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RESEARCH ARTICLE

Hawking radiation from a Vaidya black hole by Hamilton–Jacobi method

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Using the Hamilton–Jacobi method, Hawking radiation from the apparent horizon of a dynamical Vaidya black hole is calculated. The black hole thermodynamics can be built successfully on the apparent horizon. If a relativistic perturbation is given to the apparent horizon, a similar calculation can also lead to a purely thermal spectrum, which corresponds to a modified temperature from the former. The first law of thermodynamics can also be constructed successfully at a new supersurface which has a small deviation from the apparent horizon. When the event horizon is thought as such a deviation from the apparent horizon, the expressions of the characteristic position and temperature are consistent with the previous result that asserts that thermodynamics should be built on the event horizon. It is concluded that the thermodynamics should be constructed on the apparent horizon exactly while the event horizon thermodynamics is just one of the perturbations near the apparent horizon.

Keywords Vaidya black hole, Hawking radiation, apparent horizon, event horizon, thermodynamics

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1 Introduction

As is well known, Hawking radiation [1, 2] of black holes is exactly black body spectrum, which takes nothing out of the black hole. According to black hole thermodynamics [3, 4], the surface gravity is regarded as temperature and the area of event horizon is regarded as entropy. Thinking of a static or stationary black hole, the thermodynamics can be constructed on the event horizon successfully. Because the apparent horizon and the event horizon are coincident with each other in static or stationary circumstances, we cannot exactly tell where Hawking radiation comes from; apparent horizon or event horizon? Thinking of a dynamical Vaidya black hole, where does Hawking radiation come from after all? There are two main viewpoints in the previous works. Balbinot, Zhao, and Vagenas *et al.* [5–7], have proposed that Hawking radiation is from the event horizon. On the other hand, Fodor, Hajicek, Collins [8–10] considered that the radiation is from the apparent horizon.

In this paper, using Hamilton–Jacobi method [11, 12], we will investigate a dynamical Vaidya black hole, where the apparent horizon separates from the event horizon. After calculation, we have obtained that the first law of thermodynamics can be constructed successfully at the apparent horizon. In the meantime, considering a relativistic perturbation near the apparent horizon, we can also construct successfully the first law of thermodynamics on a new supersurface near the apparent horizon. The thermodynamics near the apparent horizon can be regarded as the perturbation from the apparent horizon. Keeping in mind the thermodynamics on the event horizon, we found that the event horizon exactly can be thought of as one of such perturbations.

The paper is organized as follows. In Section 2, we will introduce the Vaidya black hole in detail. In Section 3, Hamilton–Jacobi method will be used to calculate Hawking radiation from the apparent horizon of a Vaidya black hole. In Section 4, we will try to obtain Hawking radiation near the apparent horizon after perturbation. In the end, some conclusions and comments are given. We will use units $c = G = \hbar = \kappa_{\rm B} = 1$ throughout this paper.

2 The Vaidya black hole

The line element of a Vaidya black hole is as follows:

$$\mathrm{d}s^2 = -\left[1-\frac{2m(v)}{r}\right]\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{1}$$

where $m(v)$ is the mass of black hole and v is Eddington–Finkelstein advanced time coordinate.

It is easy to calculate the inverse matrix of $g_{\mu\nu}$ as:

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-2m(v)/r & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{2}$$

The event horizon r_{EH} satisfies the null supersurface condition

$$g^{\mu\nu}\frac{\partial f(r,v)}{\partial x^\mu}\frac{\partial f(r,v)}{\partial x^\nu} = 0 \tag{3}$$

Considering Eq. (3) and $f(r,v)\,|_{r_{\mathrm{EH}}} = 0$, we have

$$r_{\mathrm{EH}} = \frac{2m(v)}{1-2\dot{r}_{\mathrm{EH}}}, \quad \kappa\,|_{r_{\mathrm{EH}}} = \frac{(1-2\dot{r}_{\mathrm{EH}})^2}{4m(v)}, \quad T\,|_{r_{\mathrm{EH}}} = \frac{(1-2\dot{r}_{\mathrm{EH}})^2}{8\pi m(v)} \tag{4}$$

The time-like limit surface r_{TLS} satisfies

$$g_{vv} = -\left[1-\frac{2m(v)}{r}\right] = 0$$

so we get

$$r_{\mathrm{TLS}} = 2m(v) \tag{5}$$

The apparent horizon is the outermost marginally trapped surface, and it is the beginning of one-way space time area. According to the derivative geometry, it can be defined by the expansion

$$\Theta = l^\mu_{;\mu} - \kappa = 0 \tag{6}$$

where $\kappa = n^\mu l^\nu l_{\mu;\nu} = n^\mu l^\nu \left(l_{\mu,\nu} - \Gamma^\sigma_{\mu\nu} l_\sigma\right)$ is surface gravity of the horizon.

In order to calculate the apparent horizon, we need to choose a null tetrad frame as:

$$n_\mu = (1,0,0,0), \quad l_\mu = \left(\frac{1}{2}\left(1-\frac{2m}{r}\right), -1, 0, 0\right)$$
$$m_\mu = \sqrt{\frac{1}{2}}\,(0,0,r,\mathrm{i}r\sin\theta) \tag{7}$$
$$\overline{m}_\mu = \sqrt{\frac{1}{2}}\,(0,0,r,-\mathrm{i}r\sin\theta)$$

So the contravariant forms of the basis vectors are

$$n^\mu = (0,-1,0,0), \quad l^\mu = \left(1, \frac{1}{2}\left(1-\frac{2m}{r}\right), 0, 0\right)$$
$$m^\mu = \sqrt{\frac{1}{2}}r^{-1}\left(0,0,1,\frac{\mathrm{i}}{\sin\theta}\right) \tag{8}$$
$$\overline{m}^\mu = \sqrt{\frac{1}{2}}r^{-1}\left(0,0,1,-\frac{\mathrm{i}}{\sin\theta}\right)$$

It is obvious that they satisfy the condition of null tetrad frame

$$n_\mu n^\mu = l_\mu l^\mu = m_\mu m^\mu = \overline{m}_\mu \overline{m}^\mu = 0$$
$$n_\mu l^\mu = -m_\mu \overline{m}^\mu = 1$$
$$n_\mu m^\mu = n_\mu \overline{m}^\mu = l_\mu m^\mu = l_\mu \overline{m}^\mu = 0$$

Putting Eq. (7) and Eq. (8) into Eq. (6), we can obtain

$$r_{\mathrm{AH}} = 2m(v) \tag{9}$$

According to Ref. [8], the surface gravity at the apparent horizon is

$$\kappa\,|_{r_{\mathrm{AH}}} = \frac{1}{4m(v)} \tag{10}$$

3 Hamilton–Jacobi method

Thinking of a scalar particle moving in spacetime, under semi-classical approximation, the classical action I of the particle satisfies the relativistic Hamilton–Jacobi equation [11, 12]:

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0 \tag{11}$$

Usually, due to the symmetries of the metric, one is looking for a solution in the form

$$I = -Ev + W(r) + J(\theta,\varphi)$$

As a consequence, we have

$$\partial_v I = -E, \quad \partial_r I = W'(r), \quad \partial_\theta I = J_\theta, \quad \partial_\varphi I = J_\varphi \tag{12}$$

where J_θ and J_φ are constants (some of them may be chosen to be zero).

Putting Eq. (2) and Eq. (12) into Eq. (11), the Hamilton–Jacobi equation can be rewritten as:

$$\left[1-\frac{2m(v)}{r}\right]W'^2(r) - 2EW'(r) + r^{-2}J_\theta^2 + r^{-2}\sin^{-2}\theta J_\varphi^2 + m^2 = 0$$

so we get

$$W'(r) = \frac{E \pm \sqrt{E^2 - \left[1-\frac{2m(v)}{r}\right](r^{-2}J_\theta^2 + r^{-2}\sin^{-2}\theta J_\varphi^2 + m^2)}}{1-2m(v)/r} \tag{13}$$

Considering $1-2m(v)/r = 0$ near the apparent horizon and E is positive, we can only consider the "+" case in Eq. (13) as the "−" case is not meaningful, so we have

$$I = -Ev + \int \frac{2E}{1-2m(v)/r}\mathrm{d}r + J(\theta,\varphi)$$

The residue theorem tells us that

$$\mathrm{Im}\, I = \mathrm{Im}\, W = 4\pi m(v)E = 2\pi r_{\mathrm{AH}}E \tag{14}$$

and the semi-classical emission rate, including the leading term linear in E, reads

$$\varGamma \equiv \mathrm{e}^{-2\mathrm{Im}I} = \mathrm{e}^{-8\pi m(v)E} = \mathrm{e}^{-\beta E}$$

We can easily get Hawking temperature as: follows:

$$T_{\mathrm{AH}} = \frac{1}{\beta} = \frac{1}{8\pi m(v)} = \frac{\kappa}{2\pi} \tag{15}$$

4 Fluctuation around the apparent horizon

To investigate Hawking radiation from the supersurfaces near the apparent horizon $r' = r_{\mathrm{AH}}(1+\delta)$, where δ is an infinitesimal parameter, we define a new radial coordinate as follows:

$$R = r - \frac{1}{2}\delta \cdot v$$

so Eq. (1) can be changed into

$$\mathrm{d}s^2 = -\left[1 - \frac{2m(v)}{r} - \delta\right]\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{16}$$

So we get the new supersurface

$$r' = r_{\mathrm{AH}}(1+\delta)$$

If we use Hamilton–Jacobi method to calculate the classical action I of a scalar particle as Section 3, we obtain

$$\mathrm{Im}\, I = \mathrm{Im}\, W = \frac{4\pi m(v)E}{(1+\delta)^2}$$
$$\kappa' = \frac{(1+\delta)^2}{4m(v)}, \quad T' = \frac{(1+\delta)^2}{8\pi m(v)} \tag{17}$$

The area of the hypersurface r' can be regarded as the entropy of the black hole thermal system

$$S' = \frac{1}{4}A' = \pi r'^2 = \frac{4\pi m^2(v)}{(1+\delta)^2} \tag{18}$$

Obviously, they obey the first law of thermodynamics as:

$$T'\mathrm{d}S' = \frac{(1+\delta)^2}{8\pi m(v)}\frac{8\pi m(v)}{(1+\delta)^2}\mathrm{d}m(v) = \mathrm{d}m(v) \tag{19}$$

Therefore, we can construct thermodynamics on the new surface near the apparent horizon.

Letting $\delta = -2\dot{r}_{\mathrm{EH}}$, we have

$$r' = \frac{2m(v)}{1-2\dot{r}_{\mathrm{EH}}} = r_{\mathrm{EH}}, \quad \kappa' = \frac{(1-2\dot{r}_{\mathrm{EH}})^2}{4m(v)} = \kappa_{\mathrm{EH}}$$
$$T' = \frac{(1-2\dot{r}_{\mathrm{EH}})^2}{8\pi m(v)} = T_{\mathrm{EH}} \tag{20}$$

Apparently, they are the same with the thermodynamics on the event horizon [5, 6].

5 Conclusions and comments

Using the Hamilton-Jacobi method, we have calculated the emission rate from the apparent horizon of a Vaidya black hole. We found that the emission rate satisfies Boltzmann distribution $\mathrm{e}^{-\beta\omega}$ with inverse temperature $\beta = 8\pi m(v)$. In addition, the black hole thermodynamics can be founded successfully on the apparent horizon and this is consistent with the conclusions in previous works [8–10]. When considering a relativistic perturbation on the apparent horizon, we have concluded that black hole thermodynamics can also be constructed successfully on the new supersurface. If $\delta = 2\dot{r}_{\mathrm{EH}}$, the new supersurface under relativistic perturbation $r' = 2m(v)(1+\delta)$ is exactly the event horizon of a Vaidya black hole.

The event horizon thermodynamics is just one of the perturbations near the apparent horizon. In fact, Hawking radiation comes from the apparent horizon. For a Vaidya black hole, the apparent horizon is also the infinite red-shift surface. Therefore, we need to study more general and complicated cases, such as the Kerr–Newman black hole.

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