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RESEARCH ARTICLE

Entropy majorization, thermal adiabatic theorem, and quantum phase transitions

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Let a general quantum many-body system at a low temperature adiabatically cross through the vicinity of the system's quantum critical point. We show that the system's temperature is significantly suppressed due to both the entropy majorization theorem in quantum information science and the entropy conservation law in reversible adiabatic processes. We take the one-dimensional transverse-field Ising model and the spinless fermion system as concrete examples to show that the inverse temperature might become divergent around the systems' critical points. Since the temperature is a measurable quantity in experiments, it can be used, via reversible adiabatic processes at low temperatures, to detect quantum phase transitions in the perspectives of quantum information science and quantum statistical mechanics.

Keywords quantum phase transition, entropy majorization

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1 Introduction

In recent years, quantum phase transitions [1] have been paid considerable attentions in various fields of physics because they can provide valuable information about novel types of matter that emerge from the vicinity of the absolute zero temperature. Examples that have been studied both theoretically and experimentally include high-T_c superconductors, fractional quantum Hall liquids, quantum magnets, Mott-insulators etc. Unlike thermal phase transitions that occur at finite temperatures and are driven by thermal fluctuations, quantum phase transitions occur at the absolute zero temperature and are driven solely by quantum fluctuations. Conventionally, quantum phase transitions are characterized by singularities of the ground-state energy. The order of transitions is defined by discontinuities (or singularities) occurring in the nth-order derivative of the energy. For instance, a second-order quantum phase transition corresponds to the discontinuity (or singularity) in the second derivative of the energy. Such a characterization is inherent in traditional studies on thermal phase transitions since the finite-temperature free energy becomes the ground-state energy in the limit of zero temperature. Theoretical frameworks, such as Landau–Ginzburg–Wilson spontaneous symmetry-breaking theory, have been used to understand continuous quantum phase transitions too. In Landau's paradigm, local order operators and their correlation functions play a central role in the studies of quantum phase transitions. The nonvanishing value of the order parameter at a long distance characterizes a symmetry-breaking phase, a unique feature which exists only for a system with infinite degrees of freedom.

Recently, a huge interest was raised in the attempt of characterizing quantum phase transitions in terms of the concepts borrowed from quantum information science [2]. Due to the mutual relation between fluctuations and correlations, people are curious about the role of entanglement, a pure quantum correlation existing uniquely in quantum systems, in those transitions that are driven solely by quantum fluctuations. Hundreds of papers (For a review, see Ref. [3]) have been published since the two original work finished respectively by Osterloh *et al.* [4], and Osborne and Nielsen [5]. Though a unified theory on the role of entanglement in quantum phase transitions is still unavailable, some definitive conclusions have been commonly accepted [3]. Another promising approach to quantum phase transitions is based on the quantum fidelity, a concept emerging also from quantum information science. The fidelity measures the similarity between two ground states, hence is expected to show a dramatic change across the transition points [6–8]. This motivated people to start exploring its role in quantum phase transitions (For a review, see Ref. [9]). Moreover, as the fidelity is a space geometrical quantity, no a priori knowledge of the order parameter and symmetry breaking of the system is assumed. It is a great advantage to study quantum phase transitions using the fidelity approach.

Experimentally, however, to study the ground-state entanglement and the fidelity in quantum many-body systems is still a challenging problem. The nuclear-magnetic-resonance quantum simulator, though is promising, can only measure the entanglement [10] and the fidelity [11–13] in few-body systems (for instance the spin dimer). The critical phenomena, such as the scaling properties and various critical exponents, are not able to be studied in experiments yet.

In this paper, we are going to study the quantum criticality based on the entropy majorization theorem [14], which is an important theorem in quantum information science. At low temperatures, the entropy majorization theorem manifests that the entropy is significantly enhanced around the critical point due to a higher density of state nearby. Meanwhile the entropy is also required to be conserved in reversible adiabatic processes because there is no heat transfer during the process. In order to satisfy the entropy conservation law in the reversible adiabatic process, we show that the temperature has to be suppressed and the *adiabatic inverse temperature* (denoting the inverse temperature in the adiabatic process) might even become singular at the critical point.

In previous studies, the entropy majorization theorem has been used to study the ground-state in quantum many-body systems [21, 22]. Meanwhile, the thermal entropy has been observed to be enhanced around the quantum critical point. For instance, the entropy can be divergent in the saturated region of the one-dimensional XY model [15]; similar properties were found in recent works on Magnetocaloric effect close to quantum critical points [17–19] too. Therefore, our work might provide a new angle to interpret the enhancement of thermal entropy in a perspective of quantum information science.

This work is organized as follows. In Section 2, we give a warm-up example to show explicitly that the adiabatic inverse temperature will become singular when a ground-state level-crossing occurs. In Section 3, we show that, based on the entropy majorization theorem, the significant changes in the low-energy spectra might lead to a singular adiabatic inverse temperature around the quantum critical point. In Section 4, we take the one-dimensional transverse-field Ising model as an example to show that the adiabatic inverse temperature becomes singular around the critical point. In Section 5, we study the thermodynamics of the one-dimensional spinless fermion system at low temperatures, and show explicitly that the adiabatic inverse temperature is divergent at the quantum phase transition point. In Section 6, we discuss the potential application of our motivation to experiments. Finally, we give a summary in Section 7.

2 A warm-up two-level model

To begin with, we first have a look at a two-level toy model describing a free spin in an external field as a warm-up example. The model's Hamiltonian reads as

$$H = -h\sigma^z \tag{1}$$

where σ^z is the z-component of Pauli matrices $\{\sigma^x, \sigma^y, \sigma^z\}$. In the 1/2 spin basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, the Hamiltonian is already diagonal and the corresponding eigenenergies are

$$E_1 = -h, \quad E_2 = h \tag{2}$$

Therefore, if $h > 0$, the ground state is $|\uparrow\rangle$; while if $h < 0$, the ground state becomes $|\downarrow\rangle$. A simple first-order quantum phase transition induced by the ground-state level-crossing occurs at the transition point $h_c = 0$.

The quantum phase transition occurring in such a two-level system looks rather trivial. However, it captures one of the most important properties in quantum critical phenomena, which is the vanishing of the energy gap (or more generally a significant change in the density of state). Suppose the absolute zero temperature is unavailable, the thermal equilibrium state, according to quantum statistical mechanics, can only be described by a thermal density matrix. For the present system, the thermal density matrix at a temperature $\beta = 1/T$, with the Boltzmann constant $k_{\rm B} = 1$, is

$$\rho = \frac{1}{Z}\begin{pmatrix} \mathrm{e}^{\beta h} & 0 \\ 0 & \mathrm{e}^{-\beta h} \end{pmatrix} \tag{3}$$

where

$$Z = \mathrm{e}^{\beta h} + \mathrm{e}^{-\beta h} \tag{4}$$

is the partition function. From the thermal density matrix, we can see that the thermal probability of the two states are

$$p_1 = \frac{\mathrm{e}^{\beta h}}{Z}, \quad p_2 = \frac{\mathrm{e}^{-\beta h}}{Z} \tag{5}$$

respectively. Therefore, the entropy of the system reads

$$\begin{aligned} S &= -p_1 \ln p_1 - p_2 \ln p_2 \\ &= \ln(1 + \mathrm{e}^{-2\beta h}) + \frac{2\beta h \mathrm{e}^{-2\beta h}}{1 + \mathrm{e}^{-2\beta h}} \end{aligned} \tag{6}$$

Now we suppose the system adiabatically envolves $h = h_0$ and $\beta = \beta_0$ through $h = -h_0$ with $h_0 > 0$. During the adiabatic process, the system's entropy is required to be conserved, i.e., $S = \text{const.}$, hence

$$\ln(1 + \mathrm{e}^{-2\beta h}) + \frac{2\beta h \mathrm{e}^{-2\beta h}}{1 + \mathrm{e}^{-2\beta h}} = \text{const.} \tag{7}$$

In this expression, the only relevant parameter that changes the entropy is βh. Therefore, to fix the entropy, we need only to fix βh, i.e.,

$$h\beta = h_0\beta_0 \tag{8}$$

such that

$$\beta = \frac{h_0\beta_0}{h} \tag{9}$$

We can see clearly that the adiabatic inverse temperature β becomes divergent as h tends to zero. That is, if we change the external field slowly enough, the temperature will become zero at the transition point.

As a numerical demonstration, we show, in Fig. 1, the thermal entropy of the two-level system as a function of h and β, and its contour map on the h–β plane. The lines on the contour map denote adiabatic paths along which the system evolves, from which we can judge that the adiabatic inverse temperature is divergent around $h = 0$.

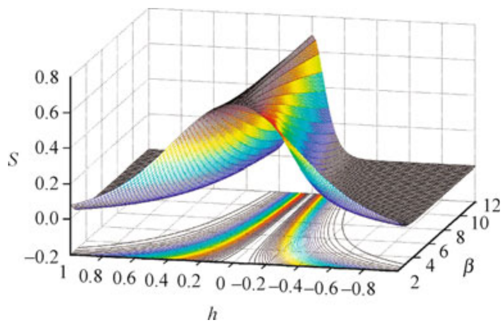

Fig. 1 The thermal entropy of the two-level system as a function of h and β. Contour lines in the h–β plane denote the adiabatic evolution paths.

3 Entropy majorization theorem and quantum criticality

The discussion in the above two-level system shows a clear picture that the changes in the energy spectra around the critical point might lead to a divergent adiabatic inverse temperature. In this section, we want to generalize this picture to an arbitrary quantum many-body system.

We consider a general quantum many-body system described by the Hamiltonian

$$H = H_0 + \lambda H_I \tag{10}$$

where H_I is the driving Hamiltonian and λ denotes its strength. Without loss of generality, a quantum phase transition is supposed to occur in the system's ground state $|\phi_0(\lambda)\rangle$ at the critical point λ_c. Since there is no rigorous ground state in experiments, according to quantum statistical physics, the thermal equilibrium state of the system at a large β (hence small T) can be expressed as a density matrix

$$\rho(\beta,\lambda) = \frac{1}{Z}\sum_n \mathrm{e}^{-\beta E_n}\,|\phi_n\rangle\,\langle\phi_n| \tag{11}$$

Here $|\phi_n(\lambda)\rangle$, which satisfies

$$H(\lambda)|\phi_n(\lambda)\rangle = E_n(\lambda)|\phi_n(\lambda)\rangle \tag{12}$$

define a set of orthogonal complete basis, and

$$Z = \sum_n \mathrm{e}^{-\beta E_n} \tag{13}$$

is the partition function. The entropy of the system can then be expressed as

$$S = \beta\left(\langle E\rangle - F\right) = -\sum_n p_n \ln p_n \tag{14}$$

Here $\langle E\rangle$ and F are the internal energy and the free energy respectively, and

$$p_n = \frac{1}{\bar{Z}}\mathrm{e}^{-\beta(E_n - E_0)} \tag{15}$$

where $\bar{Z} = Z\mathrm{e}^{\beta E_0}$ and with

$$\sum_n p_n = 1 \tag{16}$$

$$p_0 \geqslant p_1 \geqslant p_2 \cdots \tag{17}$$

$p_i = p_j$ occurs if $E_i = E_j$.

In the thermodynamic limit, the entropy of the system can be written as

$$S = -N\int p(\epsilon)\log\left[p(\epsilon)\right]\rho(\epsilon)\mathrm{d}\epsilon \tag{18}$$

where $\epsilon = E/N$ with N being the system size, and $\rho(\epsilon)$ denotes the density of state at ϵ. Now we are going to show that a significant change in the density of state around the ground state will change the temperature or the entropy, depending on which one keeps constant. The change in the density of state include, but are not limited to, the vanishing of the energy gap around the critical point [1] and the emergence of a Van Hove singularity in the ground state [20].

3.1 Quantum phase transitions induced by a vanishing gap

We consider the system at two points λ and λ' in the parameter space. Without loss of generality, we assume that λ' is closer to the critical point λ_c, i.e., $|\lambda'-\lambda_c| < |\lambda-\lambda_c|$, and the temperature is the same. So there are two thermal probability distributions $\{p_n\}$ and $\{p'_n\}$ at λ and λ', respectively. The low-temperature properties of the system are determined by both the ground state and the low-lying excitations, so we need to consider only their contributions to the entropy of the system. Since the system is gapped and the gap is assumed to vanish at the critical point, we have $E'_1 - E'_0 < E_1 - E_0$. Defining

$$\bar{Z}_k = \sum_{n=0}^{k} \mathrm{e}^{-\beta(E_n - E_0)}, \quad \bar{Z}'_k = \sum_{n=0}^{k} \mathrm{e}^{-\beta\left(E'_n - E'_0\right)} \tag{19}$$

we have

$$\begin{aligned} &\bar{Z}_0 = \bar{Z}'_0 \\ &\bar{Z}_1 < \bar{Z}'_1 \\ &\quad \vdots \\ &\bar{Z}_\Lambda < \bar{Z}'_\Lambda \end{aligned} \tag{20}$$

where Λ is a cutoff. Therefore, there exists a range in the energy space, in which

$$\begin{aligned} &p_0 > p'_0 \\ &p_1 < p'_1 \\ &\quad \vdots \\ &p_\Lambda < p'_\Lambda \end{aligned} \tag{21}$$

such that

$$\sum_j^k p_j > \sum_j^k p'_j \tag{22}$$

for $0 < k \leqslant \Lambda$. Here Λ can be finite or even as large as the dimension of the whole Hilbert space depending on both the low temperature cupoff and the normalization condition of the Boltzmann distribution. If the temperature is low enough, contributions from those eigenstates above E_Λ are neglectable. Therefore, in such a region, we can say the distribution $\{p_n\}$ majorizes $\{p'_n\}$ [23]. Since $\{p_n\}$ and $\{p'_n\}$ denote the probability distribution, we have, according to the entropy majorization theorem [14],

$$S(p) < S(p') \tag{23}$$

The inequality manifests that the entropy increases as the energy gap vanishes.

On the other hand, the thermal entropy is a monotonically increasing function of the temperature. In order to ensure that the entropy is conserved during the adiabatic process, the temperature T at λ' must be lower than that at λ. As a result, the adiabatic inverse temperature β increases. To see the singularity of the adiabatic inverse temperature, the starting temperature before the adiabatic evolution must be low enough such that the Boltzmann probability of the ground state $1/\bar{Z}$ is finite, say 1/10 which does not vanish even in the thermodynamic limit. In this case, the system's entropy is usually intensive given that there is no singularity in the density of state below E_Λ. If the energy gap is closed, the density of state becomes continuous around the ground state (see the top right of Fig. 2), and the adiabatic inverse temperature becomes divergent. Otherwise, the entropy would be extensive (divergent as $\sim N$) if β is still finite.

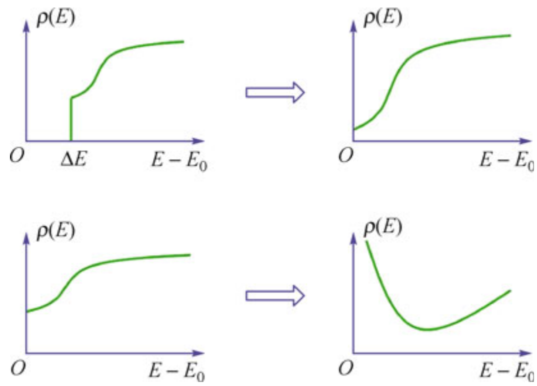

Fig. 2 Possible changes of density of state when quantum phase transitions occur. *Upper*: A gap vanishes as the system tends yo a critical point. A typical case is the one-dimensional transverse-field Ising model. *Lower*: The density of state in the low-energy region becomes singular around the critical point. A typical case is the the one-dimensional spinless fermion system.

3.2 Quantum phase transitions induced by a Van Hove singularity

Generally, the significant change in the density of state is not limited to the case of a vanishing gap. A straightforward example is the emergence of a Van Hove singularity at the ground state (see the bottom right of Fig. 2). This case usually occurs as the low-lying excitations' dispersion is, though gapless, intrinsically changed around the critical point, for instance, from linear to quadratic.

Assuming at λ, the density of state is a continuous function in the vicinity of the ground state, then

$$S = -N \int_0^\Lambda p(\epsilon) \ln\left[p(\epsilon)\right] \rho(\epsilon) \mathrm{d}\epsilon \tag{24}$$

where Λ is a cut-off due to $p(\epsilon) \sim \mathrm{e}^{-\beta N \epsilon}/Z$; and at another point λ' which is closer to the critical point, the entropy becomes

$$S' = -N \int_0^\Lambda p(\epsilon) \ln\left[p(\epsilon)\right] \rho'(\epsilon) \mathrm{d}\epsilon \tag{25}$$

Since $\rho(\epsilon = 0)$ becomes more and more singular as λ tends to λ_c, the integrand in Eq. (24) majorizes that of Eq. (25) due to

$$Z = N \int_0^\Lambda \mathrm{e}^{-\beta N \epsilon} \rho(\epsilon) \mathrm{d}\epsilon \tag{26}$$

Therefore, according to the entropy majorization theorem, we again have $S < S'$. To ensure the entropy conservation during adiabatic processes, the adiabatic inverse temperature in Eq. (25) must be enhanced. Moreover, if $\rho(\epsilon = 0)$ is divergent at the critical point, β is expected to be divergent too.

3.3 Potential scaling issues based on the Landau–Zener model

Quantum phase transitions occurring in few-body systems are usually induced by the ground-state level-crossing. Nevertheless, most of the interesting critical phenomena occur in thermodynamic systems where the singularity results from the infinite degrees of freedom in the thermodynamic limit. That is, in these cases, the ground-state properties do not show any singular behaviors for a finite sample. Then the scaling analysis becomes very important. Here we take the two-level Landau–Zener model [24, 25] to illustrate such a picture. The model Hamiltonian reads as

$$H = \begin{pmatrix} \lambda & \omega(N) \\ \omega(N) & -\lambda \end{pmatrix} \tag{27}$$

in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, where $\omega(N)$ denotes the N-dependent hoping amplitude between two levels. The eigenstates of the Hamiltonian are

$$\psi_1 = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle \tag{28}$$
$$\psi_2 = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$$

with $\cos\theta = \varepsilon/\sqrt{1+\varepsilon^2}, \sin\theta = 1/\sqrt{1+\varepsilon^2}$, and $\varepsilon = \lambda/\omega$. The energy gap between the two states are $\Delta E = 2\sqrt{\omega^2+\lambda^2}$. If $\omega(N)$ is finite, the energy gap is always finite. This case corresponds to a finite system. However, if $\omega(N)$ vanishes in the infinite N limit, for instance, $\omega(N) \sim 1/N^\mu$ with $\mu > 0$, a quantum phase transition then occurs at $\lambda = 0$.

Let the system adiabatically evolve from λ_0 and at temperature β_0. The adiabatic process requires that

$$\beta\sqrt{\omega^2+\lambda^2} = \beta_0\sqrt{\omega^2+\lambda_0^2} \tag{29}$$

hence,

$$\beta = \frac{\beta_0\sqrt{\omega^2+\lambda_0^2}}{\sqrt{\omega^2+\lambda^2}} \tag{30}$$

which reaches a maximum (for a finite N) at $\lambda = 0$. In the large N limit, we have

$$\beta \sim N^\mu \tag{31}$$

at $\lambda = 0$. From the above relation, we can see that the adiabatic inverse temperature diverges as the inverse of energy gap according to the Landau–Zener model.

4 The one-dimensional transverse-field Ising model

In this section, we consider a continuous quantum phase transition occurring in the ground state of the one-dimensional transverse-field Ising model. Such a phase transition is believed to be well characterized by the Landau–Ginzburg–Wilson spontaneous symmetry-breaking paradigm. Moreover, due to its simplicity and exact solvability, the one-dimensional transverse-field Ising model becomes one of the most popular models for studying quantum phase transitions. The model's Hamiltonian reads as

$$H = -\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + h\sigma_j^z\right) \tag{32}$$
$$\sigma_1^x = \sigma_{N+1}^x \tag{33}$$

where h is the transverse field and N is the number of spins. As implied from the model's name, the Hamiltonian describes a chain of spins with the nearest-neighboring Ising interaction along the x-direction, and all the spins are subjected to a transverse magnetic field h along the z-direction. If h is infinite, all the spins are polarized along the z-direction. The ground state is paramagnetic in the x-direction. While if $h = 0$, the system becomes a classical Ising chain whose ground state is ferromagnetic and presents a true long-range order in the x-direction. A quantum phase transition from a ferromagnetic phase to a paramagnetic phase occurs at the critical point $h_c = 1$.

The Ising model can be diagonalized exactly through three transformations together, i.e. the Jordan-Wigner transformation, Fourier transformation, and Bogoliubov transformation. The Hamiltonian finally becomes a quasi-free fermion system in momentum space,

$$H = \sum_k \epsilon(k)\left(2b_k^\dagger b_k - 1\right) \tag{34}$$

where

$$\epsilon(k) = \sqrt{1 - 2h\cos k + h^2} \tag{35}$$

is the dispersion relation of the quasi particles defined by $b_k^\dagger$. The dispersion relation shows that, in the thermodynamic limit, the system is gapless only at $h = 1$, and gapped in both the phases of $0 < h < 1$ and $h > 1$.

According to quantum statistical mechanics, the partition function of the system at β can be calculated as

$$Z = \prod_k \mathrm{e}^{\beta\epsilon(k)}\left[1 + \mathrm{e}^{-2\beta\epsilon(k)}\right] \tag{36}$$

Then according to thermodynamics, $F = -T\ln Z$ and $S = -\partial F/\partial T$, the free energy and the thermal entropy are

$$F = E_0 - \frac{NT}{2\pi}\int_{-\pi}^{\pi}\ln\left[1+\mathrm{e}^{-2\beta\epsilon(k)}\right]\mathrm{d}k \tag{37}$$
$$S = \frac{N}{2\pi}\int_{-\pi}^{\pi}\left(\ln\left[1+\mathrm{e}^{-2\beta\epsilon(k)}\right] + \frac{2\beta\epsilon(k)}{1+\mathrm{e}^{2\beta\epsilon(k)}}\right)\mathrm{d}k \tag{38}$$

where E_0 is the ground-state energy.

It is not easy to get the final expression of the entropy explicitly due to the presence of the energy gap. Nevertheless, we can calculate the system's entropy as a function of h and β numerically. The results are shown in Fig. 3. From the figure, we can first see that the thermal entropy is a monotonically decreasing function of β. This observation is consistent with thermodynamics. Secondly, the thermal entropy decreases more quickly in both the gapped phases, while rather slowly around the critical point. Such a difference comes from the gap protection in the non-critical region. Thirdly, if we fix β, we can see that the entropy increases significantly at the transition point. The physics behind this phenomenon can be illuminated with the entropy majorization theorem as we discussed in Section 3. Finally, what is of significant importance is the contour map of the thermal entropy on the h–β plane. Clearly, a single line on the contour map denotes an isoline of the thermal entropy. If the transverse field is changed slowly enough, these lines are just paths along which the system evolves. Then if the initial inverse temperature is comparable with the inverse energy gap, the inverse temperature becomes divergent at the critical point.

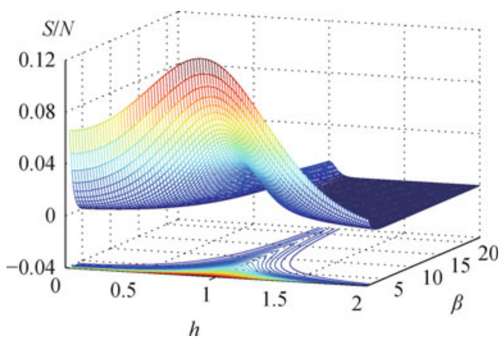

Fig. 3 The thermal entropy per site of the one-dimensional transverse-field Ising model as a function of h and β. Contour lines in the h–β plane denote the adiabatic evolution paths.

Meanwhile, we would like to point out that the eigenstates of the Ising model changes as the system evolves adiabatically because the driving term does not commute with the Hamiltonian. This observation means that the low-temperature adiabatic process here involve not only thermal fluctuations, but also quantum fluctuations.

5 The one-dimensional spinless fermion system

In this section, we consider a quantum phase transition induced by continuous level-crossing, as illustrated by the one-dimensional spinless fermion system subjected to a chemical potential. Such a phase transition is basically different from that occuring in the one-dimensional transverse-field Ising model. The latter is due to the avoided level-crossing, while the former is caused by the continuous ground-state level-crossing. The model's Hamiltonian can be written as

$$H = -t\sum_j \left(c^\dagger_{j+1}c_j + c^\dagger_j c_{j+1}\right) + \mu\sum_j c^\dagger_j c_j \tag{39}$$

where $c^\dagger_j$ and c_j are creation and annihilation operators for fermions at site j; t is the hoping integral; and μ is the chemical potential. The model is equivalent to the one-dimensional XY model via a Jordan-Wigner transition. The Hamiltonian can be diagonalized via a simple Fourier transformation. In the momentum space, it reads as

$$H = \sum_k (\mu - 2t\cos k)\, c^\dagger_k c_k - \frac{N\mu}{2} \tag{40}$$

where $N = \sum_j c^\dagger_j c_j$. Besides the global energy shift $N\mu/2$, the ground state of the system is also determined by the chemical potential μ. If $\mu < \mu_c (= 2t)$, the ground state of the system is partially filled with fermions. The Fermi point is nonzero and determined by $\cos k_{\rm F} = \mu/(2t)$. In this case, the dispersion of the low-lying excitations is linear without any gaps. If $\mu > \mu_c$, the ground state becomes a vacuum state with no fermions. Moreover, a finite energy will be introduced if one adds a fermion to the ground state. A quantum phase transition occurs at $\mu = \mu_c$ at which the ground state is still gapless, while the dispersion of the low-lying excitations is quadratic. The transition is of the second order.

The partition function of the system can be calculated as

$$Z = z^{-N/2}\prod_k [1 + z\exp(2t\beta\cos k)] \tag{41}$$

where $z = \exp(\mu\beta)$. According to thermodynamics, the free energy can be calculated as

$$F = -N\mu - \frac{N}{2\pi\beta}\int_{-\pi}^{\pi} \ln[1 + \exp(-\beta(\mu - 2t\cos k))]\,{\rm d}k \tag{42}$$

and the entropy

$$S = \frac{N}{2\pi}\int_{-\pi}^{\pi} u[\beta(h - 2t\cos k)]\,{\rm d}k \tag{43}$$

$$u(x) = \ln\left(2\cosh\frac{x}{2}\right) - \frac{x}{2}\tanh\frac{x}{2} \tag{44}$$

At low temperatures, the integration range in x space becomes infinite. The entropy can be evaluated as

$$S \simeq \frac{\pi N}{6t\beta\cos^{-1}(\mu/2t)} \tag{45}$$

Therefore, in the adiabatic process, if the entropy is conserved, then

$$\beta \simeq \frac{\pi N}{6tS\cos^{-1}(\mu/2t)} \tag{46}$$

which diverges at the critical point $\mu_c = 2t$. When $\mu \to \mu_c^-$, the ground state keeps gapless so the transition has nothing to do with a vanishing gap. A careful scrutiny reveals that the density of state shows a singularity at $E = \mu - 2t$. That is,

$$\rho(E) = \frac{1}{\sqrt{4t^2 - (\mu - E)^2}} \tag{47}$$

If $\mu = \mu_c$, the density of state is singular at $E = 0$. Such a singularity makes a huge number of low-lying excitations contribute to the entropy, hence suppresses the temperature significantly at the transition point.

Nevertheless, though the entropy majorization theorem works equally well for both the quantum Ising model and the spinless fermion system, there is a clear difference between them. For the Ising model, both quantum fluctuations and thermal fluctuations play important roles. For the spinless fermion system, however, quantum fluctuations are absent because the two terms in the Hamiltonian commute with each other. This observation holds for the toy model discussed in Section 2 too. From this point of view, the entropy majorization theorem provides a unique angle to understand both types of quantum phase transitions that occur at the zero temperature limit.

6 Experimental detection of quantum criticality

Armed with the above results, we now try to clarify the experimental possibility to detect quantum criticality in quantum many-body systems. A sketch is shown in Fig. 4. We assume that a quantum many-body system described by the Hamiltonian [Eq. (10)] is isolated adiabatically [see Fig. 4(a)]. A thermometer is used to measure the temperature of the system. Moreover, we assume that the thermometer is small enough so that the heat transfer between the thermometer and the system can be neglected. Now, we first prepare the system in the gapped region and try to cool the system down to a temperature β_0 that is comparable to the energy gap $\Delta E(\lambda)$. Secondly, let the system adiabatically evolves to the another phase. According to the above motivation, the vanishing gap will make the temperature reach a minimum at the critical point, and the adiabatic inverse temperature might be divergent (See the right of Fig. 4 for a sketch).

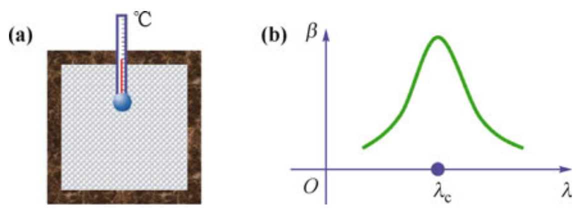

Fig. 4 A sketch of the adiabatic detection of quantum critical point. **(a)** a quantum many-body system is adiabatically isolated from the environment except for a thermometer that can measure the temperature of the system but is assume to have no obvious interference to the system. **(b)** the expected behavior of the adiabatic inverse temperature around the critical point.

On the other hand, according to thermodynamics, for an isolated ideal gas, an adiabatic cooling occurs when the pressure of the gas decreases adiabatically as it works on its surroundings. Besides, a low-temperature adiabatic process in the vicinity of the critical point can also adiabatically cool the system. Compared with what occurs in the isolated ideal gas, the adiabatic cooling here is quantum-like, and is not because of the mechanism of working on the surroundings. For instance, for the toy model discussed in Section 2, the internal energy is

$$\langle E \rangle = -h \tanh [\beta \mathrm{sgn}(h)] \tag{48}$$

where $\mathrm{sgn}(x)$ is the sign function of x. Clearly $\langle E \rangle$ reaches a maximum at $h = 0$ if $\beta \mathrm{sgn}(h)$ =const., so the internal energy increases as the system tends to the critical point. We interpret it as that the ground-state energy increases a lot during such a process. The surrounding should still act on the system. The discrepancy can be solved if we regard the ground-state energy as the reference potential energy. The internal energy becomes

$$\langle E - E_0 \rangle = -h \tanh [\beta \mathrm{sgn}(h)] + h \mathrm{sgn}(h) \tag{49}$$

which reaches a minimum at $h = 0$. Therefore, if we extract the ground-state energy from the internal energy, the system does act on the surroundings. In any case, a potential issue might be that if such a cooling technique is experimentally useful in getting ultra-low temperatures.

7 Summary and discussion

In summary, we have studied the quantum criticality in terms of a low-temperature adiabatic process. The combination of the entropy majorization theorem in quantum information science and the adiabatic theorem in statistical physics manifests that the adiabatic inverse temperature might become singular at the critical point due to the significant change of the low-lying energy spectra around the critical point. Such a straightforward motivation has been illustrated by two simple quantum phase transitions occurring in the ground state of the one-dimensional transverse-field Ising model and the spinless fermion system respectively. Since temperature is a measurable quantity in experiments, it can be used, via adiabatic processes, to detect quantum phase transitions in a perspective of quantum information science and quantum statistical mechanics.

Though we have given two examples to show that the adiabatic inverse temperature might diverge around the critical point, we should point out that the reversible adiabatic process might be destroyed at the singular point of the free energy, especially in the case that a symmetry breaking occurs. The singularity in the free energy usually occurs in three-dimensional systems. In this case, one can study the reversible adiabatic process only at one side of the critical point. In one- or two-dimensional systems, there is no symmetry-breaking phase transition at finite temperatures due to the Mermin–Wagner theorem [26]. In any case, according to the entropy majorization theorem, it is expected that the temperature should be significantly suppressed as the system tends adiabatically to the quantum critical point.

On top of all that we would like to emphasize here that the starting temperature should be low enough. For a quantum phase transition from one gapped phase to another gapped phase, the temperature should be comparatively lower than the gap. For the quantum phase transition discussed in Section 5, since the transition is related to the singularity in the density of state, the starting temperature in the non-polarized phase should not be larger than to excite the state around the singular point.

Finally, though it is very difficult to realize a reversible adiabatic process in an infinite system around its quantum critical point, the experiments based on finite systems, such as a 8-qubit (or 10, 12, 14, etc.) Ising chain, can still be used to check our results. In the latter cases, a maximum adiabatic inverse temperature is expected to be observed around the critical point. Then the exact critical point in the thermodynamic limit can be obtained with a scaling analysis on these maximum points.

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