger a} |f\rangle \quad (250)$$

From Eqs. (249) and (246) one can see that the FrFT in Eq. (246) corresponds actually to the rotating operator ($e^{i\theta a^\dagger a}$) transform in Eq. (249) between two quantum states, which is just the quantum version of FrFT.

In fact, using quantum version of FrFT, one can directly derive various properties of the FrFTs. An important feature of the FrFT is that they are composed

according to $\mathcal{F}_{\theta'} \mathcal{F}_\theta = \mathcal{F}_{\theta'+\theta}$ (the additivity property). Without losing generality, we examine

$$\mathcal{F}_{\theta+\theta'} [f(x)] \equiv \int_{-\infty}^{\infty} \frac{d^2 \eta}{\pi} \langle y | e^{i(\theta+\theta')a^\dagger a} | x \rangle f(x) \quad (251)$$

According to the completeness relation of coordinate eigenvector, $\int_{-\infty}^{\infty} dx' \langle x'|x\rangle \langle x'| = 1$, Eq. (251) yields

$$\begin{aligned} \mathcal{F}_{\theta+\theta'} [f(x)] &= \int_{-\infty}^{\infty} dx' \langle y | e^{i\theta a^\dagger a} | x' \rangle \int_{-\infty}^{\infty} dx \langle x' | e^{i\theta' a^\dagger a} | x \rangle f(x) \\ &= \int_{-\infty}^{\infty} dx' \langle y | e^{i\theta a^\dagger a} | x' \rangle \mathcal{F}_\theta [f(x)] = \mathcal{F}_\theta \mathcal{F}_{\theta'} [f(x)] \end{aligned} \quad (252)$$

which is just the additivity of FrFT.

In particular, when $|f\rangle$ is the number state, $|f\rangle = |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle$, its wavefunction in coordinate representation is

$$f(x) = \langle x | n \rangle = \frac{1}{\sqrt{2^n n!} \sqrt{\pi}} e^{-x^2/2} H_n(x) \quad (253)$$

The FrFT of $\langle x | n \rangle$ is $\mathcal{F}_\theta [\langle x | n \rangle] = \langle y | e^{i\theta a^\dagger a} | n \rangle = e^{in\theta} \langle y | n \rangle$, or

$$\mathcal{F}_\theta \left[e^{-x^2/2} H_n(x) \right] = e^{in\theta} e^{-y^2/2} H_n(y) \quad (254)$$

which indicates that the eigenfunction is Hermite-Gaussian function with the corresponding eigenvalue being $e^{in\theta}$.

10.2 On the scaled FrFT operator

In studying various optical transformations the optical operator method is proposed [77–79] as mapping of ray-transfer ABCD matrix, such that the ray transfer through optical instruments and the diffraction can be discussed by virtue of the commutative relations of operators and the matrix algebra. The square phase operators, scaling operator, Fourier transform operator and the propagation operator in free space have been proposed in the literature. Two important questions thus naturally arise: (i) What is the scaled FrFT (SFrFT) operator which corresponds to the SFrFT's integration kernel [36]

$$\frac{1}{\sqrt{2\pi i f_e \sin \phi}} \exp \left[\frac{i(x^2 + x'^2)}{2f_e \tan \phi} - \frac{ix'x}{f_e \sin \phi} \right] \quad (255)$$

where f_e is standard focal length (or a scaled parameter)? (ii) If this operator is found, can it be further decomposed into simpler operators and what are their physical meaning? Since SFrFT has wide application in optical information detection and can be implemented even by using a thick lens [44], so our questions are worth paying attention to Ref. [102].

Let us start with a thick lens (shown in Fig. 1) which represents a transfer matrix [44]

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 1 - \frac{(1-1/n)l}{R_1} & \frac{l}{n} \\ -[(n-1)\frac{R_1+R_2}{R_1R_2} - \frac{l(n-1)^2}{nR_1R_2}] & 1 - \frac{(1-1/n)l}{R_2} \end{pmatrix} \times \exp\left(\frac{f_e \tan \phi}{2i} P^2\right) \quad (261)$$

where n is the reflective index; l is the thickness of thick lens; R_1 and R_2 denotes the curvature radius of the two surfaces of the lens, respectively. When we choose $R_1 = R_2 = R$, Eq. (256) is reduced to

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 1 - \frac{(1-1/n)l}{R} & \frac{l}{n} \\ -[(n-1)\frac{2}{R} - \frac{l(n-1)^2}{nR^2}] & 1 - \frac{(1-1/n)l}{R} \end{pmatrix} \quad (257)$$

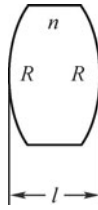


Fig. 1 A thick lens as a kind of fractional Fourier transform device.

By defining $1 - \frac{(1-1/n)l}{R} = \cos \phi$, $\frac{l}{n} = f_e \sin \phi$, and $\frac{l}{R} = \frac{n(1-\cos \phi)}{n-1}$, $l = n f_e \sin \phi$, we can recast (257) into the simple form

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \cos \phi & f_e \sin \phi \\ -\sin \phi / f_e & \cos \phi \end{pmatrix}$$

$$\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = 1 \quad (258)$$

According to Eq. (98) we immediately know that the operator of SFrFT is

$$F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \exp\left[\frac{i(f_e - 1/f_e) \tan \phi}{2V} a^{\dagger 2}\right]$$

$$\times \exp\left[\left(a^\dagger a + \frac{1}{2}\right) \ln \frac{2\phi}{V}\right]$$

$$\times \exp\left[\frac{i(f_e - 1/f_e) \tan \phi}{2V} a^2\right]$$

$$V = [2 + i(f_e + 1/f_e) \tan \phi] \quad (259)$$

Noting that the matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ can be decomposed into

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_e} \tan \phi & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & 0 \\ 0 & \cos \phi \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & f_e \tan \phi \\ 0 & 1 \end{pmatrix} \quad (260)$$

according to the previous section, we have

$$F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \exp\left(\frac{\tan \phi}{2i f_e} Q^2\right)$$

$$\times \exp\left[-\frac{i}{2}(QP + PQ) \ln \cos \phi\right]$$

where $Q = (a + a^\dagger)/\sqrt{2}$, $P = (a - a^\dagger)/(\sqrt{2}i)$ and $\exp\left(-\frac{i \tan \phi}{2 f_e} Q^2\right)$, $\exp\left[-\frac{i}{2}(QP + PQ) \ln \cos \phi\right]$ and $\exp\left(-\frac{i f_e \tan \phi}{2} P^2\right)$ are the quadrature phase operator, the squeezing operator and the free propagation operator, respectively. On the other hand, from $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{D} & -\mathcal{B} \\ -\mathcal{C} & \mathcal{A} \end{pmatrix}$, and Eq. (261) we see

$$F_1^{-1}(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \exp\left(-\frac{i\mathcal{C}}{2\mathcal{D}} Q^2\right)$$

$$\times \exp\left[-\frac{i}{2}(QP + PQ) \ln \mathcal{D}\right]$$

$$\times \exp\left(\frac{i\mathcal{B}}{2\mathcal{D}} P^2\right) \quad (262)$$

It then follows that

$$F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \exp\left(\frac{f_e \tan \phi}{2i} P^2\right)$$

$$\times \exp\left[\frac{i}{2}(QP + PQ) \ln \cos \phi\right]$$

$$\times \exp\left(\frac{\tan \phi}{2i f_e} Q^2\right) \quad (263)$$

Using the canonical operator form (261) or (263) of $F_1(\mathcal{A}, \mathcal{B}, \mathcal{C})$ we can deduce its matrix element in the coordinate states $|x\rangle$ (its conjugate state is $|p\rangle$)

$$\langle x' | F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) | x \rangle$$

$$= \frac{1}{\sqrt{2\pi i f_e \sin \phi}} \exp\left[\frac{i(x^2 + x'^2)}{2f_e \tan \phi} - \frac{ix'x}{f_e \sin \phi}\right] \quad (264)$$

which is just the kernel of SFrFT. Thus we name $F_1(\mathcal{A}, \mathcal{B}, \mathcal{C})$ SFrFT operator. Noticing that the $Q^2/2$, $P^2/2$ and $\frac{i}{4}(QP + PQ)$ construct a close $SU(2)$ Lie algebra, we can put Eq. (263) into a more compact form, i.e.,

$$F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \exp\left[-i\frac{\phi f_e}{2}(P^2 + \frac{Q^2}{f_e^2})\right] \quad (265)$$

Eqs. (261), (263) and (265) are different forms of the same operator of SFrFT; especially when $f_e = 1$, $F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) \rightarrow \exp(-i\phi a^\dagger a)$, which is the usual FrFT operator.

Using Eq. (264) the SFrFT of $f(x) = \langle x | f \rangle$, denoted as $\mathcal{F}_{f_e}^\phi[f(x)]$, can be expressed as an matrix element in quantum optics context,

$$\mathcal{F}_{f_e}^\phi[f(x)] = \int dx \langle x' | F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) | x \rangle \langle x | f \rangle$$

$$= \langle x' | F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) | f \rangle \quad (266)$$

The above discussions are useful since any unimodular matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ can be decomposed into [103]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathcal{P} & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \times \begin{pmatrix} \cos \phi & f_e \sin \phi \\ -\sin \phi / f_e & \cos \phi \end{pmatrix} \quad (267)$$

where the parameters m, \mathcal{P}, ϕ are all real,

$$m^2 = A^2 + \frac{B^2}{f_e^2}, \tan \phi = \frac{B}{A f_e}, \mathcal{P} = -\frac{AC + DB / f_e^2}{A^2 + \frac{B^2}{f_e^2}} \quad (268)$$

Correspondingly, the operator of FrFT is given by

$$\begin{aligned} F_1(A, B, C) &= \exp\left(-\frac{i}{2}\mathcal{P}Q^2\right) \\ &\times \exp\left[-\frac{i}{2}(QP + PQ) \ln m\right] \\ &\times \exp\left[\frac{\phi f_e}{2i}\left(P^2 + \frac{Q^2}{f_e^2}\right)\right] \end{aligned} \quad (269)$$

where $F_1(1, 0, -\mathcal{P}) = \exp(-\frac{i}{2}\mathcal{P}Q^2)$ is the quadratic phase operator. Thus the general Fresnel transform can always be expressed by SFrFT as follows:

$$\begin{aligned} g(x') &= \int dy \langle x' | F_1(1, 0, -\mathcal{P}) F_1(m, 0, 0) | y \rangle \\ &\times \int dx \langle y | F_1(\mathcal{A}, \mathcal{B}, \mathcal{C}) | x \rangle f(x) \\ &= \frac{1}{\sqrt{m}} \exp\left(-\frac{i}{2}\mathcal{P}x'^2\right) \mathcal{F}_{f_e}^\phi(f)\left(\frac{x'}{m}\right) \end{aligned} \quad (270)$$

i.e., the output $g(x')$ is the SFrFT of the input $f(x)$ plus a quadratic phase term $\exp(-\frac{i}{2}\mathcal{P}x'^2)$.

10.3 An integration transformation from Chirplet to FrFT kernel

In the history of the development of optics we have known that each optical setup corresponds to an optical transformation. For example, thick lens as a fractional Fourier transformer. In turn, once a new integration transform is found, its experimental implementation is expected. In this subsection we report a new integration transformation which can convert chirplet function to FrFT kernel [104]. As this new transformation is invertible and obeys Parseval theorem, we expect it be realized by experimentalists.

The new transform we propose here is

$$\iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} h(p, q) \equiv f(x, y) \quad (271)$$

which differs from the usual two-fold Fourier transformation $\iint_{-\infty}^{\infty} \frac{dx dy}{4\pi^2} e^{ipx+iqy} f(x, y)$. In particular, when $h(p, q) = 1$, Eq. (271) is reduced to

$$\iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} = 1 \quad (272)$$

so $e^{2i(p-x)(q-y)}$ can be considered a basis function in $p-q$ phase space, or Eq. (271) can be looked as an expansion of $f(x, y)$ with the expansion coefficient being $h(p, q)$. We can prove that the reciprocal transformation of Eq. (271) is

$$\iint_{-\infty}^{\infty} \frac{dx dy}{\pi} e^{-2i(p-x)(q-y)} f(x, y) = h(p, q) \quad (273)$$

In fact, substituting Eq. (271) into the left-hand side of Eq. (273), we get

$$\begin{aligned} &\iint_{-\infty}^{\infty} \frac{dp' dq'}{\pi} h(p', q') \iint \frac{dx dy}{\pi} \\ &\times e^{2i[(p'-x)(q'-y)-(p-x)(q-y)]} \\ &= \iint_{-\infty}^{\infty} dp' dq' h(p', q') e^{2i(p'q' - pq)} \\ &\times \delta(p - p') \delta(q - q') = h(p, q) \end{aligned} \quad (274)$$

This transformation's Parseval-like theorem can be proved

$$\begin{aligned} &\iint_{-\infty}^{\infty} \frac{dpdq}{\pi} |h(p, q)|^2 \\ &= \iint \frac{dx dy}{\pi} |f(x, y)|^2 \iint \frac{dx' dy'}{\pi} e^{2i(x'y' - xy)} \\ &\times \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i[(-y'p - x'q) + (py + xq)]} \\ &= \iint \frac{dx dy}{\pi} |f(x, y)|^2 \end{aligned} \quad (275)$$

Now we apply Eq. (271) to phase space transformation in quantum optics. Recall that a signal $\psi(q)$'s Wigner transform [86, 90–94, 105] is

$$\psi(q) \rightarrow \int \frac{du}{2\pi} e^{ipu} \psi^*\left(q + \frac{u}{2}\right) \psi\left(q - \frac{u}{2}\right) \quad (276)$$

Use Dirac's symbol [7] to write $\psi(q) = \langle q | \psi \rangle$. $|q\rangle$ is the eigenvector of coordinate Q , and the Wigner operator emerges from Eq. (276),

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{-ipu} |q - \frac{u}{2}\rangle \langle q + \frac{u}{2}| = \Delta(p, q), \hbar = 1 \quad (277)$$

If $h(q, p)$ is quantized as the operator $\hat{H}(P, Q)$ through the Weyl–Wigner correspondence [51]

$$H(P, Q) = \iint_{-\infty}^{\infty} dpdq \Delta(p, q) h(q, p) \quad (278)$$

then

$$h(q, p) = \int_{-\infty}^{\infty} du e^{-ipu} \langle q + \frac{u}{2} | \hat{H}(Q, P) | q - \frac{u}{2} \rangle \quad (279)$$

which in the literature is named the Weyl transform. $h(q, p)$ is the Weyl classical correspondence of the operator $\hat{H}(Q, P)$. Substituting Eq. (279) into Eq. (271), we have

$$\begin{aligned} & \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} h(p, q) \\ &= \int_{-\infty}^{\infty} du e^{-ixu} \langle y + u | \hat{H}(Q, P) | y \rangle \end{aligned} \quad (280)$$

Using $\langle y + u | = \langle u | e^{iPy}$ and $(\sqrt{2\pi})^{-1} e^{-ixu} = \langle p=x | u \rangle$, where $\langle p |$ is the momentum eigenvector, and

$$\int_{-\infty}^{\infty} du e^{-ixu} \langle y + u | = \sqrt{2\pi} \langle p=x | e^{ixy}$$

then Eq. (280) becomes

$$\begin{aligned} & \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} h(p, q) \\ &= \sqrt{2\pi} \langle p=x | \hat{H}(Q, P) | y \rangle e^{ixy} \end{aligned} \quad (281)$$

Thus through the new integration transformation a new relationship between a phase space function $h(p, q)$ and its Weyl–Wigner correspondence operator $\hat{H}(Q, P)$ has been revealed. The inverse of Eq. (281), according to Eq. (273), is

$$\iint_{-\infty}^{\infty} \frac{dx dy}{\sqrt{\pi/2}} e^{-2i(p-x)(q-y)} \langle p=x | \hat{H}(Q, P) | y \rangle e^{ixy} = h(p, q) \quad (282)$$

For example, when $\hat{H}(Q, P) = e^{f(P^2+Q^2-1)/2}$, its classical correspondence is

$$e^{f(P^2+Q^2-1)/2} \rightarrow h(p, q) = \frac{2}{e^f + 1} \exp \left[2 \frac{e^f - 1}{e^f + 1} (p^2 + q^2) \right] \quad (283)$$

Substituting Eq. (283) into Eq. (281), we have

$$\begin{aligned} & \frac{2}{e^f + 1} \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} \exp \left[2 \frac{e^f - 1}{e^f + 1} (p^2 + q^2) \right] \\ &= \sqrt{2\pi} \langle p=x | e^{f(P^2+Q^2-1)/2} | y \rangle e^{ixy} \end{aligned} \quad (284)$$

Using the Gaussian integration formula

$$\begin{aligned} & \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} e^{-\lambda(p^2+q^2)} \\ &= \frac{1}{\sqrt{\lambda^2+1}} \exp \left[\frac{-\lambda(x^2+y^2)}{\lambda^2+1} + \frac{2i\lambda^2}{\lambda^2+1} xy \right] \end{aligned} \quad (285)$$

in particular, when $\lambda = -i \tan(\frac{\pi}{4} - \frac{\alpha}{2})$, with $\frac{-\lambda}{\lambda^2+1} = \frac{i}{2 \tan \alpha}$, $\frac{2\lambda^2}{\lambda^2+1} = 1 - \frac{1}{\sin \alpha}$, Eq. (285) becomes

$$\begin{aligned} & \frac{2}{ie^{-i\alpha} + 1} \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} \\ & \times \exp \left[i(p^2 + q^2) \tan \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \right] \\ &= \frac{1}{\sqrt{ie^{-i\alpha} \sin \alpha}} \exp \left[\frac{i(x^2 + y^2)}{2 \tan \alpha} - \frac{ixy}{\sin \alpha} \right] e^{ixy} \end{aligned} \quad (286)$$

where $\exp[i \tan(\frac{\pi}{4} - \frac{\alpha}{2})(p^2 + q^2)]$ represents an infinite long chirplet function. Comparing Eq. (286) with Eq. (284) we see $ie^{-i\alpha} = e^f$, $f = i(\frac{\pi}{2} - \alpha)$, it then follows

that

$$\begin{aligned} & \langle p=x | e^{i(\frac{\pi}{2}-\alpha)(P^2+Q^2-1)/2} | y \rangle \\ &= \frac{1}{\sqrt{2\pi} ie^{-i\alpha} \sin \alpha} \exp \left[\frac{i(x^2 + y^2)}{2 \tan \alpha} - \frac{ixy}{\sin \alpha} \right] \end{aligned} \quad (287)$$

where the right-hand side of Eq. (287) is just the FrFT kernel. Therefore the new integration transformation (271) can convert spherical wave to FrFT kernel. We expect that this transformation could be implemented by experimentalists.

Moreover, this transformation can also be used to solve some operator ordering problems. We notice

$$\begin{aligned} & \frac{1}{\pi} \exp[2i(p-x)(q-y)] \\ &= \int_{-\infty}^{\infty} \frac{dv}{2\pi} \delta(q-y-\frac{v}{2}) \exp[i(p-x)v] \end{aligned} \quad (288)$$

so the transformation (271) is equivalent to

$$\begin{aligned} h(p, q) &\rightarrow \iint_{-\infty}^{\infty} \frac{dpdq}{\pi} e^{2i(p-x)(q-y)} h(p, q) \\ &= \iint_{-\infty}^{\infty} \frac{dpdq}{2\pi} h(p+x, y+\frac{q}{2}) e^{ipq} \end{aligned} \quad (289)$$

For example, using Eqs. (277) and (288), we have

$$\begin{aligned} \Delta(p, q) &\rightarrow \iint_{-\infty}^{\infty} \frac{dpdq}{2\pi} \Delta(p+x, y+\frac{q}{2}) e^{ipq} \\ &= \delta(y-Q) \delta(x-P) \end{aligned} \quad (290)$$

so

$$\frac{1}{\pi} \iint dp' dq' \Delta(q', p') e^{2i(p-p')(q-q')} = \delta(q-Q) \delta(p-P) \quad (291)$$

Thus this new transformation can convert the Wigner operator to $\delta(q-Q)\delta(p-P)$. Similarly, we have

$$\frac{1}{\pi} \iint dp' dq' \Delta(q', p') e^{-2i(p-p')(q-q')} = \delta(p-P) \delta(q-Q)$$

Then for the Wigner function of a density operator ρ , $W_\psi(p, q) \equiv \text{Tr} [\rho \Delta(p, q)]$, we have

$$\begin{aligned} & \iint_{-\infty}^{\infty} \frac{dp' dq'}{\pi} \text{Tr} [\rho \Delta(p', q')] e^{2i(p-p')(q-q')} \\ &= \text{Tr} [\rho \delta(q-Q) \delta(p-P)] \\ &= \int \frac{dudv}{4\pi^2} \text{Tr} \left[\rho e^{i(q-Q)u} e^{i(p-P)v} \right] \end{aligned} \quad (292)$$

We may define $\text{Tr} [\rho e^{i(q-Q)u} e^{i(p-P)v}]$ as the Q - P characteristic function. Similarly,

$$\begin{aligned} & \iint_{-\infty}^{\infty} \frac{dp' dq'}{\pi} \text{Tr} [\rho \Delta(p', q')] e^{-2i(p-p')(q-q')} \\ &= \text{Tr} [\rho \delta(p-P) \delta(q-Q)] \\ &= \int \frac{dudv}{4\pi^2} \text{Tr} \left[\rho e^{i(p-P)v} e^{i(q-Q)u} \right] \end{aligned} \quad (293)$$

We name $\text{Tr} [\rho e^{i(p-P)v} e^{i(q-Q)u}]$ as the P - Q characteristic function.

11 Complex fractional Fourier transformation

In this section, we extend 1D FrFT to the complex fractional Fourier transformation (CFrFT).

11.1 Quantum version of CFrFT

According to Ref. [106], based on the entangled state $|\eta\rangle$ in two-mode Fock space and its orthonormal property, we can take the matrix element of $\exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)]$ in the entangled state $|\eta\rangle$,

$$\mathcal{K}^F(\eta', \eta) = \langle \eta' | \exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)] | \eta \rangle \quad (294)$$

as the integral transform kernel of CFrFT,

$$\mathcal{F}_\alpha[f](\eta) = \int \frac{d^2\eta}{\pi} \mathcal{K}^F(\eta', \eta) f(\eta) \quad (295)$$

Using the normally ordered expansion of $e^{\lambda a_1^\dagger a_1} = : \exp[(e^\lambda - 1)a_1^\dagger a_1] :$ and the completeness relation of the coherent state representation, $|z_i\rangle = \exp(-\frac{1}{2}|z_i|^2 + z_i a_i^\dagger) |0\rangle_i$, we calculate that $\mathcal{K}^F(\eta', \eta)$ is

$$\mathcal{K}^F(\eta', \eta) = \frac{e^{i(\alpha - \frac{\pi}{2})}}{2 \sin \alpha} \exp \left[\frac{i(|\eta'|^2 + |\eta|^2)}{2 \tan \alpha} - \frac{i(\eta'^* \eta + \eta^* \eta')}{2 \sin \alpha} \right] \quad (296)$$

which is just the integral kernel of the CFrFT in Ref. [107]. Thus we see that the matrix element of $\exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)]$ between two entangled state representations $|\eta\rangle$ and $|\eta'\rangle$ corresponds to CFrFT. This is a new route from quantum optical transform to classical CFrFT transform. Let $\eta = x_2 + iy_2, \eta' = x_1 + iy_1$, Eq. (295) becomes

$$\begin{aligned} \mathcal{F}_\alpha[f](x_2, y_2) &= \frac{e^{i(\alpha - \frac{\pi}{2})}}{2 \sin \alpha} \exp \left[\frac{i(x_2^2 + y_2^2)}{2 \tan \alpha} \right] \\ &\times \int \frac{dx_1 dy_1}{\pi} \exp \left[\frac{i(x_1^2 + y_1^2)}{2 \tan \alpha} - i \frac{(x_1 x_2 + y_1 y_2)}{\sin \alpha} \right] \\ &\times f(x_1, y_1) \end{aligned} \quad (297)$$

In fact, let $f(\eta) = \langle \eta | f \rangle$. Then use Eqs. (30) and (296), and we have

$$\begin{aligned} &\langle \eta' | \exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)] | f \rangle \\ &= \int \frac{d^2\eta}{\pi} \mathcal{K}^F(\eta', \eta) f(\eta) = \frac{e^{i(\alpha - \frac{\pi}{2})}}{2 \sin \alpha} \int \frac{d^2\eta}{\pi} \\ &\times \exp \left[\frac{i(|\eta'|^2 + |\eta|^2)}{2 \tan \alpha} - \frac{i(\eta'^* \eta + \eta^* \eta')}{2 \sin \alpha} \right] f(\eta) \end{aligned} \quad (298)$$

Thus the quantum mechanical version of CFrFT is given by

$$\mathcal{F}_\alpha[f](\eta') \equiv \langle \eta' | \exp[-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)] | f \rangle \quad (299)$$

The standard complex Fourier transform is $\mathcal{F}_{\pi/2}$. \mathcal{F}_0 is the identity operator.

11.2 Additivity property and eigenmodes of CFrFT

We will show later that CFrFT can help us reveal some new property which has been overlooked in the formulation of the direct product of two real FrFTs [108]. The definition (309) is of course required to satisfy the basic postulate that $\mathcal{F}_\alpha \mathcal{F}_\beta[f](\eta') = \mathcal{F}_{\alpha+\beta}[f(\eta)]$ (the additivity property). For this purpose, using Eq. (309) and Eq. (30), we see

$$\begin{aligned} \mathcal{F}_{\alpha+\beta}[f(\eta)] &\equiv \langle \eta' | e^{-i(\alpha+\beta)(a_1^\dagger a_1 + a_2^\dagger a_2)} | f \rangle \\ &= \int_{-\infty}^{\infty} \frac{d^2\eta''}{\pi} \langle \eta' | e^{-i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} | \eta'' \rangle \mathcal{F}_\beta[f(\eta)] \\ &= \mathcal{F}_\alpha \mathcal{F}_\beta[f(\eta)] \end{aligned} \quad (300)$$

This derivation is clear and concise by employing the $|\eta\rangle$ representation and quantum mechanical version of CFrFT.

On the other hand, the formula (299) can help us to derive CFrFT of some wave functions easily. For example, when $|f\rangle$ is a two-mode number state $|m, n\rangle = a_1^{\dagger m} a_2^{\dagger n} / \sqrt{m!n!} |00\rangle$, then the CFrFT of the wave function $\langle \eta | m, n \rangle$ is

$$\begin{aligned} \mathcal{F}_\alpha[\langle \eta | m, n \rangle] &= \langle \eta' | e^{i(\alpha+\beta)(a_1^\dagger a_1 + a_2^\dagger a_2)} | m, n \rangle \\ &= e^{i(\alpha+\beta)(m+n)} \langle \eta' | m, n \rangle \end{aligned} \quad (301)$$

To calculate $\langle \eta' | m, n \rangle$, let us recall the definition of two-variable Hermite polynomial $H_{m,n}(\xi, \xi^*)$, and we can expand $\langle \eta' |$ as

$$\langle \eta' | = \langle 00 | \sum_{m,n=0}^{\infty} i^{m+n} \frac{a_1^m a_2^n}{m!n!} H_{m,n}(-i\eta'^*, i\eta') e^{-|\eta'|^2/2} \quad (302)$$

thus

$$\langle \eta' | m, n \rangle = \frac{i^{m+n}}{\sqrt{m!n!}} H_{m,n}(-i\eta'^*, i\eta') e^{-|\eta'|^2/2} \quad (303)$$

As a result of Eq. (303), we see that Eq. (301) becomes

$$\begin{aligned} \mathcal{F}_\alpha \left[H_{m,n}(-i\eta'^*, i\eta') e^{-|\eta'|^2/2} \right] \\ = e^{i(\alpha+\beta)(m+n)} H_{m,n}(-i\eta'^*, i\eta') e^{-|\eta'|^2/2} \end{aligned} \quad (304)$$

If we consider the operation \mathcal{F}_α as an operator, one can say that the eigenfunction of \mathcal{F}_α (the eigenmodes of CFrFT) is the two-variable Hermite polynomials $H_{m,n}$ with the eigenvalue being $e^{i(\alpha+\beta)(m+n)}$. This is a new property of CFrFT. Since the function space spanned by $H_{m,n}(\eta, \eta^*)$ is complete,

$$\int \frac{d^2\eta}{\pi} e^{-|\eta|^2} H_{m,n}(\eta, \eta^*) [H_{m,n}(\eta, \eta^*)]^* = \sqrt{m!n!m'!n'!} \delta_{m,m'} \delta_{n,n'} \quad (305)$$

and

$$\sum_{m,n=0}^{\infty} \frac{1}{m!n!} H_{m,n}(\eta, \eta^*) [H_{m,n}(\eta', \eta'^*)]^* e^{-|\eta|^2} = \pi \delta(\eta - \eta') \delta(\eta^* - \eta'^*) \quad (306)$$

one can confirm that the eigenmodes of CFrFT form an orthogonal and complete basis set [109]. Note that the two variable Hermite polynomial $H_{m,n}(\eta, \eta^*)$ is not the direct product of two independent ordinary Hermite polynomials, so CFrFT differs from the direct product of two FrFTs.

11.3 From chirplet to CFrFT kernel

In this subsection, by developing Eq. (271) to a more general case which can be further related to the transformation between two mutually conjugate entangled state representations $|\xi\rangle$ and $|\eta\rangle$, we shall propose a new integration transformation in ξ - η phase space [see Eq. (307) below] and its inverse transformation. We find that Eq. (307) also possesses some well-behaved transformation properties and can be used to obtain the CFrFT kernel from a chirplet [110].

11.3.1 New complex integration transformation

Corresponding to the structure of phase space spanned by $|\xi\rangle$ and $|\eta\rangle$ and enlightened by Eq. (271), we propose a new complex integration transformation in ξ - η phase space

$$\int \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} \mathcal{F}(\eta, \xi) \equiv D(\nu, \mu) \quad (307)$$

When $\mathcal{F}(\xi, \eta) = 1$, Eq. (307) becomes

$$\int \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} = \int d^2\xi \delta(\xi - \mu) \delta(\xi^* - \mu^*) e^{\nu(\xi^* - \mu^*) - \nu^*(\xi - \mu)} = 1 \quad (308)$$

so $e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}$ can be considered a basis function in $\xi - \eta$ phase space, or Eq. (307) can be looked as an expansion of $D(\nu, \mu)$ in terms of $e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}$, with the expansion coefficient being $\mathcal{F}(\eta, \xi)$.

We can prove that the inverse transform of Eq. (307) is

$$\int \frac{d^2\mu d^2\nu}{\pi^2} e^{(\xi^*-\mu^*)(\eta-\nu)-(\eta^*-\nu^*)(\xi-\mu)} D(\nu, \mu) \equiv \mathcal{F}(\eta, \xi) \quad (309)$$

In fact, substituting Eq. (307) into the left-hand side of Eq. (309), and we obtain

$$\int \frac{d^2\xi' d^2\eta'}{\pi^2} \mathcal{F}(\eta', \xi') \int \frac{d^2\mu d^2\nu}{\pi^2} \times e^{(\xi'-\mu)(\eta'^*-\nu^*)-(\eta'-\nu)(\xi'^*-\mu^*)+(\xi^*-\mu^*)(\eta-\nu)-(\eta^*-\nu^*)(\xi-\mu)} = \int d^2\xi' d^2\eta' \mathcal{F}(\eta', \xi') e^{(\xi'\eta'^* - \eta'\xi'^* + \xi^*\eta - \eta^*\xi)} \times \delta^{(2)}(\eta' - \eta) \delta^{(2)}(\xi - \xi') = \mathcal{F}(\eta, \xi) \quad (310)$$

The Parseval-like theorem for this transformation can also be demonstrated as

$$\int \frac{d^2\xi d^2\eta}{\pi^2} |\mathcal{F}(\eta, \xi)|^2 = \int \frac{d^2\mu d^2\nu}{\pi^2} |D(\nu, \mu)|^2 \quad (311)$$

11.3.2 Complex integration transformation and complex Weyl transformation

In Ref. [53, 54] for correlated two-body systems, we have successfully established the so-called entangled Wigner operator, expressed in the entangled state $\langle \eta |$ representation as Eq. (43),

$$\Delta(\sigma, \gamma) \rightarrow \Delta(\eta, \xi) = \int \frac{d^2\sigma}{\pi^3} |\eta - \sigma\rangle \langle \eta + \sigma| e^{\sigma\xi^* - \sigma^*\xi} \quad (312)$$

The advantage of introducing $\Delta(\eta, \xi)$ can be seen in Ref. [80]. The corresponding Wigner function for a density matrix ρ is

$$W_\rho(\eta, \xi) = \int \frac{d^2\eta}{\pi^3} \langle \eta + \sigma | \rho | \eta - \sigma \rangle e^{\sigma\xi^* - \sigma^*\xi} \quad (313)$$

If $F(\eta, \xi)$ is quantized as the operator $F(Q_1, Q_2, P_1, P_2)$ through the Weyl-Wigner correspondence

$$F(Q_1, Q_2, P_1, P_2) = \int d^2\eta d^2\xi \mathcal{F}(\eta, \xi) \Delta(\eta, \xi) \quad (314)$$

then using Eq. (313) we see

$$\begin{aligned} \mathcal{F}(\eta, \xi) &= 4\pi^2 \text{Tr} [F(Q_1, Q_2, P_1, P_2) \Delta(\eta, \xi)] \\ &= 4 \int \frac{d^2\sigma}{\pi} e^{\sigma\xi^* - \sigma^*\xi} \langle \eta + \sigma | F(Q_1, Q_2, P_1, P_2) | \eta - \sigma \rangle \end{aligned} \quad (315)$$

which is named as the complex Weyl transform, and $\mathcal{F}(\eta, \xi)$ is the Weyl classical correspondence of $F(Q_1, Q_2, P_1, P_2)$. Substituting Eq. (315) into Eq. (307), we get

$$\begin{aligned} &\int \int \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} \mathcal{F}(\eta, \xi) \\ &= 4 \int \frac{d^2\sigma}{\pi} e^{\mu^*\sigma - \mu\sigma^*} \langle \nu + 2\sigma | F(Q_1, Q_2, P_1, P_2) | \nu \rangle \end{aligned} \quad (316)$$

Using Eq. (29), we have

$$\begin{aligned} \langle \nu + 2\sigma | &= \langle 2\sigma | \exp \left\{ \frac{i}{\sqrt{2}} [\nu_1(P_1 - P_2) - \nu_2(Q_1 + Q_2)] \right\} \\ \nu &= \nu_1 + i\nu_2 \end{aligned} \quad (317)$$

As a result of Eq. (317) and $\frac{1}{2}e^{\mu^* \sigma - \mu \sigma^*} = \langle \xi_{=\mu} | 2\sigma \rangle$, we see

$$4 \int d^2\sigma e^{\mu^* \sigma - \mu \sigma^*} \langle \nu + 2\sigma | = 2\pi \langle \xi_{=\mu} | e^{i(\mu_2\nu_1 - \mu_1\nu_2)} \quad (318)$$

Using Eq. (318), we convert Eq. (316) as

$$\begin{aligned} \iint \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} F(\eta, \xi) \\ = 2\pi \langle \xi_{=\mu} | F(Q_1, Q_2, P_1, P_2) | \nu \rangle e^{i(\nu_1\mu_2 - \nu_2\mu_1)} \end{aligned} \quad (319)$$

The inverse of Eq. (319), according to Eq. (309), is

$$\begin{aligned} \mathcal{F}(\eta, \xi) &= \iint \frac{2d^2\mu d^2\nu}{\pi} e^{(\xi^*-\mu^*)(\eta-\nu)-(\eta^*-\nu^*)(\xi-\mu)} \\ &\times \langle \xi_{=\mu} | F(Q_1, Q_2, P_1, P_2) | \nu \rangle e^{i(\nu_1\mu_2 - \nu_2\mu_1)} \end{aligned} \quad (320)$$

Thus through the new integration transformation, a new relationship between a phase space function $\mathcal{F}(\eta, \xi)$ and its Weyl-Wigner correspondence operator $F(Q_1, Q_2, P_1, P_2)$ is revealed.

For example, from the following Weyl-Wigner correspondence

$$\frac{4}{(e^f + 1)^2} \exp \left[\frac{e^f - 1}{e^f + 1} (|\eta|^2 + |\xi|^2) \right] \rightarrow \exp \{ f [K_+ + K_- - 1] \} \quad (321)$$

[K_+ and K_- are defined in Eqs. (193) and (196)] and Eq. (320) we have

$$\begin{aligned} \frac{4}{(e^f + 1)^2} \iint \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} \\ \times \exp \left[\frac{e^f - 1}{e^f + 1} (|\eta|^2 + |\xi|^2) \right] \\ = 2\pi \langle \xi_{=\mu} | F(Q_1, Q_2, P_1, P_2) | \nu \rangle e^{i(\nu_1\mu_2 - \nu_2\mu_1)} \end{aligned} \quad (322)$$

Using the Gaussian integration formula

$$\begin{aligned} \iint \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} e^{-\lambda(|\eta|^2 + |\xi|^2)} \\ = \frac{1}{1 + \lambda^2} \exp \left[-\frac{\lambda(|\mu|^2 + |\nu|^2)}{1 + \lambda^2} + \frac{\lambda^2(\mu\nu^* - \mu^*\nu)}{1 + \lambda^2} \right] \end{aligned} \quad (323)$$

in particular, when $\lambda = -i \tan(\frac{\pi}{4} - \frac{\alpha}{2})$, with $\frac{-\lambda}{\lambda^2 + 1} = \frac{i}{2 \tan \alpha}$, $\frac{\lambda^2}{\lambda^2 + 1} = \frac{1}{2} - \frac{1}{2 \sin \alpha}$, Eq. (323) becomes

$$\begin{aligned} \iint \frac{d^2\xi d^2\eta}{\pi^2} e^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)} \\ \times \exp \left[i \tan \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) (|\eta|^2 + |\xi|^2) \right] \end{aligned}$$

$$\begin{aligned} = \frac{e^{i\alpha}}{i \sin \alpha} \exp \left[\frac{i(|\mu|^2 + |\nu|^2)}{2 \tan \alpha} - \frac{\mu\nu^* - \mu^*\nu}{2 \sin \alpha} \right. \\ \left. + i\mu_2\nu_1 - i\mu_1\nu_2 \right] \end{aligned} \quad (324)$$

where $\exp[i \tan(\frac{\pi}{4} - \frac{\alpha}{2})(|\eta|^2 + |\xi|^2)]$ represents an infinite long chirplet function. By taking $f = i(\frac{\pi}{2} - \alpha)$ in Eq. (322), such that $ie^{-i\alpha} = e^f$, and comparing with Eq. (324) we obtain

$$\begin{aligned} \langle \xi_{=\mu} | F(Q_1, Q_2, P_1, P_2) | \nu \rangle \\ = \frac{-ie^{i\alpha}}{2\pi \sin \alpha} \exp \left[\frac{i(|\mu|^2 + |\nu|^2)}{2 \tan \alpha} - \frac{\mu\nu^* - \mu^*\nu}{2 \sin \alpha} \right] \end{aligned} \quad (325)$$

where the right-hand side of Eq. (325) is just the CFrFT kernel whose properties can be seen in Ref. [111]. (One may compare the forms (296) and (325) to see their slight difference. For the relation between them we refer to Refs. [110, 111]). Dragoman has shown that the kernel of the CFrFT can be classically produced with rotated astigmatic optical systems that mimic the quantum entanglement. Therefore the new integration transformation (307) can convert spherical wave to CFrFT kernel. We expect this transformation could be implemented by experimentalists.

11.4 Squeezing for the generalized scaled FrFT

In some practical applications it is necessary to introduce input and output scale parameters [44, 103] into FrFT, i.e., scaled FrFT. The reason lies in two facts: (i) the scaled FrFT may be more useful and convenient for optical information processing due to the scale parameters (free parameters) introduced into FrFT; (ii) it can be reduced to the conventional FrFT under a given condition. In this subsection, by establishing the relation between the optical scaled FrFT and quantum mechanical squeezing-rotating operator transform in a one-mode case, we employ the IWOP technique and the bipartite entangled state representation of two-mode squeezing operator to extend the scaled FrFT to more general cases, such as scaled complex FrFT and entangled scaled FrFT. The properties of scaled FrFTs can be seen more clearly from the viewpoint of representation transform in quantum mechanics.

11.4.1 Quantum correspondence of the scaled FrFT

The scaled FrFT [44] of α -order is defined in such a manner that the usual FrFT is its special case, i.e.,

$$\begin{aligned} \mathcal{F}_\alpha [f(x)] &= \sqrt{\frac{e^{i(\frac{\pi}{2}-\alpha)}}{2\pi\mu\nu \sin \alpha}} \int_{-\infty}^{\infty} \exp \left[-i \frac{x^2/\mu^2 + y^2/\nu^2}{2 \tan \alpha} \right. \\ &\quad \left. + \frac{ixx'}{\mu\nu \sin \alpha} \right] f(x) dx \end{aligned} \quad (326)$$

where the exponential function is an integral kernel. In a similar way to deriving the quantum correspondence of FrFT in Eq. (248), and using the natural repression of single-mode squeezing operator S_1 in coordinate representation [112–114],

$$S_1(\mu) = \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dx \left| \frac{x}{\mu} \right\rangle \langle x| \quad (327)$$

we have

$$\begin{aligned} \exp\left(-i \frac{x^2/\mu^2 + y^2/\nu^2}{2 \tan \alpha} + \frac{ixx'}{\mu\nu \sin \alpha}\right) \\ = \sqrt{-2\pi i \mu \nu e^{i\alpha} \sin \alpha} \langle y | S_1^\dagger(\nu) \exp(i\alpha a^\dagger a) S_1(\mu) | x \rangle \end{aligned} \quad (328)$$

which implies that the integral kernel in Eq. (326) is just the matrix element of operator $S_1^\dagger(\nu) \exp(i\theta a^\dagger a) S_1(\mu)$ in coordinate states. From Eq. (328) it then follows that

$$\begin{aligned} \mathcal{F}_\alpha[f(x)] &= \int_{-\infty}^{\infty} dx \langle y | S_1^\dagger(\nu) e^{i\alpha a^\dagger a} S_1(\mu) | x \rangle f(x) \\ &= \langle y | S_1^\dagger(\nu) e^{i\alpha a^\dagger a} S_1(\mu) | f \rangle \equiv g(y) \end{aligned} \quad (329)$$

which suggests

$$|g\rangle = S_1^\dagger(\nu) e^{i\alpha a^\dagger a} S_1(\mu) | f \rangle \quad (330)$$

From Eqs. (329) and (326) one can see that the scaled FrFT in Eq. (329) corresponds actually to the squeezing-rotating operator ($S_1^\dagger(\nu) e^{i\theta a^\dagger a} S_1(\mu)$) transform in Eq. (329) between two quantum states.

11.4.2 The Scaled CFrFT

On the basis of quantum mechanical version of one-mode scaled FrFT, we generalize it a two-mode case, i.e., we can introduce the integral

$$\begin{aligned} \mathcal{F}_\alpha^C[f(\eta)] &\equiv \langle \eta' | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} S_2(\mu) | f \rangle \\ &= \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \langle \eta' | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} S_2(\mu) | \eta \rangle f(\eta) \end{aligned} \quad (331)$$

where $f(\eta) = \langle \eta | f \rangle$. Using the natural expression of the two-mode squeezing operator S_2 [Eq. (34)], and noticing that $\langle \eta' | e^{i\theta(a_1^\dagger a_1 + a_2^\dagger a_2)} | \eta \rangle$ is just the integral kernel of CFrFT (296), we can reform (331) as

$$\begin{aligned} \mathcal{F}_\alpha^C[f(\eta)] &= \frac{e^{i(\frac{\pi}{2}-\alpha)}}{2\mu\nu \sin \alpha} \int \frac{d^2\eta}{\pi} f(\eta) \\ &\times \exp\left[-\frac{i(|\eta'|^2/\nu^2 + |\eta|^2/\mu^2)}{2 \tan \alpha} + \frac{i(\eta'^* \eta + \eta^* \eta')}{2\mu\nu \sin \alpha}\right] \end{aligned} \quad (332)$$

It is obvious that Eq. (332) is just a generalized CFrFT with squeezing parameters. We name it the scaled CFrFT. Thus we link a two-mode squeezing-rotating op-

erator transform to the scaled CFrFT of complex functions.

11.4.3 Entangled scaled FrFT

On the other hand, recall that the entangled state $|\eta\rangle$ can be Schmidt-decomposed as [33]

$$|\eta\rangle = e^{-i\eta_1 \eta_2} \int_{-\infty}^{\infty} dx |x\rangle_1 \otimes |x - \sqrt{2}\eta_1\rangle_2 e^{i\sqrt{2}x\eta_2} \quad (333)$$

We see that

$$\begin{aligned} \langle x'_1, x'_2 | \eta' \rangle &= e^{-i\eta'_1 \eta'_2} \delta(\sqrt{2}\eta'_1 + x'_2 - x'_1) e^{i\sqrt{2}x'_1 \eta'_2} \\ \langle \eta | x_1, x_2 \rangle &= e^{i\eta_1 \eta_2} \delta(\sqrt{2}\eta_1 + x_2 - x_1) e^{-i\sqrt{2}x_1 \eta_2} \end{aligned} \quad (334)$$

Using Eq. (30), we have

$$\begin{aligned} K(x'_1, x'_2, x_1, x_2) &= \langle x'_1, x'_2 | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} S_2(\mu) | x_1, x_2 \rangle \\ &= \int \frac{d^2\eta d^2\eta'}{\pi^2} \langle x'_1, x'_2 | \eta' \rangle \langle \eta' | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} \\ &\times S_2(\mu) | \eta \rangle \langle \eta | x_1, x_2 \rangle \end{aligned} \quad (335)$$

where $|x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$. Substituting Eqs. (333) and (334) into Eq. (335), we can derive

$$\begin{aligned} K(x'_1, x'_2, x_1, x_2) &= \sqrt{\frac{e^{i(\frac{\pi}{2}-\alpha)}}{2\pi \sin \alpha}} \exp\left(-i \frac{\lambda_\nu'/2 + \lambda_\mu^2}{2 \tan \alpha} + \frac{i\lambda_\mu \lambda_\nu'}{\sin \alpha}\right) \\ &\times \sqrt{\frac{e^{i(\frac{\pi}{2}-\alpha)}}{2\pi \sin \alpha}} \exp\left(-i \frac{\kappa_\nu'/2 + \kappa_\mu^2}{2 \tan \alpha} + \frac{i\kappa_\mu \kappa_\nu'}{\sin \alpha}\right) \end{aligned} \quad (336)$$

where $\lambda_\mu = \frac{x_1-x_2}{\sqrt{2}\mu}$, $\lambda_\nu' = \frac{x'_1-x'_2}{\sqrt{2}\nu}$; $\kappa_\mu = \frac{\mu(x_2+x_1)}{\sqrt{2}}$, $\kappa_\nu' = \frac{\nu(x'_1+x'_2)}{\sqrt{2}}$.

From Eq. (336) one can see that a new 2D scaled FrFT can be composed of one 1D scaled FFT in its space domain and the other in its “frequency” domain, with the transform variables being the combination of two coordinates. Hence Eq. (336) is quite different from the direct product of the two 1D scaled FrFTs that are both in “space domain” indicated in Eq. (329). Note that the new 2D scaled FFT is still characterized by only 3-parameter. Therefore, for any function $f(x_1, x_2) = \langle x_1, x_2 | f \rangle$ we can define an entangled scaled FrFT, i.e.,

$$\begin{aligned} \mathcal{F}_\alpha^E[f(x_1, x_2)] &= \int_{-\infty}^{\infty} K(x'_1, x'_2, x_1, x_2) f(x_1, x_2) dx_1 dx_2 \\ &= \langle x'_1, x'_2 | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} S_2(\mu) | f \rangle \end{aligned} \quad (337)$$

Next we examine the properties of these scaled FrFTs in the quantum optics context. Without losing generality, for the additivity property, we consider the scaled

CFrFT,

$$\mathcal{F}_{\alpha+\beta}^C [f(\eta)] \equiv \int \frac{d^2\eta}{\pi} \langle \eta' | S_2^\dagger(\nu) e^{i(\alpha+\beta)(a_1^\dagger a_1 + a_2^\dagger a_2)} \times S_2(\mu) | \eta \rangle f(\eta) \quad (338)$$

Inserting the completeness relation of $|\eta\rangle$ into Eq. (338), we get

$$\mathcal{F}_{\alpha+\beta}^C [f(\eta)] = \int \frac{d^2\eta''}{\pi} \langle \eta' | S_2^\dagger(\nu) e^{i\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)} \times S_2(\tau) | \eta'' = \tau \rangle \mathcal{F}_\beta [f(\eta)] = \mathcal{F}_\alpha \mathcal{F}_\beta [f(\eta)] \quad (339)$$

which is just the additivity property. It should be pointed out that the condition of additive operator for the scaled FrFTs is that the parameter ν' of the prior cascade should be equal to the parameter μ' of the next one, i.e., $\mu' = \nu'$. For other scaled FrFTs, the properties can also be discussed in the similar way (according to their quantum versions).

To this end, we should emphasize that different scaled FrFTs correspond to different quantum mechanical squeezing operators or representations. That is to say, it is possible that some other scaled FrFT can be presented by using different quantum mechanical squeezing operators or representations.

12 Adaption of Collins diffraction formula and CFrFT

The connection between the Fresnel diffraction in free space and the FrFT had been bridged by Pellat-Finet [115] who found that FrFTs are adapted to the mathematical expression of Fresnel diffraction, just as the standard Fourier transform is adapted to Fraunhofer diffraction. In previous sections, a new formulation of the CFrFT and the Collins diffraction formula are respectively derived in the context of representation transform of quantum optics. In this section we inquire if the adaption problem of Collins diffraction formula to the CFrFT can also be tackled in the context of quantum optics. We shall treat this topic with the use of two-mode (3 parameters) squeezing operator and in the entangled state representation of continuous variables. In so doing the quantum mechanical version of the associated theory of classical diffraction and classical CFrFT is obtained, which connects classical optics and quantum optics in this aspect.

For Gaussian beam, the $ABCD$ rule is equally derived via optical diffraction integral theory — the Collins integral formula. As shown in Fig. 2, if $f(\eta)$ represents the input field amplitude at point η on S_1 , and $g(\eta')$ denotes the diffraction field amplitude at point η' on S_2 . Then Collins formula in complex form takes the form (181) and (182). Next we shall examine the adaption of the Collins

formula to the CFrFT by virtue of the entangled state representation in quantum optics [108].

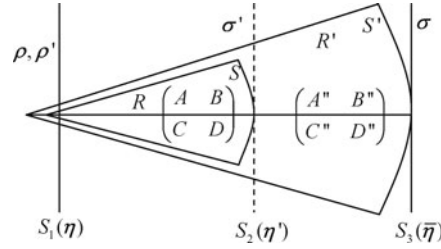


Fig. 2 The Fresnel diffraction through ABCD optical system.

12.1 Adaption of the Collins formula to CFrFT

Using the completeness relation of $|\eta\rangle$, we can further put Eq. (181) into

$$g(\eta') = \langle \eta' | U_2(r, s) | f \rangle = \langle \eta' | U_2(r, s) \mu_1^2 \int \frac{d^2\eta}{\pi} |\mu_1\eta\rangle \langle \mu_1\eta | f \rangle \quad (340)$$

Take $\eta' = \sqrt{\frac{B}{D}} \frac{\sigma}{K}$, $\mu_1 = \sqrt{\frac{B}{A}}/L$ and write out

$$g(\eta') \rightarrow \langle \sqrt{\frac{B}{D}} \frac{\sigma}{K} | g \rangle \equiv G(\sigma), f(\mu_1\eta) \equiv F(\eta) \quad (341)$$

where K and L are two constants to be determined later. Then according to Eqs. (181) and (340) we have

$$G(\sigma) = \mu_1^2 \int \frac{d^2\eta}{\pi} \langle \eta' | U_2(r, s) | \mu_1\eta \rangle F(\eta) = \frac{1}{2iAL^2} \exp \left[\frac{i|\sigma|^2}{2K^2} \right] \int \frac{d^2\eta}{\pi} \times \exp \left[\frac{i|\eta|^2}{2L^2} - \frac{i(\sigma^*\eta + \sigma\eta^*)}{2LK\sqrt{AD}} \right] F(\eta) \quad (342)$$

Comparing Eq. (342) with Eq. (182) leads us to choose

$$L^2 = \tan \alpha, K = \sqrt{\sin 2\alpha / (2AD)} \quad (343)$$

Then Eq. (342) becomes

$$G(\sigma) = \frac{\cos \alpha}{i2A \sin \alpha} \exp \left(i \frac{AD - \cos^2 \alpha}{\sin 2\alpha} |\sigma|^2 \right) \times \int \frac{d^2\eta}{\pi} \exp \left[\frac{i(|\eta|^2 + |\sigma|^2)}{2 \tan \alpha} - \frac{i(\sigma^*\eta + \sigma\eta^*)}{2 \sin \alpha} \right] F(\eta) = \frac{\cos \alpha}{A} e^{-i\alpha} \exp \left(i \frac{AD - \cos^2 \alpha}{\sin 2\alpha} |\sigma|^2 \right) \mathcal{F}_\alpha [F](\sigma) \quad (344)$$

so Eq. (344) is a standard CFrFT up to a quadratic phase term $\exp \left(i \frac{AD - \cos^2 \alpha}{\sin 2\alpha} |\sigma|^2 \right)$, according to Eq. (343) and $\sqrt{\frac{B}{D}} \frac{\sigma}{K} = \eta'$, it can also be written as

$$\exp \left(i \frac{AD - \cos^2 \alpha}{\sin 2\alpha} |\sigma|^2 \right) = \exp \left(\frac{i}{R} |\eta'|^2 \right) \quad (345)$$

which represents a quadratic approximation to a sphere wave diverging from a luminous point at the distance

$$R = \frac{2AB}{AD - \cos^2 \alpha} \tag{346}$$

from S_2 . Let S be the sphere tangent to S_2 with radius R (see Fig. 2). A point on S is located by its projection on S_2 , which means that the coordinates on S_2 can also be used as those on S . Therefore, the quadratic phase term can be compensated if the output field is observed on S instead of S_2 . Then, after considering the phase compensation, the field transforms from S_1 to S is

$$G_S(\sigma) = \frac{\cos \alpha}{A} e^{-i\alpha} \mathcal{F}_\alpha [F](\sigma) \tag{347}$$

In this way, the field amplitude on S is the perfect α -th CFrFT of the field amplitude on S_1 .

12.2 Adaption of the additivity property of CFrFT to the Collins formula for two successive Fresnel diffractions

The most important property of FrFT is that \mathcal{F}_α obeys the additivity rule, i.e., two successive FrFT of order α and β makes up the FrFT of order $\alpha + \beta$. For the CFrFT, its additivity property is proven in Eq. (300). For Collins diffraction from S_1 to S' (see Fig. 1), the additivity means that the diffraction pattern observed on $S'(\bar{\eta})$ (the sphere tangent to S_3 with radius R') and associated with $\mathcal{F}_{\alpha+\beta}$ should be the result of a first diffraction phenomenon (associated with \mathcal{F}_α) on S (with η'), followed by a second diffraction phenomenon (associated with \mathcal{F}_β) from S to S' . This is a necessary consequence of the Huygens principle. Next we prove that such is indeed the case.

First, let us consider the field transform from S_1 (with η) to S' (see Fig. 2) described by the ray transfer matrix $[A', B', C', D']$. Similar to deriving Eq. (346), after the squeezing transform and the phase compensation,

$$R' = \frac{2A'B'}{A'D' - \cos^2 \alpha'} \rightarrow \exp\left(\frac{i}{R'} |\bar{\eta}|^2\right) \tag{348}$$

Thus we can obtain the expression of CFrFT for Collins diffraction from S_1 to S' (not S_3),

$$G_{S'}(\sigma') = \frac{\cos \alpha'}{A'} e^{-i\alpha'} \mathcal{F}_{\alpha'} [f(\mu'_1 \eta)](\sigma') \tag{349}$$

where $\bar{\eta} = \sqrt{\frac{B'}{D'}} \frac{\sigma'}{K'}$ and $\mu'_1 = \sqrt{\frac{B'}{A'}} / L'$,

$$L'2 = \tan \alpha', \quad K' = \sqrt{\sin 2\alpha' / (2A'D')} \tag{350}$$

Eq. (349) is the same in form as Eq. (347) but with primed variables. Using Eqs. (348) and (181) one can prove that the transform from S_1 to S' is [see Eqs. (181), (344)–(346)]

$$g_{S'}(\bar{\eta}) = \exp\left(-\frac{i}{R'} |\bar{\eta}|^2\right) \langle \bar{\eta} | U_2(r', s') | f \rangle \equiv G_{S'}(\sigma')$$

$$(351)$$

In Eq. (351) we have taken the phase compensation term (348) into account.

Second, let us consider the second diffraction from S to S_3 determined by the ray transfer matrix $[A'', B'', C'', D'']$. For this purpose, using the group multiplication rule of $F_2(r, s)$, we can decompose the diffraction from S_1 to S' into two parts: one is described as the matrix $[A, B, C, D]$ from the plane S_1 (with η) to $S_2(S)$ (with η'), the other is $[A'', B'', C'', D'']$ from the plane S_2 to $S_3(S')$ (with $\bar{\eta}$), then the total matrix from S_1 to S_3 is

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{352}$$

Using Eq. (30) and the group multiplication rule of $F_2(r', s')$, we can further put Eq. (351) into another form

$$\begin{aligned} G_{S'}(\sigma') &= \exp\left(-\frac{i}{R'} |\bar{\eta}|^2\right) \langle \bar{\eta} | U_2(r'', s'') U_2(r, s) | f \rangle \\ &= \frac{B}{DK^2} \int \frac{d^2\sigma}{\pi} \exp\left(-\frac{i}{R'} |\bar{\eta}|^2\right) \\ &\quad \times \langle \bar{\eta} | U_2(r'', s'') | \mu'_2 \sigma \rangle G(\sigma) \end{aligned} \tag{353}$$

where $\mu'_2 = \sqrt{\frac{B}{D}} \frac{1}{K}$ and we have made a reasonable assumption that $\mu'_1 = \mu_1$ [hence $f(\mu'_1 \eta) = F(\eta)$], which means that there are same scaled variants for the input field amplitudes on S_1 of the diffractions from S_1 to S and from S_1 to S' .

In order to examine the second diffraction domain from S to S' (not S_3), we need to translate the output field amplitude $G(\sigma)$ observed on the plane S_2 to the field amplitude observed on sphere plane S , i.e., putting $G(\sigma)$ into $G_S(\sigma)$ [see Eq. (347)] by taking the phase compensation [see Eq. (342)] into account. Thus the field transform from S to S' is $[G_S(\sigma) \rightarrow G_{S'}(\sigma')]$

$$\begin{aligned} G_{S'}(\sigma') &= \frac{B}{DK^2} \int \frac{d^2\sigma}{\pi} \exp\left(\frac{i}{R} \frac{B|\sigma|^2}{DK^2} - \frac{i}{R'} |\bar{\eta}|^2\right) \\ &\quad \times \langle \bar{\eta} | U_2(r'', s'') | \mu'_2 \sigma \rangle G_S(\sigma) \\ &= \frac{B}{DK^2} \frac{1}{2iB''} \int \frac{d^2\sigma}{\pi} \exp\left[\frac{iB'}{K'2D'} \left(\frac{D''}{2B''} - \frac{1}{R'}\right) |\sigma'|^2\right. \\ &\quad \left. + \frac{iB|\sigma|^2}{DK^2} \left(\frac{A''}{2B''} + \frac{1}{R}\right) - \frac{i(\sigma\sigma'^* + \sigma'\sigma^*)}{2B''K'K\sqrt{\frac{DD'}{BB'}}}\right] G_S(\sigma) \end{aligned} \tag{354}$$

Comparing Eq. (354) with Eq. (182), we have

$$\sin \beta = B'' K' K \sqrt{\frac{DD'}{BB'}} = \frac{B''}{2} \sqrt{\frac{\sin 2\alpha' \sin 2\alpha}{A'ABB'}} \tag{355}$$

and noticing that $\mu'_1 = \mu_1$ yields $\frac{B'}{A' \tan \alpha'} = \frac{B}{A \tan \alpha}$, we

have

$$A' = \frac{B'' \cos \alpha' \sin \alpha}{B \sin \beta}, A = \frac{B'' \sin \alpha' \cos \alpha}{B' \sin \beta} \quad (356)$$

combine Eqs. (340) and (356), and it then follows that (let $\alpha' = \alpha + \beta$)

$$\frac{B}{DK^2} \frac{1}{2iB''} = \frac{AB}{\sin 2\alpha} \frac{1}{iB''} = \frac{A}{iA' \cos \alpha} \frac{\cos \alpha'}{2 \sin \beta} \quad (357)$$

and

$$\begin{aligned} \frac{B'}{K'2D'} \left(\frac{D''}{2B''} - \frac{1}{R'} \right) &= \frac{1}{2} \cot \beta \\ \frac{B}{DK^2} \left(\frac{A''}{2B''} + \frac{1}{R} \right) &= \frac{1}{2} \cot \beta \end{aligned} \quad (358)$$

where we have used Eqs. (346), (348) and (352).

Substitution of Eqs. (347), (355), (357) and (358) into Eq. (354) yields

$$\begin{aligned} G_{S'}(\sigma') &= \frac{A}{iA' \cos \alpha} \frac{\cos \alpha'}{2 \sin \beta} \int \frac{d^2\sigma}{\pi} \exp \left[\frac{i(|\sigma'|^2 + |\sigma|^2)}{2 \tan \beta} - \frac{i(\sigma\sigma'^* + \sigma'\sigma^*)}{2 \sin \beta} \right] G_S(\sigma) \\ &= \frac{\cos \alpha' e^{-i\alpha}}{iA'} \frac{1}{2 \sin \beta} \int \frac{d^2\sigma}{\pi} \exp \left[\frac{i(|\sigma'|^2 + |\sigma|^2)}{2 \tan \beta} - \frac{i(\sigma\sigma'^* + \sigma'\sigma^*)}{2 \sin \beta} \right] \mathcal{F}_\alpha [F](\sigma) \\ &= \frac{\cos \alpha'}{A'} e^{-i(\alpha+\beta)} \mathcal{F}_\beta \mathcal{F}_\alpha [F](\sigma') \end{aligned} \quad (359)$$

The first equation of Eq. (359) indicates that it is just a CFrFT of $G_S(\sigma)$ from S to S' . Comparing Eq. (359) with Eq. (349), we see

$$\mathcal{F}_\beta \mathcal{F}_\alpha [F](\sigma') = \mathcal{F}_{\alpha+\beta} [F](\sigma') \quad (360)$$

So far we have completed the study of adaption of CFrFT to the mathematical representation of Collins diffraction formula in quantum optics context.

13 Wavelet transformation and the IWOP technique

In recent years wavelet transforms [116, 117] have been developed which can overcome some shortcomings of the classical Fourier analysis and therefore has been widely used in Fourier optics and information science since 1980s. Here we present a quantum optical version of classical wavelet transform (WT) by virtue of the IWOP technique.

13.1 Quantum optical version of classical WTs

A wavelet has its energy concentrated in time to give a tool for the analysis of transient, nonstationary, or time-varying phenomena. (It is a wavelet because it is localized and it resembles a wave because it oscillates.) Mathematically, wavelets are defined by starting with a function ψ of the real variable x , named a mother wavelet which is required to decrease rapidly to zero as $|x|$ tends to infinity,

$$\int_{-\infty}^{\infty} \psi(x) dx = 0 \quad (361)$$

A more general requirement for a mother wavelet is for demanded $\psi(x)$ to have vanishing moments $\int_{-\infty}^{\infty} x^l \psi(x) dx = 0$, $l = 0, 1, 2, \dots, L$. (A greater degree of

smoothness than continuity also leads to vanishing moments for the mother wavelet). The theory of wavelets is concerned with the representation of a function in terms of a two-parameter family of dilates and translates of a fixed function. The mother wavelet ψ generates the other wavelets of the family $\psi_{(\mu,s)}$ (μ is scaling parameter, s is a translation parameter, $s \in \mathbb{R}$), and the dilated-translated function is defined as

$$\psi_{(\mu,s)}(x) = \frac{1}{\sqrt{|\mu|}} \psi \left(\frac{x-s}{\mu} \right) \quad (362)$$

while the wavelet integral transform of a signal function $f(x) \in L^2(\mathbb{R})$ by ψ is defined by

$$W_\psi f(\mu, s) = \frac{1}{\sqrt{|\mu|}} \int_{-\infty}^{\infty} f(x) \psi^* \left(\frac{x-s}{\mu} \right) dx \quad (363)$$

We can express Eq. (363) as

$$W_\psi f(\mu, s) = \langle \psi | U(\mu, s) | f \rangle \quad (364)$$

where $\langle \psi |$ is the state vector corresponding to the given mother wavelet, $|f\rangle$ is the state to be transformed, and

$$U(\mu, s) \equiv \frac{1}{\sqrt{|\mu|}} \int_{-\infty}^{\infty} \left| \frac{x-s}{\mu} \right\rangle \langle x | dx \quad (365)$$

is the squeezing-translating operator [116, 118, 119], $\langle x |$ is the eigenvector of coordinate operator. In order to combine the wavelet transform with quantum states transform more tightly, using the IWOP technique, we can directly perform the integral in Eq. (365) ($Q = (a + a^\dagger)/\sqrt{2}$, $\mu > 0$)

$$\begin{aligned} U(\mu, s) &= \exp \left[\frac{-s^2}{2(1+\mu^2)} - \frac{a^{\dagger 2}}{2} \tanh \lambda - \frac{a^\dagger s}{\sqrt{2}} \operatorname{sech} \lambda \right] \\ &\times \exp \left[\left(a^\dagger a + \frac{1}{2} \right) \ln \operatorname{sech} \lambda \right] \\ &\times \exp \left(\frac{a^2}{2} \tanh \lambda + \frac{sa}{\sqrt{2}} \operatorname{sech} \lambda \right) \end{aligned} \quad (366)$$

where $\mu = e^\lambda$, $\operatorname{sech}\lambda = \frac{2\mu}{1+\mu^2}$, $\tanh\lambda = \frac{\mu^2-1}{\mu^2+1}$, and the operator identity $e^{ga^\dagger a} = \exp[(e^g - 1)a^\dagger a]$ is used. In particular, when $s = 0$, it is reduced to the well-known squeezing operator,

$$U(\mu, 0) = \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} \left| \frac{x}{\mu} \right\rangle \langle x| dx = \exp \left[\frac{\lambda}{2} (a^2 - a^{\dagger 2}) \right] \tag{367}$$

For a review of the squeezed state theory we refer to Ref. [14].

13.2 The condition of mother wavelet in the context of quantum optics

Now we analyze the condition (361) for mother wavelet from the point of view of quantum optics. Due to

$$\int_{-\infty}^{\infty} |x\rangle dx = |p = 0\rangle \tag{368}$$

where $|p\rangle$ is the momentum eigenstate, we can recast the condition into quantum mechanics as

$$\int_{-\infty}^{\infty} \psi(x) dx = 0 \rightarrow \langle p = 0 | \psi \rangle = 0 \tag{369}$$

which indicates that the probability of a measurement of $|\psi\rangle$ by the projection operator $|p\rangle \langle p|$ with value $p = 0$ being zero. Without a loss of generality, we suppose

$$|\psi\rangle_M = G(a^\dagger) |0\rangle = \sum_{n=0}^{\infty} g_n a^{\dagger n} |0\rangle \tag{370}$$

where g_n are chosen as to let $|\psi\rangle$ obey the condition (361). Using the coherent states' overcompleteness relation, we have

$$\begin{aligned} \langle p = 0 | \psi \rangle &= \langle p = 0 | \int \frac{d^2z}{\pi} |z\rangle \langle z| \sum_n g_n a^{\dagger n} |0\rangle \\ &= \sum_n g_{2n} = 0 \end{aligned} \tag{371}$$

Eq. (371) provides a general formalism in finding the qualified wavelets. For example, assuming $g_{2n} = 0$ for $n > 3$, the coefficients of the survived terms should satisfy

$$g_0 + g_2 + 3g_4 + 15g_6 = 0 \tag{372}$$

and $|\psi\rangle$ becomes

$$|\psi\rangle = (g_0 + g_2 a^{\dagger 2} + g_4 a^{\dagger 4} + g_6 a^{\dagger 6}) |0\rangle \tag{373}$$

Projecting it onto the coordinate representation, we get the qualified wavelets

$$\begin{aligned} \psi(x) &= \pi^{-1/4} e^{-x^2/2} [g_0 + g_2(2x^2 - 1) \\ &\quad + g_4(4x^4 - 12x^2 + 3) \\ &\quad + g_6(8x^6 - 60x^4 + 90x^2 - 15)] \end{aligned} \tag{374}$$

where we have used $\langle x | n \rangle = (2^n n! \sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2}$,

and $H_n(x)$ is the Hermite polynomials. Now we take some examples.

Case 1: In Eq. (371), by taking $g_0 = \frac{1}{2}$, $g_2 = -\frac{1}{2}$, $g_{2n} = 0$ (otherwise), we have

$$|\psi\rangle_M = \frac{1}{2} (1 - a^{\dagger 2}) |0\rangle \tag{375}$$

It then follows that

$$\psi_M(x) \equiv \frac{1}{2} \langle x | (1 - a^{\dagger 2}) |0\rangle = \pi^{-1/4} e^{-x^2/2} (1 - x^2) \tag{376}$$

which is just the Mexican hat wavelet, satisfying the condition $\int_{-\infty}^{\infty} e^{-x^2/2} (1 - x^2) dx = 0$. Hence $\frac{1}{2} (1 - a^{\dagger 2}) |0\rangle$ is the state vector corresponding to the Mexican hat mother wavelet (see Fig. 3). Once the state vector $|\psi\rangle$ corresponding to mother wavelet is known, for any state $|f\rangle$ the matrix element ${}_M \langle \psi | U(\mu, s) | f \rangle$ is just the wavelet transform of $f(x)$ with respect to $\langle \psi |$.

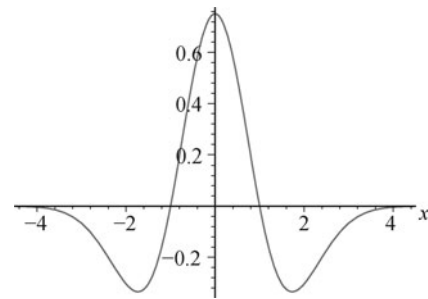


Fig. 3 Traditional Mexican hat wavelet.

Case 2: When $g_0 = -2$, $g_2 = -1$, $g_4 = 1$ and $g_6 = 0$, from Eq. (374) we obtain (see Fig. 4)

$$\psi_2(x) = 2\pi^{-1/4} e^{-x^2/2} (2x^4 - 7x^2 + 1) \tag{377}$$

which also satisfies $\int_{-\infty}^{\infty} \psi_2(x) dx = 0$. Note that when $g_0 = -1$, $g_2 = -2$, $g_4 = 1$ and $g_6 = 0$, we obtain a slightly different wavelet (see Fig. 5). Therefore, as long as the parameters g_{2n} conforms to condition (372), we can adjust their values to control the shape of the wavelets.

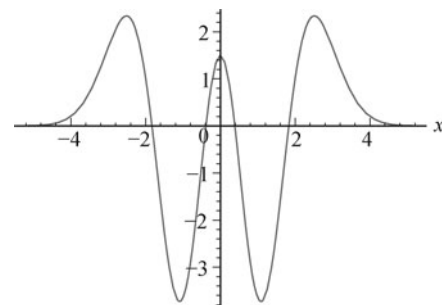


Fig. 4 Generalized Mexican hat wavelet $\psi_2(x)$ when $g_0 = -2$, $g_2 = -1$, $g_4 = 1$ and $g_6 = 0$.

Case 3: When $g_0 = 1$, $g_2 = 2$, $g_4 = 4$ and $g_6 = -1$, we get (see Fig. 6)

$$\psi_3(x) = \pi^{-1/4} e^{-x^2/2} (-8x^6 + 76x^4 - 134x^2 + 26) \tag{378}$$

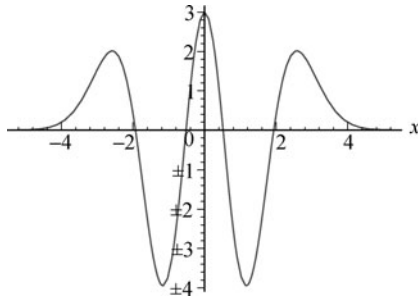


Fig. 5 Generalized Mexican hat wavelet when $g_0 = -1, g_2 = -2, g_4 = 1$ and $g_6 = 0$.

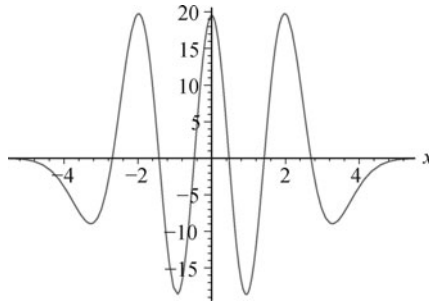


Fig. 6 Generalized Mexican hat wavelet $\psi_3(x)$ when $g_0 = 1, g_2 = 2, g_4 = 4$ and $g_6 = -1$.

and $\int_{-\infty}^{\infty} \psi_3(x)dx = 0$. From these figures we observe that the number of the nodes of the curves at the x -axis is equal to the highest power of the wavelet functions.

To further reveal the properties of the newly found wavelets, we compare the wavelet transform computed with the well-known Mexican hat wavelet $\psi_1(x)$ and that with our new wavelet $\psi_2(x)$. Concretely, we map a simple cosine signal $\cos(\pi x)$ by performing the wavelet transforms with $\psi_i[T(x - X)], i = 1, 2$, into a 2D space (X, T) , where X denotes the location of a wavelet and a its size. The resulting wavelet transforms by $\psi_1(x)$ ($=\psi_M(x)$) and $\psi_2(x)$ are

$$\Omega_1(X, T) = \frac{2}{\sqrt{3}} \int_{-\infty}^{\infty} dx \psi_1 [T(x - X)] \cos(\pi x) \quad (379)$$

$$\Omega_2(X, T) = \frac{1}{\sqrt{30}} \int_{-\infty}^{\infty} dx \psi_2 [T(x - X)] \cos(\pi x) \quad (380)$$

where $2/\sqrt{3}$ and $1/\sqrt{30}$ are the normalization factors for ψ_1 and ψ_2 respectively. The wavelet integral $\Omega_i(X, T)$ are also called wavelet coefficient which measures the variation of $\cos \pi x$ in a neighborhood of X , whose size is proportional to $1/T$. The contour line representation of $\Omega_1(X, T)$ and $\Omega_2(X, T)$ are depicted in Fig. 7 and Fig. 8, respectively, where the transverse axis is X -axis (time axis), while the longitudinal axis (T -axis) is the frequency axis.

It is remarkable that although two overall shapes of the two contour lines look similar, there exist two notable differences between these two figures: (i) Along T -axis $\Omega_1(X, T)$ has one maximum, while $\Omega_2(X, T)$ has one main maximum and one subsidiary maximum (“two

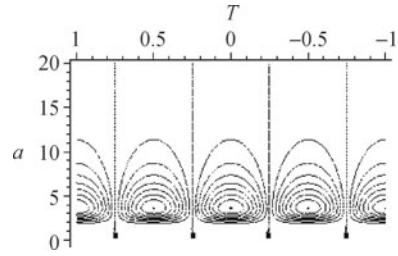


Fig. 7 Contour line representation of $\Omega_1(X, a)$.

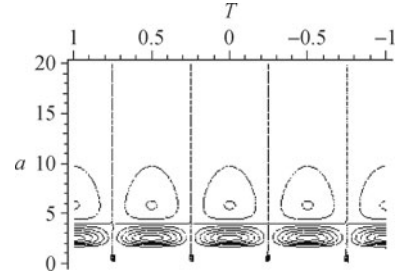


Fig. 8 Contour line representation of $\Omega_2(X, a)$.

islands”), so when ψ_2 scales its size people have one more chance to identify the frequency information of the cosine wave than using ψ_M . Interestingly enough, the “two islands” of $\Omega_2(X, T)$ in Fig. 8 can be imagined as if they were produced while the figure of $\Omega_1(X, T)$ deforms into two sub-structures along a -axis. (ii) Near the maximum of $\Omega_2(X, T)$ the density of the contour lines along a -axis is higher than that of $\Omega_1(X, T)$, which indicates that the new wavelet ψ_2 is more sensitive in detecting frequency information of the signal at this point. Therefore, $\psi_2(x)$ may be superior to $\psi_M(x)$ in analyzing some signals. Finally, we mention that there exist some remarkable qualitative similarities between the mother wavelets presented in Figs. 3 through 6 and some of the amplitude envelopes of higher order laser spatial modes and spatial supermodes of phase locked diode laser arrays [120–122], which are due to spatial coherence.

13.3 Quantum mechanical version of Parseval theorem for WT

In this subsection, we shall prove that the Parseval theorem of 1D WT [118, 119, 123]:

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds W_\psi f_1(\mu, s) W_\psi^* f_2(\mu, s) = 2C_\psi \int f_1(x) f_2^*(x) dx \quad (381)$$

where $\psi(x)$ is a mother wavelet whose Fourier transform is $\psi(p), C_\psi = 2\pi \int_0^\infty \frac{|\psi(p)|^2}{p} dp < \infty$. In the context of quantum mechanics, according to Eq. (364), we see that the quantum mechanical version of Parseval theorem should be

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds \langle \psi | U(\mu, s) | f_1 \rangle \langle f_2 | U^\dagger(\mu, s) | \psi \rangle$$

$$= 2C_\psi \langle f_2 | f_1 \rangle \tag{382}$$

and since $\psi(x) = \langle x | \psi \rangle$, $\psi(p)$ involved in C_ψ is $\langle p | \psi \rangle$, and $\langle p |$ is the momentum eigenvector

$$\psi(p) = \langle p | \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi(x) e^{-ipx} \tag{383}$$

Eq. (382) indicates that once the state vector $\langle \psi |$ corresponding to mother wavelet is known, for any two states $|f_1\rangle$ and $|f_2\rangle$, their overlap up to the factor C_ψ [determined by Eq. (396)] is just their corresponding overlap of WTs in the (μ, s) parametric space.

Proof of Equation (382): In order to show Eq. (382), we calculate

$$U^\dagger(\mu, s) |p\rangle = \frac{1}{\sqrt{|\mu|}} e^{-i\frac{ps}{\mu}} \left| \frac{p}{\mu} \right\rangle \tag{384}$$

which leads to

$$\int_{-\infty}^{\infty} ds U^\dagger(\mu, s) |p'\rangle \langle p| U(\mu, s) = 2\pi \delta(p - p') \left| \frac{p'}{\mu} \right\rangle \left\langle \frac{p}{\mu} \right| \tag{385}$$

where we have used the formula

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix(p-p')} = \delta(p - p') \tag{386}$$

Inserting the completeness relation $\int_{-\infty}^{\infty} dp |p\rangle \langle p| = 1$ into the left side of Eq. (381) and then using Eq. (385), we have

$$\begin{aligned} & \text{L.H.S of Eq. (381)} \\ &= 2\pi \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} dp \psi^*(p) \psi(p) \langle f_2 | \frac{p}{\mu} \rangle \left\langle \frac{p}{\mu} \right| f_1 \rangle \\ &\equiv I_1 + I_2 \end{aligned} \tag{387}$$

where

$$I_1 = 2\pi \int_{-\infty}^{\infty} dp \left[\int_0^{\infty} |\psi(\mu p)|^2 \frac{d\mu}{\mu} \right] \langle f_2 | p \rangle \langle p | f_1 \rangle \tag{388}$$

and

$$I_2 = 2\pi \int_{-\infty}^{\infty} dp \left[\int_0^{\infty} |\psi(-\mu p)|^2 \frac{d\mu}{\mu} \right] \langle f_2 | p \rangle \langle p | f_1 \rangle \tag{389}$$

Further, we can put Eqs. (388) and (389) into the following forms:

$$\begin{aligned} I_1 &= C_\psi \int_0^{\infty} dp \langle f_2 | p \rangle \langle p | f_1 \rangle \\ &+ C_{\psi'} \int_0^{\infty} dp \langle f_2 | p \rangle \langle p | f_1 \rangle \end{aligned} \tag{390}$$

and

$$\begin{aligned} I_2 &= C_{\psi'} \int_0^{\infty} dp \langle f_2 | p \rangle \langle p | f_1 \rangle \\ &+ C_\psi \int_{-\infty}^0 dp \langle f_2 | p \rangle \langle p | f_1 \rangle \end{aligned} \tag{391}$$

where

$$\begin{aligned} C_\psi &= 2\pi \int_0^{\infty} |\psi(\mu p)|^2 \frac{d\mu}{\mu} = 2\pi \int_0^{\infty} |\psi(p)|^2 \frac{dp}{p} \\ C_{\psi'} &= 2\pi \int_0^{-\infty} |\psi(p')|^2 \frac{dp'}{p'} = 2\pi \int_0^{\infty} |\psi(-p)|^2 \frac{dp}{p} \end{aligned} \tag{392}$$

Thus, when the definite integration satisfies the admissible condition, i.e.,

$$\int_0^{\infty} |\psi(p)|^2 \frac{dp}{p} = \int_0^{\infty} |\psi(-p)|^2 \frac{dp}{p} \tag{393}$$

which leads to

$$2\pi \int_{-\infty}^{\infty} |\psi(p)|^2 \frac{dp}{|p|} = 2C_\psi \tag{394}$$

Eq. (387) can be transformed to

$$\begin{aligned} \text{L.H.S. of Eq. (381)} &= 2C_\psi \int_{-\infty}^{\infty} dp \langle f_2 | p \rangle \langle p | f_1 \rangle \\ &= \text{R.H.S of Eq. (381)} \end{aligned} \tag{395}$$

where

$$C_\psi \equiv 2\pi \int_0^{\infty} \frac{|\psi(p)|^2}{p} dp < \infty \tag{396}$$

Thus, the theorem is proved. Especially, when $|f_1\rangle = |f_2\rangle$, Eq. (382) becomes

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds |\langle \psi | U(\mu, s) | f_1 \rangle|^2 = 2C_\psi \langle f_1 | f_1 \rangle \tag{397}$$

which is named isometry of energy.

13.4 Inversion formula of WT

Now we can directly derive the inversion formula of WT. In other words, we take $\langle f_2 | = \langle x |$ in Eq. (382), then using Eq. (364), we see that Eq. (382) is reduced to

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds W_\psi f_1(\mu, s) \langle x | U^\dagger(\mu, s) | \psi \rangle \\ &= 2C_\psi \langle x | f_1 \rangle \end{aligned} \tag{398}$$

Due to Eq. (365) we have

$$\begin{aligned} \langle x | U^\dagger(\mu, s) &= \frac{1}{\sqrt{|\mu|}} \langle x | \int_{-\infty}^{\infty} dx' |x'\rangle \left\langle \frac{x' - s}{\mu} \right| \\ &= \frac{1}{\sqrt{|\mu|}} \left\langle \frac{x - s}{\mu} \right| \end{aligned} \tag{399}$$

It then follows that

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds W_\psi f_1(\mu, s) \frac{1}{\sqrt{|\mu|}} \left\langle \frac{x - s}{\mu} \right| \psi \rangle \\ &= 2C_\psi \langle x | f_1 \rangle \end{aligned} \tag{400}$$

which means

$$f_1(x) = \frac{1}{2C_\psi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2 \sqrt{|\mu|}} \int_{-\infty}^{\infty} ds \psi \left(\frac{x-s}{\mu} \right) W_\psi f_1(\mu, s) \tag{401}$$

This is the inversion formula of WT.

13.5 New orthogonal property of mother wavelet in parameter space

Form the Parserval theorem (381) of WT in quantum mechanics we can derive some new properties of the mother wavelet [124]. Taking $|f_1\rangle = |x\rangle$, $|f_2\rangle = |x'\rangle$ in Eq. (381) one can see that

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2 |\mu|} \int_{-\infty}^{\infty} ds \psi \left(\frac{x-s}{\mu} \right) \psi^* \left(\frac{x'-s}{\mu} \right) = 2C_\psi \delta(x-x') \tag{402}$$

which is a new orthogonal property of mother wavelet in parameter space spanned by (μ, s) . In a similar way, we take $|f_1\rangle = |f_2\rangle = |n\rangle$, a number state, since $\langle n | n \rangle = 1$, then we have

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds |\langle \psi | U(\mu, s) | n \rangle|^2 = 2C_\psi \tag{403}$$

Or take $|f_1\rangle = |f_2\rangle = |z\rangle$, and $|z\rangle = \exp(-|z|^2/2 + za^\dagger) |0\rangle$ is the coherent state, then

$$\int_{-\infty}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} ds |\langle \psi | U(\mu, s) | z \rangle|^2 = 2C_\psi \tag{404}$$

This indicates that C_ψ is $|f_1\rangle$ -independent, which coincides with the expression in Eq. (396). Next, we consider a special example. When the mother wavelet is the Mexican hat (376), we have

$$\begin{aligned} \psi_M(p) &\equiv \langle p | \psi_M \rangle = \frac{1}{2} (\langle p | 0 \rangle - \sqrt{2} \langle p | 2 \rangle) \\ &= \pi^{-1/4} p^2 e^{-\frac{1}{2}p^2} \end{aligned} \tag{405}$$

where

$$\langle p | n \rangle = \frac{(-i)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{-p^2/2} H_n(p) \tag{406}$$

Here $H_n(p)$ is the single-variable Hermit polynomial. Substituting Eq. (405) into Eq. (396), we have

$$C_\psi \equiv 2\pi \int_0^\infty \frac{|\psi_M(p)|^2}{p} dp = \sqrt{\pi} \tag{407}$$

Thus, for the Mexican hat wavelet (376), we see

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2 |\mu|} \int_{-\infty}^{\infty} ds \psi_M \left(\frac{x-s}{\mu} \right) \psi_M^* \left(\frac{x'-s}{\mu} \right) \\ = 2\sqrt{\pi} \delta(x-x') \end{aligned} \tag{408}$$

Eq. (408) can be checked as follows. Using Eq. (376) and noticing that $\psi_M(x) = \psi_M(-x)$, we can put the left-hand side of Eq. (408) into

L.H.S. of Eq. (408)

$$\begin{aligned} &= 2 \int_0^\infty du \int_{-\infty}^{\infty} ds \psi_M(ux-s) \psi_M^*(ux'-s) \\ &= \begin{cases} 0, & x \neq x' \\ \frac{3}{2} \int_0^\infty du \rightarrow \infty, & x = x' \end{cases} = \text{R.H.S. of Eq. (408)} \end{aligned} \tag{409}$$

where we have used the integration formulas

$$\int_{-\infty}^{\infty} (1-s^2)^2 \exp(-s^2) ds = \frac{3}{4} \sqrt{\pi} \tag{410}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} (1-s^2) [1-(s-b)^2] e^{-s^2/2-(b-s)^2/2} ds \\ = \frac{\sqrt{\pi}}{16} e^{-\frac{b^2}{4}} [12 + b^2(b^2-12)] \end{aligned} \tag{411}$$

Next, we examine if the Morlet wavelet obey the formalism (402). The Morlet wavelet is defined as [125–127]

$$\psi_{mor}(x) = \pi^{-1/4} (e^{ifx} - e^{-f^2/2}) e^{-x^2/2} \tag{412}$$

Substituting Eq. (412) into the left hand side of (402), we get

$$\begin{aligned} I &\equiv \int_{-\infty}^{\infty} \frac{d\mu}{\mu^2 |\mu|} \int_{-\infty}^{\infty} ds \psi_{mor} \left(\frac{x-s}{\mu} \right) \psi_{mor}^* \left(\frac{x'-s}{\mu} \right) \\ &= \begin{cases} 0, & x \neq x' \\ 2(1 + e^{-f^2} - 2e^{-3f^2/4}) \int_0^\infty \frac{d\mu}{\mu^2} \rightarrow \infty, & x = x' \end{cases} \end{aligned} \tag{413}$$

Thus the Morlet wavelet satisfies Eq. (402).

13.6 WT and Wigner–Husimi distribution function

Phase space technique has been proved very useful in various branches of physics. Distribution functions in phase space have been a major topic in studying quantum mechanics and quantum statistics. Among various phase space distributions the Wigner function $F_w(q, p)$ [90–94] is the most popular, since its two marginal distributions lead to measuring probability density in coordinate space and momentum space, respectively. But the Wigner distribution function itself is not a probability distribution because it is both positive and negative. In spite of its attractive formal properties, it needs improving. To overcome this inconvenience, the Husimi distribution function $F_h(q', p')$ is introduced [128], which is defined in a manner that guarantees its nonnegative ness. Its definition is smoothing out the Wigner function by averaging over a “coarse graining” function,

$$\begin{aligned} F_h(q, p, \kappa) &= \int \int_{-\infty}^{\infty} dq' dp' F_w(q', p') \\ &\times \exp \left[-\kappa(q' - q)^2 - \frac{(p' - p)^2}{\kappa} \right] \end{aligned} \tag{414}$$

$\kappa > 0$ is the Gaussian spatial width parameter, which is free to be chosen and determines the relative resolution

in p -space versus q -space.

In the following, we shall employ the optical wavelet transformation to study the Husimi distribution function. That is, we shall show that the Husimi distribution function of a quantum state $|\psi\rangle$ can be obtained by making a WT of the Gaussian function $e^{-x^2/2}$, i.e.,

$$\langle\psi|\Delta_h(q,p,\kappa)|\psi\rangle = \frac{e^{-\frac{p^2}{\kappa}}}{\sqrt{\pi\kappa}} \left| \int_{-\infty}^{\infty} dx \psi^* \left(\frac{x-s}{\mu} \right) e^{-x^2/2} \right|^2 \tag{415}$$

where

$$s = \frac{-1}{\sqrt{\kappa}}(\kappa q + ip), \quad \mu = \sqrt{\kappa} \tag{416}$$

and $\langle\psi|\Delta_h(q,p)|\psi\rangle$ is the Husimi distribution function as well as $\Delta_h(q,p,\kappa)$ is the Husimi operator,

$$\Delta_h(q,p,\kappa) = \frac{2\sqrt{\kappa}}{1+\kappa} : \exp \left[\frac{-\kappa(q-Q)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa} \right] : \tag{417}$$

Here $::$ denotes normal ordering; Q and P are the coordinate and the momentum operator.

Proof of Eq. (415): According to Eqs. (363) and (364), when $|f\rangle$ is the vacuum state $|0\rangle$, $e^{-x^2/2} = \pi^{1/4} \langle x|0\rangle$, we see that

$$\pi^{-1/4} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\mu}} \psi^* \left(\frac{x-s}{\mu} \right) e^{-x^2/2} dx = \langle\psi|U(\mu,s)|0\rangle \tag{418}$$

From Eq. (366) it then follows that

$$U(\mu,s)|0\rangle = \text{sech}^{1/2} \lambda \exp \left[\frac{-s^2}{2(1+\mu^2)} - \frac{a^\dagger s}{\sqrt{2}} \text{sech} \lambda - \frac{a^{\dagger 2}}{2} \tanh \lambda \right] |0\rangle \tag{419}$$

Substituting Eq. (416) and $\tanh \lambda = \frac{\kappa-1}{\kappa+1}$, $\cosh \lambda = \frac{1+\kappa}{2\sqrt{\kappa}}$ into Eq. (419), we get

$$\begin{aligned} & e^{-\frac{p^2}{2\kappa} + \frac{ipq}{\kappa+1}} U(\mu = \sqrt{\kappa}, s = -\sqrt{\kappa}q - ip/\sqrt{\kappa}) |0\rangle \\ &= \left(\frac{2\sqrt{\kappa}}{1+\kappa} \right)^{1/2} \exp \left[\frac{-\kappa q^2}{2(1+\kappa)} - \frac{p^2}{2(1+\kappa)} + \frac{\sqrt{2}a^\dagger}{1+\kappa}(\kappa q + ip) + \frac{1-\kappa}{2(1+\kappa)} a^{\dagger 2} \right] |0\rangle \equiv |p,q\rangle_{\kappa} \end{aligned} \tag{420}$$

Then the WT of Eq. (418) can be further expressed as

$$e^{-\frac{p^2}{2\kappa} + \frac{ipq}{\kappa+1}} \int_{-\infty}^{\infty} \frac{dx}{(\kappa\pi)^{1/4}} \psi^* \left(\frac{x-s}{\mu} \right) e^{-x^2/2} = \langle\psi|p,q\rangle_{\kappa} \tag{421}$$

Using the normally ordered form of the vacuum state projector $|0\rangle\langle 0| = e^{-a^\dagger a}$, and the IWOP method as well as Eq. (420), we have

$$|p,q\rangle_{\kappa\kappa} \langle p,q| = \frac{2\sqrt{\kappa}}{1+\kappa} : \exp \left[\frac{-\kappa(q-Q)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa} \right] :$$

$$:= \Delta_h(q,p,\kappa) \tag{422}$$

Now we explain why $\Delta_h(q,p,\kappa)$ is the Husimi operator. Use the formula for converting an operator A into its Weyl ordering form [129]

$$\begin{aligned} A &= 2 \int \frac{d^2\beta}{\pi} \langle -\beta|A|\beta\rangle : \exp[2(\beta^* a - a^\dagger \beta + a^\dagger a)] : \\ d^2\beta &= d\beta_1 d\beta_2, \quad \beta = \beta_1 + i\beta_2 \end{aligned} \tag{423}$$

where the symbol $::$ denotes the Weyl ordering, and $|\beta\rangle$ is the usual coherent state. Substitute Eq. (422) into Eq. (423) and perform the integration by virtue of the technique of integration within a Weyl ordered product of operators. Then we obtain

$$|p,q\rangle_{\kappa\kappa} \langle p,q| = 2 : \exp \left[-\kappa(q-Q)^2 - \frac{(p-P)^2}{\kappa} \right] : \tag{424}$$

This is the Weyl ordering form of $|p,q\rangle_{\kappa\kappa} \langle p,q|$. Then according to Weyl quantization scheme [35] we know the classical corresponding function of a Weyl ordered operator is obtained by just replacing $Q \rightarrow q', P \rightarrow p'$,

$$\begin{aligned} & : \exp \left[-\kappa(q-Q)^2 - \frac{(p-P)^2}{\kappa} \right] : \\ & \rightarrow \exp \left[-\kappa(q-q')^2 - \frac{(p-p')^2}{\kappa} \right] \end{aligned} \tag{425}$$

and in this case the Weyl rule is expressed as

$$\begin{aligned} |p,q\rangle_{\kappa\kappa} \langle p,q| &= 2 \int dq' dp' \Delta_w(q',p') \\ & \times \exp \left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa} \right] \end{aligned} \tag{426}$$

where at the last step we used the Weyl ordering form of the Wigner operator $\Delta_w(q,p)$ [130]

$$\Delta_w(q,p) = : \delta(q-Q)\delta(p-P) : \tag{427}$$

In reference to Eq. (414) in which the relation between the Husimi function and the WF is shown, we know that the right-hand side of Eq. (426) should be just the Husimi operator, i.e.,

$$|p,q\rangle_{\kappa\kappa} \langle p,q| = \Delta_h(q,p,\kappa) \tag{428}$$

Thus Eq. (415) is proved by combining Eqs. (428) and (421).

Thus the optical WT can be used to study the Husimi distribution function in the quantum optics phase space theory [131].

14 Symplectic wavelet transformation (SWT)

In this section we shall generalize the usual wavelet trans-

form to symplectic wavelet transformation (SWT) by using the coherent state representation [132].

14.1 Single-mode SWT

First we are motivated to generalize the usual wavelet transform, which concerns dilation, to optical Fresnel transform (we will explain this in detail in section below). In other words, we shall use the symplectic-transformed — translated versions of the mother wavelet

$$\psi_{r,s;\kappa}(z) = \sqrt{s^*} \psi [s(z - \kappa) - r(z^* - \kappa^*)] \quad (429)$$

as a weighting function to synthesize the original complex signal $f(z)$,

$$W_\psi f(r, s; \kappa) = \int \frac{d^2z}{\pi} f(z) \psi_{r,s;\kappa}^*(z) \quad (430)$$

$$d^2z = dx dy, \quad z = x + iy$$

This is named the symplectic-transformed — translated wavelet transform. One can see that the mother wavelet ψ generates the other wavelets of the family $\psi^* [s(z - \kappa) - r(z^* - \kappa^*)]$ through a translating transform followed by a symplectic transform. (r, s are the symplectic transform parameter, $|s|^2 - |r|^2 = 1, \kappa$ is a translation parameter, s, r and $\kappa \in \mathbb{C}$). This will be seen more clearly if we write the second transform in a matrix form

$$\begin{pmatrix} z - \kappa \\ z^* - \kappa^* \end{pmatrix} \rightarrow M \begin{pmatrix} z - \kappa \\ z^* - \kappa^* \end{pmatrix}$$

$$M \equiv \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \quad (431)$$

where M is a symplectic matrix satisfying $M^T J M = J$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Symplectic matrices in Hamiltonian dynamics correspond to canonical transformations and keep the Poisson bracket invariant, while in matrix optics they represent ray transfer matrices of optical instruments, such as lenses and fibers.

14.1.1 Properties of symplectic-transformed — translated WT

It is straightforward to evaluate this transform and its reciprocal transform when $f(z)$ is the complex Fourier exponentials, $f(z) = \exp(z\beta^* - z^*\beta)$. (Note that $z\beta^* - z^*\beta$ is pure imaginary)

$$W_\psi f = \sqrt{s} \int \frac{d^2z}{\pi} \exp[(z + \kappa)\beta^* - (z^* + \kappa^*)\beta] \psi^*(sz - rz^*)$$

$$= \exp(\kappa\beta^* - \kappa^*\beta) \sqrt{s} \int \frac{d^2z'}{\pi}$$

$$\times \exp [z'(s^*\beta^* - r^*\beta) - z'^*(s\beta - r\beta^*)] \psi^*(z')$$

$$= \sqrt{s} \exp(\kappa\beta^* - \kappa^*\beta) \Phi(s^*\beta^* - r^*\beta) \quad (432)$$

where Φ is the complex Fourier transform of ψ^* . Then we form the adjoint operation

$$W_\psi^*(W_\psi f)(z) = \sqrt{s^*} \int \frac{d^2\kappa}{\pi} (W_\psi f)(r, s; \kappa)$$

$$\times \psi [s(z - \kappa) - r(z^* - \kappa^*)]$$

$$= |s| \Phi(s^*\beta^* - r^*\beta) \int \frac{d^2\kappa}{\pi} \exp[(\kappa + z)\beta^* - (\kappa^* + z^*)\beta] \psi [-s\kappa + r\kappa^*]$$

$$= |s| \exp(z\beta^* - z^*\beta) |\Phi(s^*\beta^* - r^*\beta)|^2 \quad (433)$$

from which we have

$$\int \frac{W_\psi^*(W_\psi f)(z) d^2s}{|s|^2}$$

$$= \exp(z\beta^* - z^*\beta) \int d^2s \frac{|\Phi(s^*\beta^* - r^*\beta)|^2}{|s|} \quad (434)$$

so we get the inversion formula

$$f(z) = e^{z\beta^* - z^*\beta} = \frac{\int d^2s W_\psi^*(W_\psi f)(z) / |s|^2}{\int d^2s |\Phi(s^*\beta^* - r^*\beta)|^2 / |s|} \quad (435)$$

Eq. (435) leads us to impose the normalization

$$\int d^2s |\Phi(s^*\beta^* - r^*\beta)|^2 / |s| = 1 \quad (436)$$

in order to get the wavelet representation

$$f(z) = \int d^2s W_\psi^*(W_\psi f)(z) / |s|^2 \quad (437)$$

Then we can have a form of Parseval's theorem for this new wavelet transform:

Proposition: For any f and f' we have

$$\iint W_\psi f(r, s; \kappa) W_\psi f'^*(r, s; \kappa) \frac{d^2\kappa d^2s}{|s|^2}$$

$$= \int \frac{d^2z}{2\pi} f(z) f'^*(z) \quad (438)$$

Proof: Let us assume $F(\beta)$ and $F'(\beta)$ are the complex Fourier transform of $f(z)$ and $f'(z)$ respectively,

$$F(\beta) = \int \frac{d^2z}{2\pi} f(z) e^{z\beta^* - z^*\beta} \quad (439)$$

Recall the convolution theorem defined on complex Fourier transform,

$$\int d^2z f(\alpha - z, \alpha^* - z^*) f'(z)$$

$$= \int d^2z \int \frac{d^2\beta}{2\pi} F(\beta) e^{(\alpha^* - z^*)\beta - (\alpha - z)\beta^*}$$

$$\times \int \frac{d^2\beta'}{2\pi} F'(\beta') e^{z^*\beta' - z\beta'^*}$$

$$= \int d^2\beta F(\beta)F'(\beta)e^{\alpha^*\beta - \alpha\beta^*} \quad (440)$$

Hence from Eqs. (440) and (429), as well as Eq. (430) we see that $W_\psi f(r, s; \kappa) = \int \frac{d^2z}{\pi} f(z)\psi_{r,s;\kappa}^*(z)$ can be considered as a convolution in the form

$$\begin{aligned} & \int d^2z f(z)\psi^*[s(z - \kappa) - r(z^* - \kappa^*)] \\ &= \int d^2\beta F(\beta)\Phi^*(s\beta - r\beta^*) \exp(\kappa\beta^* - \kappa^*\beta) \end{aligned} \quad (441)$$

It then follows from Eq. (440) that

$$\begin{aligned} & \int W_\psi f(r, s; \kappa)W_\psi f'^*(r, s; \kappa)d^2\kappa \\ &= |s| \int \frac{d^2\beta}{2\pi} F(\beta)F'^*(\beta) |\Phi(s\beta - r\beta^*)|^2 \end{aligned} \quad (442)$$

Therefore, using Eq. (436), we see that the further integration yields

$$\begin{aligned} & \int \frac{d^2s}{|s|^2} \int W_\psi f(r, s; \kappa)W_\psi f'^*(r, s; \kappa)d^2\kappa \\ &= \int \frac{d^2\beta}{2\pi} F(\beta)F'^*(\beta) = \int \frac{d^2z}{2\pi} f(z)f'^*(z) \end{aligned} \quad (443)$$

which completes the proof.

Theorem: From the Proposition (438) we have

$$\int \int W_\psi f(r, s; \kappa)\psi_{r,s;\kappa}(z) \frac{d^2\kappa d^2s}{|s|^2} = f(z) \quad (444)$$

that is, there exists an inversion formula for arbitrary function $f(z)$. In fact, in Eq. (430) when we take $f(z) = \delta(z - z')$, then

$$W_\psi f(r, s; \kappa) = \int \frac{d^2z}{\pi} f(z)\psi_{r,s;\kappa}^*(z) = \psi_{r,s;\kappa}^*(z') \quad (445)$$

Substituting Eq. (445) into Eq. (443) we obtain Eq. (444).

14.1.2 Relation between $W_\psi f(r, s; \kappa)$ and optical Fresnel transform

Now we explain why the idea of $W_\psi f(r, s; \kappa)$ is originated from the optical Fresnel transform. We can visualize the symplectic-transformed — translated wavelet transform in the context of quantum mechanics. Let $f(z) \equiv \langle z|f\rangle$, $|z\rangle$ is the coherent state, $|z\rangle = \exp[za^\dagger - z^*a] \equiv | \begin{pmatrix} z \\ z^* \end{pmatrix} \rangle$, $|0\rangle$ is the vacuum state in Fock space, then Eq. (429) can be expressed as

$$\begin{aligned} & W_\psi f(r, s; \kappa) \\ &= \sqrt{s} \int \frac{d^2z}{\pi} \langle \psi | \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \begin{pmatrix} z - \kappa \\ z^* - \kappa^* \end{pmatrix} \rangle \langle z|f\rangle \\ &= \langle \psi | F_1(r, s, \kappa) |f\rangle \end{aligned} \quad (446)$$

where $F^{(r,s,\kappa)}$ is defined as

$$F_1(r, s, \kappa) = \sqrt{s} \int \frac{d^2z}{\pi} |sz - rz^*\rangle \langle z + \kappa| \quad (447)$$

and $|sz - rz^*\rangle \equiv | \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \rangle$. Employing the normal ordering of the vacuum projector $|0\rangle\langle 0| = : \exp(-a^\dagger a) :$ and the IWOP technique to perform the integration in Eq. (447), one can get the explicit form of $F_1(r, s, \kappa)$. The transformation matrix element of $F_1(r, s, \kappa = 0)$ in the coordinate representation $|x\rangle$ is just the kernel of optical diffraction integration (86) (Fresnel transform), which explains our motivation to introduce $W_\psi f(r, s; \kappa)$. In particular, when $r^* = r \equiv \sinh \lambda$, $s = s^* \equiv \cosh \lambda$, $F_1(r, s, \kappa = 0)$ reduces to the well-known single-mode squeezing operator $\exp[\frac{\lambda}{2}(a^2 - a^{\dagger 2})]$, which corresponds to dilation in the usual WT.

14.2 Entangled SWT

In the above subsection, the mother wavelet is gained through a translating transform followed by a symplectic transform. This motivation arises from the consideration that symplectic transforms are more general than the dilated transform, and are useful in Fresnel transform of Fourier optics, e.g., ray transfer matrices of optical instruments, such as lenses and fibers in matrix optics, while in quantum optics symplectic transforms correspond to single-mode Fresnel operator (or generalized $SU(1,1)$ squeezing operator).

Recalling that in Section 9 we have introduced the 2-mode entangled Fresnel operator which is a mapping of classical mixed transformation $(z, z') \rightarrow (sz + rz'^*, sz' + rz^*)$ in a two-mode coherent state $|z, z'\rangle$ representation onto quantum operator $F_2(r, s)$, we are naturally led to develop the SWT in Eq. (430) to the so-called entangled SWT (ESWT) [133] for signals $g(z, z')$ defined in two complex planes,

$$W_\phi g(r, s; k, k') = \iint \frac{d^2z d^2z'}{\pi^2} g(z, z') \phi_{r,s;k,k'}^*(z, z') \quad (448)$$

where

$$\begin{aligned} \phi_{r,s;k,k'}(z, z') &\equiv s^* \phi[s(z - k) + r(z'^* - k'^*) \\ &\quad \times s(z' - k') + r(z^* - k^*)] \end{aligned} \quad (449)$$

is used as a weighting function to synthesize the signal $g(z, z')$ regarding two complex planes. One can see that the mother wavelet ϕ generates the family $\phi^*[s(z - k) + r(z'^* - k'^*), s(z' - k') + r(z^* - k^*)]$ through a translating transform followed by an entangled symplectic transform. We emphasize that this transform mixes the two complex planes, which is different from the tensor product of two independent transforms $(z, z') \rightarrow [s(z - k) - r(z^* - k^*), s(z' - k') - r(z'^* - k'^*)]$ given by Eq. (446). The new symplectic transform can be seen

more clearly if we write it in a matrix form:

$$\begin{pmatrix} z - k \\ z^* - k^* \\ z' - k' \\ z'^* - k'^* \end{pmatrix} \rightarrow \mathcal{M} \begin{pmatrix} z - k \\ z^* - k^* \\ z' - k' \\ z'^* - k'^* \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} s & 0 & 0 & r \\ 0 & s^* & r^* & 0 \\ 0 & r & s & 0 \\ r^* & 0 & 0 & s^* \end{pmatrix} \quad (450)$$

where \mathcal{M} is symplectic satisfying $\mathcal{M}^T \mathcal{J} \mathcal{M} = \mathcal{J}$, $\mathcal{J} = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}$, \mathcal{I} is the 2×2 unit matrix.

As Eq. (448) has been qualified as a new wavelet transform we must prove that it possesses fundamental properties of the usual wavelet transforms, such as the *admissibility condition*, the Parseval's theorem and the inversion formula. When $g(z, z')$ is the complex Fourier exponential,

$$g_1(z, z') = \exp(z\beta^* - z^*\beta + z'\gamma^* - z'^*\gamma) \quad (451)$$

According to Eqs. (448) and (449) we evaluate its ESWT

$$W_{\phi} g_1 = s \iint \frac{d^2 z d^2 z'}{\pi^2} \phi^* [sz + rz'^*, sz' + rz^*] \times \exp[(z + k)\beta^* - (z^* + k^*)\beta + (z' + k')\gamma^* - (z'^* + k'^*)\gamma] \quad (452)$$

Making the integration variables transform $sz + rz'^* \rightarrow w$, $sz' + rz^* \rightarrow w'$, Eq. (452) becomes

$$W_{\phi} g_1 = s \exp(k\beta^* - k^*\beta + k'\gamma^* - k'^*\gamma) \times \iint \frac{d^2 w d^2 w'}{\pi^2} \phi^*(w, w') \times \exp[w(s^*\beta^* + r^*\gamma) - w^*(s\beta + r\gamma^*) + w'(s^*\gamma^* + r^*\beta) - w'^*(s\gamma + r\beta^*)] \quad (453)$$

The last integration is just the complex Fourier transform (CFT) of ϕ^* . Denoting it as Φ^* , we have

$$W_{\phi} g_1 = s \exp(k\beta^* - k^*\beta + k'\gamma^* - k'^*\gamma) \times \Phi^*(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta) \quad (454)$$

Then we form the adjoint operation of Eq. (454),

$$W_{\phi}^*(W_{\phi} g_1)(z, z') = s^* \iint \frac{d^2 k d^2 k'}{\pi^2} [(W_{\phi} g)(r, s; k, k')] \phi[s(z - k) + r(z'^* - k'^*), s(z' - k') + r(z^* - k^*)] = |s|^2 \Phi^*(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta) \times \exp(z\beta^* - z^*\beta + z'\gamma^* - z'^*\gamma) \iint \frac{d^2 v d^2 v'}{\pi^2} \phi(v, v') \times \exp[-v(s^*\beta^* + r^*\gamma) + v^*(s\beta + r\gamma^*)$$

$$-v'(s^*\gamma^* + r^*\beta) + v'^*(s\gamma + r\beta^*)] \quad (455)$$

where the integration in the last line is just the CFT of ϕ [comparing with Eq. (453)], thus Eq. (455) leads to

$$W_{\phi}^*(W_{\phi} g_1)(z, z') = |s|^2 \exp(z\beta^* - z^*\beta + z'\gamma^* - z'^*\gamma) \cdot |\Phi(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta)|^2 \quad (456)$$

From Eq. (456) we have

$$\int d^2 s W_{\phi}^*(W_{\phi} g_1)(z, z') / |s|^4 = \exp(z\beta^* - z^*\beta + z'\gamma^* - z'^*\gamma) \times \int d^2 s |\Phi(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta)|^2 / |s|^2 \quad (457)$$

which together with Eq. (451) lead to

$$g_1(z, z') = \frac{\int d^2 s W_{\phi}^*(W_{\phi} g_1)(z, z') / |s|^4}{\int d^2 s |\Phi(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta)|^2 / |s|^2} \quad (458)$$

Eq. (458) implies that we should impose the normalization

$$\int d^2 s |\Phi(s^*\beta^* + r^*\gamma, s^*\gamma^* + r^*\beta)|^2 / |s|^2 = 1 \quad (459)$$

such that the reproducing process exists

$$g_1(z, z') = \int d^2 s W_{\phi}^*(W_{\phi} g_1)(z, z') / |s|^4 \quad (460)$$

Eq. (459) may be named the generalized admissibility condition. Now we have the corresponding *Parseval theorem*: For any g and g' , we have

$$\iiint W_{\phi} g(r, s; k, k') W_{\phi}^* g'(r, s; k, k') \frac{d^2 k d^2 k' d^2 s}{|s|^4} = \iint d^2 z d^2 z' g(z, z') g'^*(z, z') \quad (461)$$

Proof: Assuming $F(\beta, \gamma)$ and $F'(\beta, \gamma)$ to be CFT of $g(z, z')$ and $g'(z, z')$, respectively,

$$F(\beta, \gamma) = \iint \frac{d^2 z d^2 z'}{\pi^2} g(z, z') \times \exp(z\beta^* - z^*\beta + z'\gamma^* - z'^*\gamma) \quad (462)$$

Recall the corresponding convolution theorem

$$\iint d^2 z d^2 z' g(\alpha - z, \alpha^* - z^*; \alpha' - z', \alpha'^* - z'^*) g'(z, z') = \iint d^2 \beta d^2 \gamma F(\beta, \gamma) F'(\beta, \gamma) \times \exp(\alpha^*\beta - \alpha\beta^* + \alpha'\gamma - \alpha'\gamma^*) \quad (463)$$

and from Eqs. (448), (462) and (463), we see that $W_{\phi} g(r, s; k, k') = \iint \frac{d^2 z d^2 z'}{\pi^2} g(z, z') \phi_{r, s; k, k'}^*(z, z')$ can be considered as a convolution in the form [Noting that the CFT of ϕ^* is Φ^* , see Eqs. (453) and (454)]:

$$\begin{aligned} & \iint d^2z d^2z' g(z, z') \phi^* [s(z-k) + r(z'^* - k'^*) \\ & s(z' - k') + r(z^* - k^*)] \\ &= \iint d^2\beta d^2\gamma F(\beta, \gamma) \Phi^*(s^* \beta^* + r^* \gamma, s^* \gamma^* + r^* \beta) \\ & \times \exp(k\beta^* - k^* \beta + k' \gamma^* - k'^* \gamma) \end{aligned} \quad (464)$$

Using Eq. (464) we calculate

$$\begin{aligned} & \iint W_\phi g(r, s; k, k') W_\phi^* g'(r, s; k, k') d^2k d^2k' \\ &= |s|^2 \iiint d^2\beta d^2\gamma d^2\beta' d^2\gamma' F(\beta, \gamma) \\ & \times \Phi^*(s^* \beta^* + r^* \gamma, s^* \gamma^* + r^* \beta) \\ & \times F'^*(\beta', \gamma') \Phi(s^* \beta^* + r^* \gamma, s^* \gamma^* + r^* \beta) \\ & \times \delta(\beta - \beta') \delta(\beta^* - \beta'^*) \delta(\gamma - \gamma') \delta(\gamma^* - \gamma'^*) \\ &= |s|^2 \iint d^2\beta d^2\gamma F(\beta, \gamma) F'^*(\beta, \gamma) \\ & \times |\Phi(s^* \beta^* + r^* \gamma, s^* \gamma^* + r^* \beta)|^2 \end{aligned} \quad (465)$$

As a consequence of Eqs. (459) and (465) the further integration yields

$$\begin{aligned} & \iint \frac{d^2s}{|s|^4} \iint W_\phi g(r, s; k, k') W_\phi^* g'(r, s; k, k') d^2k d^2k' \\ &= \iint d^2\beta d^2\gamma F(\beta, \gamma) F'^*(\beta, \gamma) \\ & \times \int d^2s |\Phi(s^* \beta^* + r^* \gamma, s^* \gamma^* + r^* \beta)|^2 / |s|^2 \\ &= \iint d^2\beta d^2\gamma F(\beta, \gamma) F'^*(\beta, \gamma) \\ &= \iint d^2z d^2z' g(z, z') g'^*(z, z') \end{aligned} \quad (466)$$

which completes the proof.

Inversion formula: From Eq. (461) we have

$$g(z, z') = \iint \iint W_\phi g(r, s; k, k') \phi_{r,s;k,k'}(z, z') \frac{d^2k d^2k' d^2s}{\pi^2 |s|^4} \quad (467)$$

That is, there exists an inversion formula for $g(z, z')$ which represents the original signal $g(z, z')$ as a superposition of wavelet functions $\phi_{r,s;k,k'}$, with the value of entangled wavelet transform $W_\phi g(r, s; k, k')$ serving as coefficients. In fact, in Eq. (448) when we take

$$g(z, z') = \delta(z-u) \delta(z^* - u^*) \delta(z' - u') \delta(z'^* - u'^*) \quad (468)$$

then

$$W_\phi g(r, s; k, k') = \frac{1}{\pi^2} \phi_{r,s;k,k'}^*(u, u') \quad (469)$$

Substituting Eqs. (468) and (469) into Eq. (466), we obtain Eq. (467). We can visualize the ESWT in the context of quantum mechanics. Let $g(z, z') = \langle z, z' | g \rangle$ and use Eqs. (449) and (450), Eq. (448) is expressed as

$$\begin{aligned} W_\phi g(r, s; k, k') &= s \iint \frac{d^2z d^2z'}{\pi^2} \phi^* [s(z-k) + r(z'^* - k'^*), \\ & s(z' - k') + r(z^* - k^*)] g(z, z') \\ &= s \iint \frac{d^2z d^2z'}{\pi^2} \langle \phi | \mathcal{M} \begin{pmatrix} z - k \\ z^* - k^* \\ z' - k' \\ z'^* - k'^* \end{pmatrix} \rangle \\ & \times \langle z, z' | g \rangle = \langle \phi | F_2(r, s; k, k') | g \rangle \end{aligned} \quad (470)$$

where $F_2(r, s; k, k')$ is defined as

$$\begin{aligned} F_2(r, s; k, k') &= s \iint \frac{d^2z d^2z'}{\pi^2} |sz + rz'^*, sz' + rz^* \rangle \\ & \times \langle z + k, z' + k' | |sz + rz'^*, sz' + rz^* \rangle \\ &= |sz + rz'^* \rangle_1 \otimes |sz' + rz^* \rangle_2 \end{aligned} \quad (471)$$

When $k = 0$ and $k' = 0$, $F_2(r, s; k = k' = 0)$ is just the 2-mode Fresnel operator.

Thus, we have extended the SWT of signals in one complex plane to ESWT of signals defined in two complex planes. The latter is not the tensor product of two independent SWTs. This generalization is inevitable, since it resembles the extending from the single-mode squeezing transform (or Fresnel operator) to the two-mode squeezing transform (or entangled Fresnel operator) in quantum optics.

14.3 Symplectic-dilation mixed WT

Next we shall introduce a new kind of WT, i.e., symplectic-dilation mixed WT [134]. Recalling Ref. [135] we have constructed a new entangled-coherent state representation (ECSR) $|\alpha, x\rangle$,

$$\begin{aligned} |\alpha, x\rangle &= \exp \left[-\frac{1}{2}x^2 - \frac{1}{4}|\alpha|^2 + (x + \frac{\alpha}{2})a_1^\dagger \right. \\ & \left. + (x - \frac{\alpha}{2})a_2^\dagger - \frac{1}{4}(a_1^\dagger + a_2^\dagger)^2 \right] |00\rangle \end{aligned} \quad (472)$$

which is the common eigenvector of the operator $(X_1 + X_2)/2$ and $a_1 - a_2$, i.e., $(a_1 - a_2)|\alpha, x\rangle = \alpha|\alpha, x\rangle$ and $\frac{1}{2}(X_1 + X_2)|\alpha, x\rangle = \frac{1}{\sqrt{2}}x|\alpha, x\rangle$, where $X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger)$ is the coordinate operator, ($i = 1, 2$), and $|\alpha, x\rangle$ is complete,

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} |\alpha, x\rangle \langle \alpha, x| = 1 \quad (473)$$

and exhibits partly non-orthogonal property (for α) and orthonormal property (for x),

$$\begin{aligned} \langle \alpha', x' | \alpha, x \rangle &= \sqrt{\pi} \exp \left[-\frac{1}{4}(|\alpha|^2 + |\alpha'|^2) + \frac{1}{2}\alpha\alpha'^* \right] \\ & \times \delta(x' - x) \end{aligned} \quad (474)$$

Therefore, $|\alpha, x\rangle$ possess the behaviors of both the coher-

ent state and the entangled state. An interesting question is: Can we introduce a new kind of continuous WT which the $|\alpha, x\rangle$ representation underlies? The answer is affirmative. Our motivation of this issue comes from the mixed lens-Fresnel transform in classical optics [136].

By synthesizing Eqs. (363) and (430) and in reference to Eq. (473), we propose the mixed WT for $g(\alpha, x)$ ($\alpha = \alpha_1 + i\alpha_2$):

$$W_\psi g(s, r, \kappa; a, b) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} g(\alpha, x) \psi_{s,r,\kappa}^*(\alpha, x) \quad (475)$$

where $d^2\alpha = d\alpha_1 d\alpha_2$. The family of mother wavelet ψ involves both the the symplectic transform of α and the dilation-transform of x ,

$$\psi_{s,r,\kappa;a,b}(\alpha, x) = \sqrt{\frac{s^*}{|a|}} \psi \left[s(\alpha - \kappa) - r(\alpha^* - \kappa^*), \frac{x - b}{a} \right] \quad (476)$$

Let $g(\alpha, x) \equiv \langle \alpha, x | g \rangle$, then Eq. (475) can be expressed as the quantum mechanical version

$$W_\psi g(s, r, \kappa; a, b) = \langle \psi | U(s, r, \kappa; a, b) | g \rangle \quad (477)$$

where $U(s, r, \kappa; a, b)$ is defined as

$$U(s, r, \kappa; a, b) = \sqrt{\frac{s}{|a|}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} |\alpha + \kappa, x\rangle \langle s\alpha - r\alpha^*, \frac{x - b}{a}| \quad (478)$$

$U(s, r, \kappa = 0; a, b = 0)$ is just the generalized squeezing operator, which causes a lens-Fresnel mixed transform.

As Eq. (475) has been qualified as a new WT we must prove that it possesses some fundamental properties of the usual WTs, such as the admissibility condition, the Parseval theorem and the inversion formula. It is straightforward to evaluate the transform (475) and its reciprocal transform when $g(\alpha, x)$ is the exponential $g_1(\alpha, x) = \exp(\alpha^* \beta - \alpha \beta^* - ipx)$,

$$W_\psi g_1 = \sqrt{\frac{s}{|a|}} e^{\kappa^* \beta - \kappa \beta^* - ipb} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} \times \psi^* \left(s\alpha - r\alpha^*, \frac{x}{a} \right) e^{\alpha^* \beta - \alpha \beta^* - ipx} \quad (479)$$

Making the integration variables transform $s\alpha - r\alpha^* \rightarrow w, \frac{x}{a} \rightarrow x'$, which leads to $d^2\alpha \rightarrow d^2w$ and $\int_{-\infty}^{\infty} dx \rightarrow |a| \int_{-\infty}^{\infty} dx'$, Eq. (479) becomes

$$W_\psi g_1 = \sqrt{s|a|} \Phi^*(s^* \beta^* - r^* \beta, ap) e^{\kappa^* \beta - \kappa \beta^* - ipb} \quad (480)$$

where Φ^* is just the Fourier transform of ψ^* ,

$$\Phi^*(s^* \beta^* - r^* \beta, ap) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{\pi}} \int \frac{d^2w}{2\pi} \psi^*(w, x') \times e^{w^*(s\beta - r\beta^*) - w(s^* \beta^* - r^* \beta) - iapx'} \quad (481)$$

Then we perform the adjoint WT of Eq. (475), using Eqs. (476) and (480) we see

$$\begin{aligned} W_\psi^*(W_\psi g_1)(\alpha, x) &= \sqrt{\frac{s^*}{|a|}} \int_{-\infty}^{\infty} \frac{db}{\sqrt{\pi}} \int \frac{d^2\kappa}{2\pi} W_\psi g_1 \\ &\times \psi \left[s(\alpha - \kappa) - r(\alpha^* - \kappa^*), \frac{x - b}{a} \right] \\ &= |s| |a| g_1(\alpha, x) \Phi^*(s^* \beta^* - r^* \beta, ap) \int_{-\infty}^{\infty} \frac{db'}{\sqrt{\pi}} \\ &\times \int \frac{d^2\kappa'}{2\pi} e^{\kappa' \beta^* - \kappa'^* \beta + iapb'} \psi(s\kappa' - r\kappa'^*, b') \\ &= |s| |a| g_1(\alpha, x) |\Phi(s^* \beta^* - r^* \beta, ap)|^2 \end{aligned} \quad (482)$$

From Eq. (482) we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{da}{a^2} \int \frac{d^2s}{|s|^2} W_\psi^*(W_\psi g_1)(\alpha, x) \\ = g_1(\alpha, x) \int_{-\infty}^{\infty} \frac{da}{|a|} \int \frac{d^2s}{|s|} |\Phi(s^* \beta^* - r^* \beta, ap)|^2 \end{aligned} \quad (483)$$

which leads to

$$g_1(\alpha, x) = \frac{\int_{-\infty}^{\infty} \frac{da}{a^2} \int \frac{d^2s}{|s|^2} W_\psi^*(W_\psi g_1)(\alpha, x)}{\int_{-\infty}^{\infty} \frac{da}{|a|} \int \frac{d^2s}{|s|} |\Phi(s^* \beta^* - r^* \beta, ap)|^2} \quad (484)$$

Eq. (484) implies that we should impose the normalization

$$\int_{-\infty}^{\infty} \frac{da}{|a|} \int \frac{d^2s}{|s|} |\Phi(s^* \beta^* - r^* \beta, ap)|^2 = 1 \quad (485)$$

such that the reproducing process exists

$$g_1(\alpha, x) = \int_{-\infty}^{\infty} \frac{da}{a^2} \int \frac{d^2s}{|s|^2} W_\psi^*(W_\psi g_1)(\alpha, x) \quad (486)$$

Eq. (485) may be named the generalized *admissibility condition*. Now we have the corresponding *Parseval theorem*: For any g and g' , we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dadb}{a^2} \int \frac{d^2\kappa d^2s}{|s|^2} W_\psi g(s, r, \kappa; a, b) W_\psi^* g'(s, r, \kappa; a, b) \\ = \int_{-\infty}^{\infty} dx \int d^2\alpha g(\alpha, x) g'^*(\alpha, x) \end{aligned} \quad (487)$$

Proof: Assuming $F(\beta, p)$ and $F'(\beta, p)$ to be the Fourier transforms of $g(\alpha, x)$ and $g'(\alpha, x)$, respectively,

$$F(\beta, p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \int \frac{d^2\alpha}{\pi} g(\alpha, x) e^{\alpha \beta^* - \alpha^* \beta + ipx} \quad (488)$$

In order to prove Eq. (487), we first calculate $W_\psi g(s, r, \kappa; a, b)$. In a similar way of deriving Eq. (175), using Eqs. (170), (171) and the inversion formula of Eq. (488), we have

$$\begin{aligned} W_\psi g(s, r, \kappa; a, b) \\ = \sqrt{\frac{s}{|a|}} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \int \frac{d^2\beta}{\pi} F(\beta, p) \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} \end{aligned}$$

$$\begin{aligned}
& \times e^{\alpha^* \beta - \alpha \beta^* - ipx} \psi^* \left[s(\alpha - \kappa) - r(\alpha^* - \kappa^*), \frac{x - b}{a} \right] \\
& = \sqrt{s|a|} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \int \frac{d^2\beta}{\pi} F(\beta, p) \\
& \times \Phi^*(s^* \beta^* - r^* \beta, ap) e^{\kappa^* \beta - \kappa \beta^* - ipb} \quad (489)
\end{aligned}$$

It then follows that

$$\begin{aligned}
& \int_{-\infty}^{\infty} db \int d^2\kappa W_{\psi} g(s, r, \kappa; a, b) W_{\psi}^* g'(s, r, \kappa; a, b) \\
& = |as| \int_{-\infty}^{\infty} dp dp' \int d^2\beta d^2\beta' F(\beta, p) F'^*(\beta', p') \\
& \times \Phi^*(s^* \beta^* - r^* \beta, ap) \Phi(s^* \beta'^* - r^* \beta', ap') \\
& \times \int_{-\infty}^{\infty} \frac{db}{2\pi} \int \frac{d^2\kappa}{\pi^2} e^{\kappa^*(\beta - \beta') - \kappa(\beta^* - \beta'^*) + i(p' - p)b} \\
& = |as| \int_{-\infty}^{\infty} dp \int d^2\beta F(\beta, p) F'^*(\beta, p) \\
& \times |\Phi(s^* \beta^* - r^* \beta, ap)|^2 \quad (490)
\end{aligned}$$

Substituting Eq. (490) into the left-hand side (L.H.S) of Eq. (487) and using Eq. (485), we see

$$\begin{aligned}
\text{L.H.S of Eq. (487)} & = \int_{-\infty}^{\infty} dp \int d^2\beta F(\beta, p) F'^*(\beta, p) \\
& \times \int_{-\infty}^{\infty} \frac{da}{|a|} \int \frac{d^2s}{|s|} |\Phi^*(s^* \beta^* - r^* \beta, ap)|^2 \\
& = \int_{-\infty}^{\infty} dp \int d^2\beta F(\beta, p) F'^*(\beta, p) \quad (491)
\end{aligned}$$

Thus we complete the proof of Eq. (487).

Inversion Formula: From Eq. (487) we have

$$\begin{aligned}
g(\alpha, x) & = \int_{-\infty}^{\infty} \frac{dadb}{\sqrt{\pi}a^2} \int \frac{d^2\kappa d^2s}{2\pi|s|^2} W_{\psi} g(s, r, \kappa; a, b) \\
& \times \psi_{s,r,\kappa;a,b}(\alpha, x) \quad (492)
\end{aligned}$$

which is the inversion formula for the original signal $g(\alpha, x)$ expressed by a superposition of wavelet functions $\psi_{s,r,\kappa;a,b}(\alpha, x)$, with the value of continuous WT $W_{\psi} g(s, r, \kappa; a, b)$ serving as coefficients. In fact, in Eq. (475), when we take $g(\alpha, x) = \delta(\alpha - \alpha') \delta(\alpha^* - \alpha'^*) \delta(x - x')$, then

$$W_{\psi} g(s, r, \kappa; a, b) = \frac{1}{2\pi\sqrt{\pi}} \psi_{s,r,\kappa;a,b}^*(\alpha', x') \quad (493)$$

Substituting Eq. (493) into Eq. (491), we get (492).

We can visualize the new WT $W_{\psi} g(s, r, \kappa; a, b)$ in the context of quantum optics. Noticing that the generalized squeezing operator $U(s, r, \kappa = 0; a, b = 0)$ in Eq. (478) is an image of the combined mapping of the classical real dilation transform $x \rightarrow x/a$ ($a > 0$) and the classical complex symplectic transform $(\alpha, \alpha^*) \rightarrow (s\alpha - r\alpha^*, s^*\alpha^* - r^*\alpha)$ in $|\alpha, x\rangle$ representation, one can use the technique of integration within the normal prod-

uct of operators to perform the integration in Eq. (478) to derive its explicit form [see Eq. (15) in Ref. [136]]. The transform matrix element of $U(s, r, \kappa = 0; a, b = 0)$ in the entangled state representation $|\eta\rangle$ is

$$\begin{aligned}
& \langle \eta | U(s, r, \kappa = 0; a, b = 0) | \eta' \rangle \\
& = \sqrt{\frac{s}{a}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int \frac{d^2\alpha}{2\pi} \langle \eta | s\alpha - r\alpha^*, \frac{x}{a} \rangle \langle \alpha, x | \eta' \rangle \quad (494)
\end{aligned}$$

In Fock space $|\eta = \eta_1 + i\eta_2\rangle$ is a two-mode EPR entangled state in Eq. (28).

Then using Eqs. (472) and (28), we obtain

$$\langle \eta | \alpha, x \rangle = \frac{1}{\sqrt{2}} \exp \left[-\frac{\alpha^2 + |\alpha|^2}{4} - \frac{1}{2} \eta_1^2 + \eta_1 \alpha - i\eta_2 x \right] \quad (495)$$

Substituting Eq. (495) into Eq. (494) and using Eq. (79), we obtain

$$\begin{aligned}
& \langle \eta | U(s, r, \kappa = 0; a, b = 0) | \eta' \rangle \\
& = \frac{\pi}{\sqrt{a}} \delta(\eta_2' - \eta_2/a) \frac{1}{\sqrt{2i\pi B}} \\
& \times \exp \left[\frac{i}{2B} (A\eta_1' 2 - 2\eta_1 \eta_1' + D\eta_1^2) \right] \quad (496)
\end{aligned}$$

which is just the kernel of a mixed lens–Fresnel transform, i.e., the variable η_1 of the object experiences a generalized Fresnel transform, while η_2 undergoes a lens transformation. Thus, based on $|\alpha, x\rangle$, we have introduced SDWT which involves both the real variable dilation-transform and complex variable symplectic transform, corresponding to the lens-Fresnel mixed transform in classical optics.

15 Fresnel–Hadamard combinatorial transformation

In the theoretical study of quantum computer, the Hadamard transform is of great importance. This operation is n Hadamard gates acting in parallel on n qubits. The Hadamard transform produces an equal superposition of all computational basis states. From the point of view of Deutsch–Jozsa quantum algorithm, the Hadamard transform is an example of the $N = 2^n$ quantum Fourier transform, which can be expressed as [137, 138]

$$|j\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i j k / 2^n} |k\rangle \quad (497)$$

Now the continuous Hadamard transform, which used to go from the coordinate basis $|x\rangle$ to the momentum basis, is defined as [139]

$$\mathfrak{H} |x\rangle = \frac{1}{\sqrt{\pi\sigma}} \int_{-\infty}^{\infty} dy \exp(2ixy/\sigma^2) |y\rangle \quad (498)$$

where σ is the scale length, and \mathfrak{F} is named Hadamard operator. Using the completeness of $\int_{-\infty}^{\infty} dx |x\rangle \langle x| = 1$, we have

$$\mathfrak{F} = \frac{1}{\sqrt{\pi\sigma}} \iint_{-\infty}^{\infty} dx dy \exp(2ixy/\sigma^2) |y\rangle \langle x| \quad (499)$$

The above two transforms (Fresnel transform and Hadamard transform) are independent of each other. An interesting question thus naturally arises: can we combine the two transforms together? To put it in another way, can we construct a combinatorial operator which play the role of both Fresnel transform and Hadamard transform for two independent optical modes? The answer is affirmative. In this section we try to construct the so-called Fresnel–Hadamard combinatorial transform.

15.1 The Hadamard–Fresnel combinatorial operator

Based on the coherent-entangled representation $|\alpha, x\rangle$, and enlightened by Eqs. (81) and (498), we now construct the following ket–bra integration [140]

$$U = \frac{\sqrt{s}}{\sqrt{\pi\sigma}} \int \frac{d^2\alpha}{\pi} \iint dx dy \times \exp(2ixy/\sigma^2) |s\alpha - r\alpha^*, y\rangle \langle \alpha, x| \quad (500)$$

We name U the Hadamard–Fresnel combinatorial operator.

$$U = \frac{1}{\sqrt{s^*}} \frac{4\sqrt{\pi}\sigma}{\sqrt{\sigma^4 + 4}} : \exp \left\{ -\frac{1}{2} \frac{r}{s^*} \left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} \right)^2 + \frac{\sigma^4 - 4}{2(\sigma^4 + 4)} \left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \right)^2 + \left(\frac{1}{s^*} - 1 \right) \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} \frac{a_1 - a_2}{\sqrt{2}} \right. \\ \left. + \left(\frac{4i\sigma^2}{\sigma^4 + 4} - 1 \right) \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \frac{a_1 + a_2}{\sqrt{2}} + \frac{1}{2} \frac{r^*}{s^*} \left(\frac{a_1 - a_2}{\sqrt{2}} \right)^2 + \frac{\sigma^4 - 4}{2(\sigma^4 + 4)} \left(\frac{a_1 + a_2}{\sqrt{2}} \right)^2 \right\} : \quad (502)$$

15.2 The properties of Hadamard–Fresnel operator

Note $[a_1 - a_2, a_1^\dagger + a_2^\dagger] = 0$ and $[(a_1 - a_2)/\sqrt{2}, (a_1^\dagger - a_2^\dagger)/\sqrt{2}] = 1, [(a_1 + a_2)/\sqrt{2}, (a_1^\dagger + a_2^\dagger)/\sqrt{2}] = 1, (a_1 - a_2)/\sqrt{2}$ can be considered a mode independent of another mode $(a_1^\dagger + a_2^\dagger)/\sqrt{2}$, thus we have the operator identity

$$\exp [f(a_1^\dagger \pm a_2^\dagger)(a_1 \pm a_2)] =: \exp \left[\frac{1}{2}(e^{2f} - 1)(a_1^\dagger \pm a_2^\dagger)(a_1 \pm a_2) \right] : \quad (503)$$

Using Eq. (503), we can rewrite Eq. (502) as

$$U = U_2 U_1 = U_1 U_2 \quad (504)$$

where

$$U_1 = \frac{4\sqrt{\pi}\sigma}{\sqrt{\sigma^4 + 4}} \exp \left[\frac{\sigma^4 - 4}{2(\sigma^4 + 4)} \left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \right)^2 \right] \times \exp \left[\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \frac{a_1 + a_2}{\sqrt{2}} \ln \frac{4i\sigma^2}{(\sigma^4 + 4)} \right]$$

Substituting Eq. (498) into Eq. (500), and using the two-mode vacuum projector’s normally ordered form $|00\rangle \langle 00| =: \exp(-a_1^\dagger a_1 - a_2^\dagger a_2):$ as well as the IWOP technique, we get

$$U = \frac{\sqrt{s}}{\sqrt{\pi\sigma}} : \int \frac{d^2z}{\pi} A(z, z^*) \iint dx dy B(x, y) e^C : \quad (501)$$

where

$$C \equiv -\frac{(a_1^\dagger + a_2^\dagger)^2 + (a_1 + a_2)^2}{4} - a_1^\dagger a_1 - a_2^\dagger a_2$$

$$B(x, y) \equiv \exp \left[-\frac{y^2 + x^2}{2} + y(a_1^\dagger + a_2^\dagger) + x(a_1 + a_2) + \frac{2ixy}{\sigma^2} \right]$$

and

$$A(z, z^*) \equiv \exp \left[-\frac{|sz - rz^*|^2 + |z|^2}{4} + \frac{sz - rz^*}{2} (a_1^\dagger - a_2^\dagger) + \frac{z^*(a_1 - a_2)}{2} \right]$$

They are all within the normal ordering symbol $::$. Now perform the integration over $dx dy$ within $::$, remembering that all creation operators commute with all annihilation operators (the essence of the IWOP technique) so that they can be considered c -number during the integration. We can finally obtain the normally ordered form of Hadamard–Fresnel combinatorial operator [140]

$$\times \exp \left[\frac{\sigma^4 - 4}{2(\sigma^4 + 4)} \left(\frac{a_1 + a_2}{\sqrt{2}} \right)^2 \right] \quad (505)$$

and

$$U_2 = \exp \left[-\frac{r}{2s^*} \left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} \right)^2 \right] \times \exp \left[\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} \frac{a_1 - a_2}{\sqrt{2}} + \frac{1}{2} \right) \ln \frac{1}{s^*} \right] \times \exp \left[\frac{r^*}{2s^*} \left(\frac{a_1 - a_2}{\sqrt{2}} \right)^2 \right] \quad (506)$$

while U_2 is the Fresnel operator for mode $(a_1 - a_2)/\sqrt{2}$, and U_1 is named the Hadamard operator for mode $(a_1 + a_2)/\sqrt{2}$.

It then follows that

$$U \frac{a_1 - a_2}{\sqrt{2}} U^{-1} = s^* \frac{a_1 - a_2}{\sqrt{2}} + r \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}$$

$$U \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} U^{-1} = r^* \frac{a_1 - a_2}{\sqrt{2}} + s \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}} \quad (507)$$

from which we see that the Hadamard–Fresnel combinatorial operator can play the role of Fresnel transformation for $\frac{a_1 - a_2}{\sqrt{2}}$. Physically, $(a_1 - a_2)/\sqrt{2}$ and $(a_1 + a_2)/\sqrt{2}$ can be two output fields of a beamsplitter.

In a similar way, we have

$$\begin{aligned} U \frac{a_1 + a_2}{\sqrt{2}} U^{-1} &= \frac{1}{4i\sigma^2} \left[(\sigma^4 + 4) \frac{a_1 + a_2}{\sqrt{2}} \right. \\ &\quad \left. - (\sigma^4 - 4) \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \right] \\ U \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} U^{-1} &= \frac{1}{4i\sigma^2} \left[-(\sigma^4 + 4) \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}} \right. \\ &\quad \left. + (\sigma^4 - 4) \frac{a_1 + a_2}{\sqrt{2}} \right] \end{aligned} \quad (508)$$

which for the quadrature $X_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/(\sqrt{2}i)$, $i = 1, 2$, leads to

$$\begin{aligned} U \frac{X_1 + X_2}{2} U^{-1} &= \frac{\sigma^2}{4} (P_1 + P_2), U(P_1 + P_2)U^{-1} \\ &= -\frac{4}{\sigma^2} \frac{X_1 + X_2}{2} \end{aligned} \quad (509)$$

from which we see that the Hadamard–Fresnel combinatorial operator also plays the role of exchanging the total momentum—average position followed by a squeezing transform, with the squeezing parameter being $\frac{\sigma^2}{4}$.

The mutual transform in Eq. (509) can be realized by

$$e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} (X_1 + X_2) e^{-i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} = P_1 + P_2 \quad (510)$$

$$e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} (P_1 + P_2) e^{-i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} = -(X_1 + X_2) \quad (511)$$

while the two-mode squeezing operator is $S_2 = \exp \left[\ln \frac{2}{\sigma^2} (a_1^\dagger a_2^\dagger - a_1 a_2) \right]$, therefore

$$U_1 = S_2^{-1} e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} \quad (512)$$

From Eq. (504) and Eq. (512), we see that the Hadamard–Fresnel combinatorial operator can be decomposed as

$$U = U_2 S_2^{-1} e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} = S_2^{-1} e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)} U_2 \quad (513)$$

It can also be seen that U is unitary, and $U^+ U = U U^+ = 1$.

In this section, we have introduced the Fresnel–Hadamard combinatorial operator by virtue of the IWOP technique. This unitary operator plays the role of both Fresnel transformation for mode $\frac{a_1 - a_2}{\sqrt{2}}$ and Hadamard transformation for mode $\frac{a_1 + a_2}{\sqrt{2}}$, respectively, and the two transformations are combinatorial. We have shown that the two transformations are concisely ex-

pressed in the coherent-entangled state representation as a projective operator in an integrative form. We also found that the Fresnel–Hadamard operator can be decomposed as $U_2 S_2^{-1} e^{i\frac{\pi}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)}$, a Fresnel operator U_2 , a two-mode squeezing operator S_2^{-1} and the total momentum-average position exchanging operator. Physically, $\frac{a_1 - a_2}{\sqrt{2}}$ and $\frac{a_1 + a_2}{\sqrt{2}}$ can be two output fields of a beamsplitter. If an optical device can be designed for Fresnel–Hadamard combinatorial transform, then it can be directly applied to these two output fields of the beamsplitter.

In summary, although quantum optics and classical optics are quite different, no matter in the mathematical tools they employed or in a conceptual view-quantum optics concerns the wave–particle duality of optical field with an emphasis on its nonclassical properties, whereas classical optics works on the distribution and propagation of the light waves, it is a new exploration to link them systematically. In this review, via the route of developing Dirac’s symbolic method we have revealed some links between them by mapping classical symplectic transformation in the coherent state representation onto quantum unitary operators (GFO). Throughout our discussion the IWOP technique is indispensable for the derivation. We have resorted to the quantum optical interpretation of various classical optical transformations by adopting quantum optics concepts such as the coherent states, squeezed states, and entangled states, etc. Remarkably, we have endowed complex fractional Fourier transform, Hankel transform with quantum optical representation-transform interpretation. Our formalism, starting from quantum optics theory, not only provides quantum mechanical account of various classical optical transformations, but also have found their way back to some new classical transformations, e.g. entangled Fresnel transform, Fresnel-wavelet transform, etc, which may have realistic optical interpretation in the future. As Dirac predicted, functions that have been applied in classical optical problems may be translated in an operator language in quantum mechanics, and vice-versa. We expect that the content of this work may take its part in quantum states engineering, i.e., optical field states’ preparation and design.

Now the correspondence in this respect between the two distinct fields is established, the power of Dirac’s symbolic method can be fully displayed to solve some new problems in classical optics, e.g., to find new eigenmodes of some optical transforms; to extend the research region of classical optics theoretically by introducing new transforms (for example, the entangled Fresnel transforms), which may bring attention of experimentalists, who may get new ideas to implement these new classical optical transformations.

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