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RESEARCH ARTICLE

Different kinds of discrete breathers in a Sine–Gordon lattice

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We study a one-dimensional Sine–Gordon lattice of anharmonic oscillators with cubic and quartic nearest-neighbor interactions, in which discrete breathers can be explicitly constructed by an exact separation of their time and space dependence. DBs can stably exist in the one-dimensional Sine–Gordon lattice no matter whether the nonlinear interaction is cubic or quartic. When a parametric driving term is introduced in the factor multiplying the harmonic part of the on-site potential of the system, we can obtain the stable quasiperiodic discrete breathers and chaotic discrete breathers by changing the amplitude of the driver.

Keywords discrete breathers, quasiperiodic discrete breathers, chaotic discrete breathers, Sine–Gordon lattice

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1 Introduction

Over the last two decades, discrete breathers (DBs) (intrinsic localized modes) have attracted enormous attention in many areas of physics. They represent spatially localized and time periodic excitations in nonlinear Hamiltonian lattices. Rigorous existence proofs of DBs as strictly time periodic objects in networks of weakly coupled anharmonic oscillators can be found in Refs. [1–4]. Different aspects of these dynamical objects were analyzed in various physical systems by mathematical, numerical and experimental methods [5]. Various properties of DBs or ILMs have been addressed theoretically, i.e., discrete breathers in one-dimensional (1-D) mixed Klein–Gordon/Fermi–Pasta–Ulam chain were studied by Zhou *et al.* [6]. Because the DBs' dynamics are similar to impurity modes that can move in gap, the discrete gap breathers(DGBs) as a kind of localized modes in 1-D diatomic β–FPU chain and two-dimensional diatomic face-centered square lattice, their local characteristics and stability were presented by LÜ and Tian [7–11]. Moreover, to obtain exact DBs, various high precision numerical methods were developed. However, it is difficult to tune onto the exact breather solution in any physical and numerical experiment and therefore one should investigate the behavior of the solutions in the vicinity of a given DB rather than the behavior of the DB itself. Thus, there was a rapidly increasing interest in the investigation of the existence of the quasiperiodic and even chaotic discrete breathers in the nonlinear discrete system. Quasiperiodic discrete breathers (QDBs) that are quasiperiodic in time and localized in space have been reported in Refs. [12–14]. Chaotic discrete breathers (CDBs) were first indicated by Burlakov *et al.* [15], and the term CDBs was investigated by many people [16–19]. The most evident difference between a DB and a CDB is that the former is exactly periodic and mainly static, while the latter moves in an erratic (chaotic) way, mostly has a finite lifetime, and thus it was named as a "chaotic discrete breather".

The investigation of the behavior of QDBs and CDBs can play a significant role in dynamical systems. Some studies of QDBs and CDBs have been presented in Refs. [12–19]. In this paper, we restrict our attention to the evolution from DBs to QDBs, and eventually to CDBs. The rest of this paper is organized as follows. In Section 2, we introduce the basic equations of the motion for 1-D Sine-Gordon lattice with cubic and quartic nearest-neighbor interactions. When a parametric driving term is introduced in the factor multiplying the harmonic part of the on-site potential of the system, we can obtain the stable quasiperiodic discrete breathers and chaotic discrete breathers by changing the amplitude of the driver. Numerical results of different discrete breathers are obtained in this section. The paper is concluded by Section

3.

2 The model

We consider a 1-D Sine–Gordon lattice of coupled nonlinear oscillators with unit mass, without linear dispersion. It can be described by the Hamiltonian

$$H=\sum_n\left[\frac{1}{2}p_n^2+(1-\cos u_n)+W(u_n-u_{n-1})\right] \quad (1)$$

with $W(r)$ being the interaction potential

$$W(r)=\frac{k_3}{3}r^3+\frac{k_4}{4}r^4 \quad (2)$$

where $\dot{u}_n = p_n, \dot{p}_n = -\sin u_n - W'(u_n - u_{n-1}) + W'(u_{n+1}-u_n)$, k_3 and k_4 are the cubic parameter and quartic parameter, respectively. We point out the absence of quadratic coupling terms in Eq. (1) and hence the lack of linear dispersion in our model. The nonlinear vibration equation for this system is given as follows:

$$\begin{aligned}\ddot{u}_n &= -\sin u_n + k_3[(u_{n+1}-u_n)^2-(u_n-u_{n-1})^2]\\ &\quad +k_4[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3] \end{aligned}\quad (3)$$

The on-site anharmonic potential $V(u)=\sin u_n$ can be expanded at the lowest significant order as:

$$V(u)=\sin u_n = u_n-\frac{1}{3!}u_n^3+\frac{1}{5!}u_n^5+\cdots \quad (4)$$

so the equation of motion of the nth oscillator is thus given by

$$\begin{aligned}\ddot{u}_n &= -u_n+\frac{1}{6}u_n^3 + k_3[(u_{n+1}-u_n)^2-(u_n-u_{n-1})^2]\\ &\quad +k_4[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3] \end{aligned}\quad (5)$$

The DBs are time periodic and spatially localized excitations. If they exist in the system and they are not propagating, they have the forms as follows:

$$u_n(t)=(-1)^n\Phi_n G(t) \quad (6)$$

where Φ_n corresponds to the spatial profile of the solution and $G(t)$ describes its time evolution. It is shown that this system has a class of solutions which can be written as a product of a purely spatial (n-dependent) part and a function of time only. We seek approximate solutions for DBs via the local anharmonic approximation (LAA) and numerical method and discuss the numerical results under two cases.

2.1 Different kinds of DBs in the cubic-Sine–Gordon lattice ($k_4=0$)

Substituting Eq. (6) into Eq. (5) and writing it as an equality between an expression that depends only on spatial part n and one that is solely time dependent. Equating both of these expressions to a constant C, one thus finds that $G(t)$ satisfies the Duffing equation:

$$\ddot{G}+G+CG^2=0 \quad (7)$$

while the spatial part of the solution obeys

$$k_3[(\Phi_{n+1}+\Phi_n)^2+(\Phi_n+\Phi_{n-1})^2]+\frac{1}{6}\Phi_n^3-C\Phi_n=0 \quad (8)$$

We apply symmetric excited pattern and asymmetric excited pattern by using numerical calculation. The numerical results of Eq. (8) can be shown in Fig. 1.

Figure 1 shows the symmetric and antisymmetric DBs profiles of the 1-D cubic-Sine–Gordon lattice. The exact breather solutions are then calculated as the products of these profiles with the periodic solution of Eq. (7) having the desired frequency ω_b. For $C=1$, Eq. (7) has periodic solutions for all initial displacements $G(0)$ and $\dot{G}(0)$. We can solve Eq. (7) by the numerical method, and the phase plane of this Duffing equation and the evolution of $G(t)$ with time are shown in Fig. 2.

From Fig. 2(a) we know that Eq. (7) has a periodic solution, and from Fig. 2(b) we know that the periodic solution can stably exist for a longer time. Hence, the 1-D DBs can stably exist in the 1-D cubic-Sine–Gordon lattice.

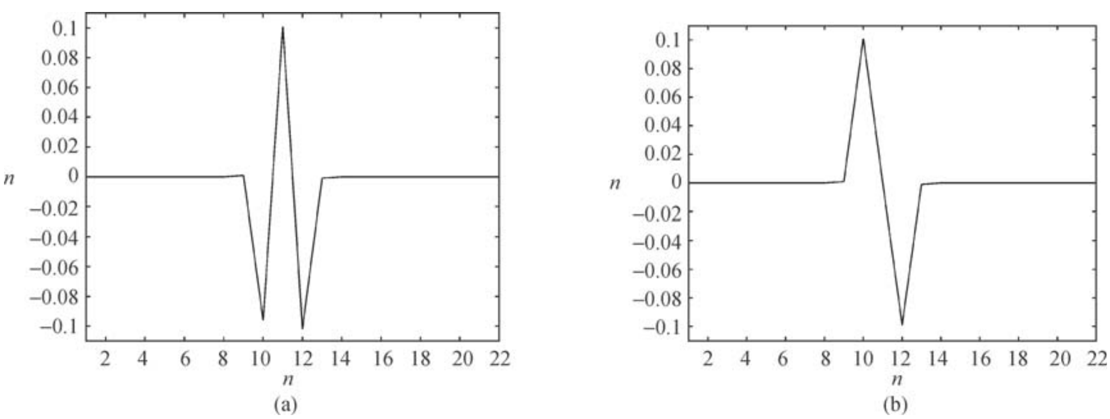

Fig. 1 Breather profiles Φ_n for $C=0.1, k_3=0.01$. **(a)** Symmetric DB; **(b)** Antisymmetric DB.

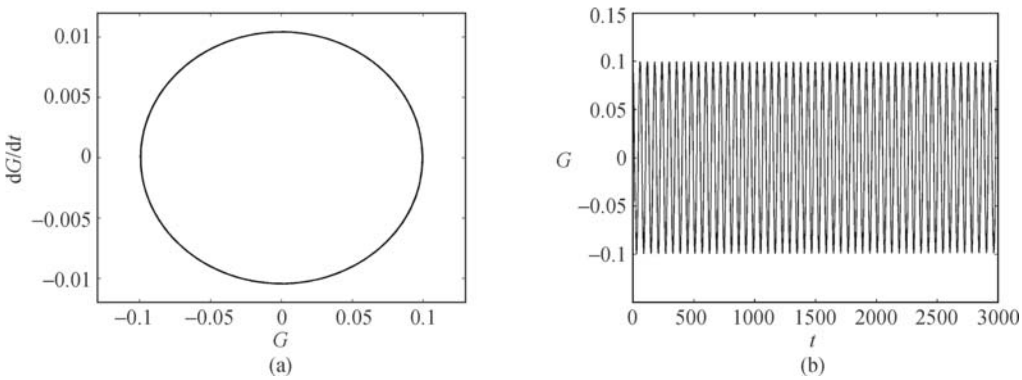

Fig. 2 Numerical results of Eq. (7) for $C = 1$. **(a)** The phase portrait of the periodic solution of Eq. (7); **(b)** The solution curve of Eq. (7).

In order to obtain the evolution from DBs to QDBs, and eventually to CDBs, we introduce a parametric driving term in the factor multiplying the harmonic part of the on-site potential of all oscillators. Equation (5) can be rewritten as:

$$\ddot{u}_n = -[1 - \lambda \cos(\omega_d t)]u_n + \frac{1}{6}u_n^3 + k_3[(u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2] + k_4[(u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3] \tag{9}$$

where λ and ω_d represent the amplitude and frequency of the driver. This type of parametric driving can be experimentally implemented, and it does not affect the mathematical property of having solutions of type (6). Thus, separating again the space and time-dependent parts as before, the Duffing equation (7) now becomes

$$\ddot{G} + [1 - \lambda \cos(\omega_d t)]G + CG^2 = 0 \tag{10}$$

while the spatial of the solution can again be constructed independently by solving the identical Eq. (8). Thus we can study breather solutions of our lattice by selecting the phase planes of $G(t)$ and $\dot{G}(t)$ with the different amplitudes of the driver and combining the corresponding solution $G(t)$ with the spatial profile Φ_n given by Eq. (8). Figure 3 shows the results of the evolution of the QDBs with time, and from Fig. 3(b) we can conclude that the QDBs can stably exist for very long periods of time in this system.

Certainly, this system is a very particular case with no linear dispersion, so there are no resonances between QDBs and normal modes and QDBs can exist for very long periods of time, because the resonances can lead to a radiation of energy outside their core when certain super positions of incommensurate frequencies, corresponding to these QDBs, get inside of the phonon band. Naturally, this process causes a decay of the QDBs as localized dynamical objects. However, in many cases, this decay can be weak enough and, therefore QDBs can exist for a very long time. If some excitations exist during a very long time, they must have a certain physical meaning and should be worth thorough investigation. The QDBs represent an example of such dynamical objects and, actually they are much more relevant entities than strictly time periodic breathers.

The evolution from DBs to CDBs is shown in Fig. 4.

From Fig. 4 we can find that the behavior of breather solutions changes with the variation of the driving parameter λ. We obtain the evolution from DBs to CDBs with the increasing λ.

2.2 Different kinds of DBs in the quartic-Sine–Gordon lattice ($k_3 = 0$)

The same as the above section, substituting Eq. (6) into

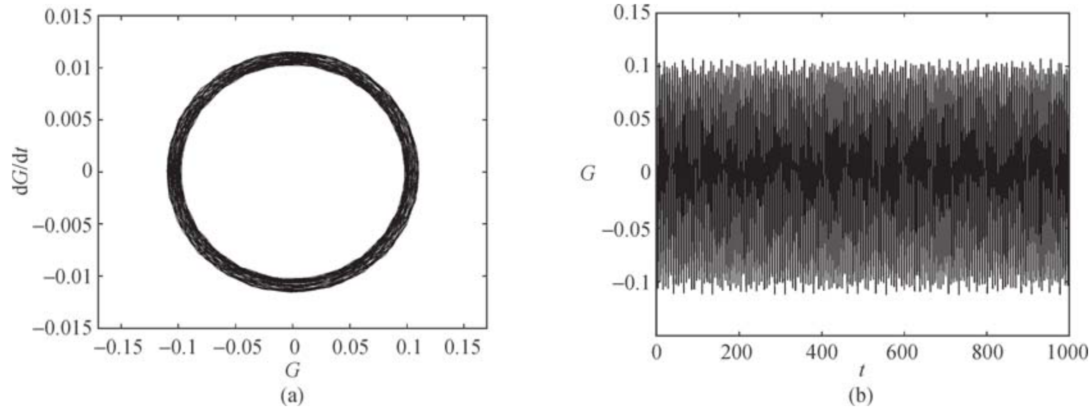

Fig. 3 Phase portrait of the evolution from DBs to QDBs with the amplitude of the driver $\lambda = 0.5$ and the driving frequency $\omega_d = 2.28$. **(a)** The phase portrait of the periodic solution of Eq. (10); **(b)** The solution curve of Eq. (10).

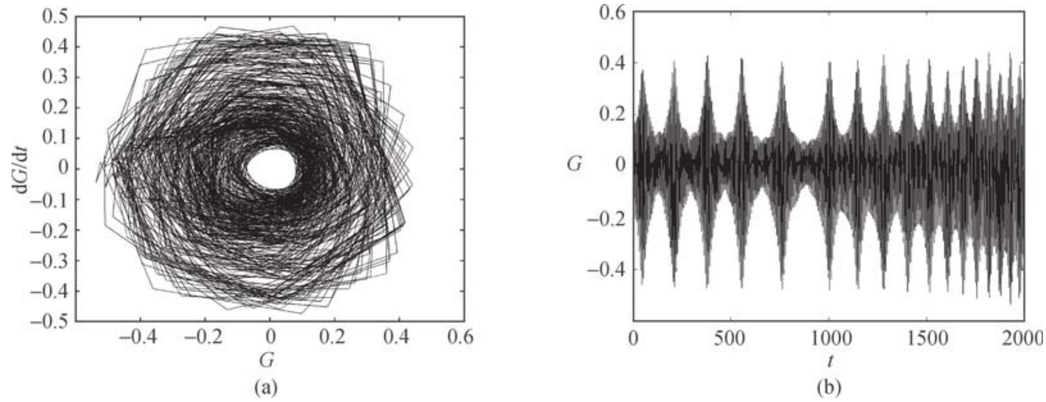

Fig. 4 Phase portrait of the evolution from DBs to CDBs with the amplitude of the driver $\lambda = 0.9$ and the driving frequency $\omega_d = 2.28$. **(a)** The phase portrait of the periodic solution of Eq. (10); **(b)** The solution curve of Eq. (10).

Eq. (5) and writing it as an equality between an expression that depends only on spatial part n and one that is solely time dependent. $G(t)$ satisfies the Duffing equation

$$\ddot{G} + G + CG^3 = 0 \tag{11}$$

while the spatial part of the solution obeys

$$k_4[(\Phi_{n+1} + \Phi_n)^3 + (\Phi_n + \Phi_{n-1})^3] + \frac{1}{6}\Phi_n^3 - C\Phi_n = 0 \tag{12}$$

We still apply symmetric excited pattern and asymmetric excited pattern by using numerical calculation. The numerical results of Eq. (12) can be shown in Fig. 5.

Figure 5 shows the symmetric and antisymmetric DBs profiles of the 1-D quartic-Sine–Gordon lattice. The exact breather solutions are then calculated as the products of these profiles with the periodic solution of Eq. (11) having the desired frequency ω_b. For $C = 1$, Eq. (11) has periodic solutions for all initial displacements $G(0)$ and $\dot{G}(0)$. We can solve Eq. (11) by the numerical method, and the phase plane of this Duffing equation and the evolution of $G(t)$ with time are shown in Fig. 6.

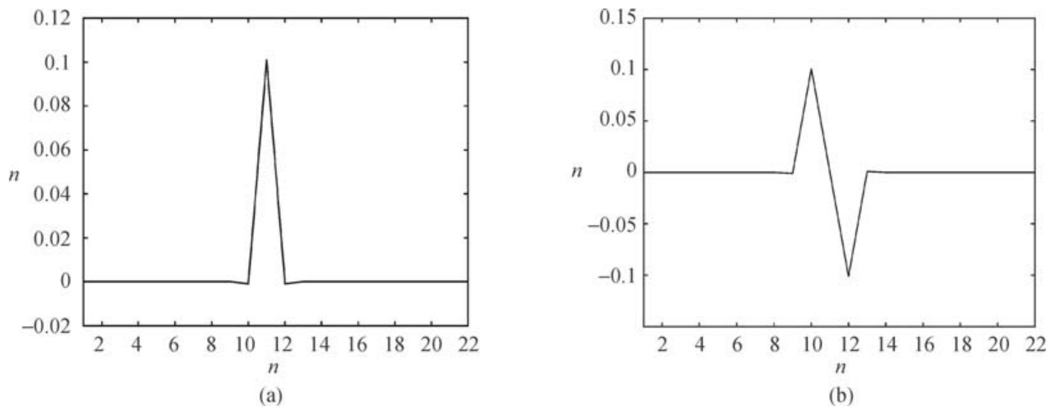

Fig. 5 Breather profiles Φ_n for $C = 0.1, k_4 = 0.01$. **(a)** Symmetric DB; **(b)** Antisymmetric DB.

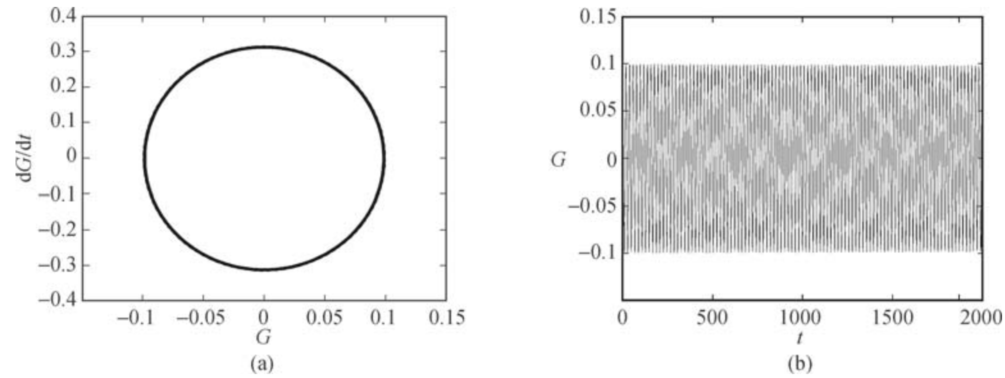

Fig. 6 Numerical results of Eq. (11) for $\boldsymbol{C} = 1$. **(a)** The phase portrait of the periodic solution of Eq. (11); **(b)** The solution curve of Eq. (11).

From Fig. 6(a) we know that Eq. (11) has a periodic solution, and from Fig. 6(b) we know that the periodic solution can stably exist for a longer time. Hence, the 1D DBs can stably exist in the 1-D quartic-Sine–Gordon lattice.

In order to obtain the evolution from DBs to QDBs, and eventually to CDBs, we introduce a parametric driving term in the factor multiplying the harmonic part of the on-site potential of all oscillators. Equation (12) can be rewritten as:

$$\ddot{u}_n = -[1-\lambda\cos(\omega_d t)]u_n + \frac{1}{6}u_n^3 + k_3[(u_{n+1}-u_n)^2 - (u_n-u_{n-1})^2] + k_4[(u_{n+1}-u_n)^3 - (u_n-u_{n-1})^3] \tag{13}$$

where λ and ω_d represent the amplitude and frequency of the driver. This type of parametric driving can be experimentally implemented, and it does not affect the mathematical property of having solutions of type (6). Thus, separating again the space and time-dependent parts as before, the Duffing equation (11) now becomes

$$\ddot{G} + [1-\lambda\cos(\omega_d t)]G + CG^3 = 0 \tag{14}$$

while the spatial of the solution can again be constructed independently by solving the identical Eq. (12). Thus, we can study breather solutions of our lattice by selecting the phase planes of $G(t)$ and $\dot{G}(t)$ with the different amplitudes of the driver and combining the corresponding solution $G(t)$ with the spatial profile Φ_n given by Eq. (12). Figure 7 shows the results of the evolution of the QDBs with time, and from Fig. 7(b) we can conclude that the QDBs can stably exist for very long periods of time in this system, because Fig. 7(b) shows that the QDBs are still stable at $t = 2000$.

From Fig. 8 we can find that the behavior of breather solutions changes with the variation of the driving parameter λ. By comparing the above two cases, we obtain that there are symmetric and antisymmetric DBs in this model and no matter whether in cubic Sine–Gordon lattice or quartic Sine-Gordon lattice, the DBs evolve into QDBs, and eventually to CDBs with the increasing driving parameter λ.

3 Summary

In this paper, we mainly investigate the possibility of different kinds of DBs in a 1-D Sine–Gordon lattice which are analyzed for two particular examples of cubic-Sine–Gordon lattice and quartic-Sine–Gordon lattice. Numerical results for the approximate DBs, the generation and the evolution of DBs in the 1-D Sine-Gordon model are presented in this passage; we prove their existence by using LAA and the numerical method. We attain the

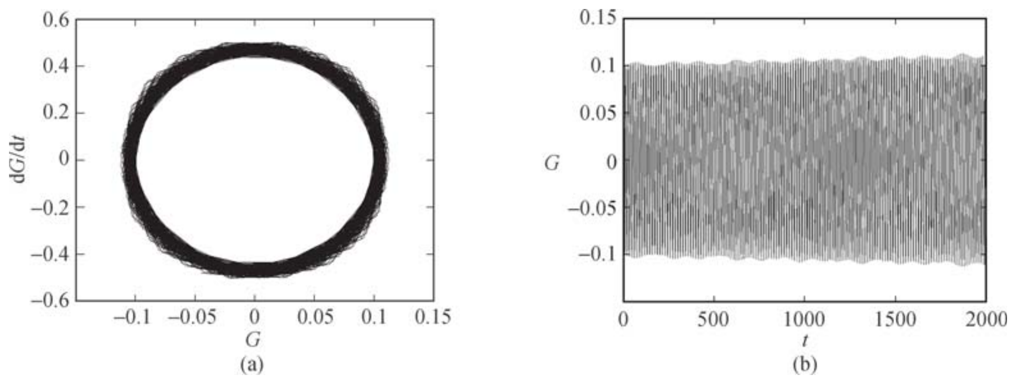

Fig. 7 Phase portrait of the evolution from DBs to QDBs with the amplitude of the driver $\lambda = 0.5$ and the driving frequency $\omega_d = 2.28$. **(a)** The phase portrait of the periodic solution of Eq. (14); **(b)** The solution curve of Eq. (14). The evolution from DBs to CDBs is shown in Fig. 8.

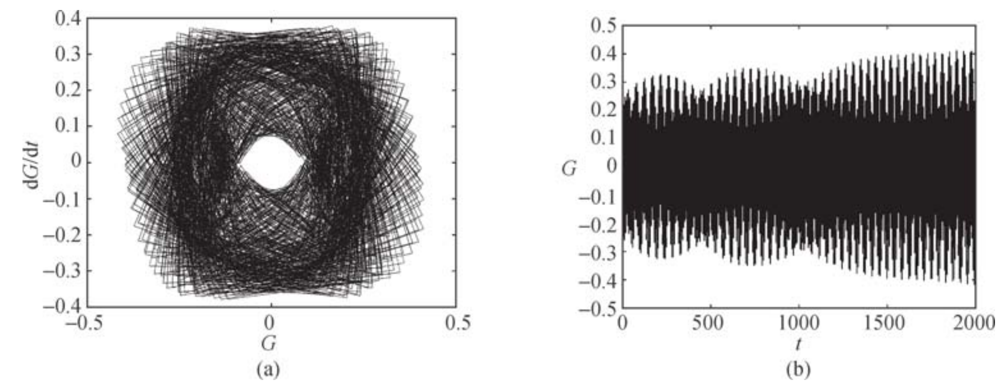

Fig. 8 Phase portrait of the evolution from DBs to CDBs with the amplitude of the driver $\lambda = 0.9$ and the driving frequency $\omega_d = 2.28$. **(a)** The phase portrait of the periodic solution of Eq. (14); **(b)** The solution curve of Eq. (14).

conclusions that there are symmetric and antisymmetric DBs in this model. In order to obtain the evolution from DBs to QDBs, eventually to CDBs, we introduce a parametric driving term in the factor multiplying the harmonic part of the on-site potential of all oscillators. The parametric driving term plays an important role in the numerical results and the resulting dynamics consists of three stages. First, when $\lambda = 0$, the solutions are stable DBs. Second, when $\lambda = 0.5$, the mode breaks up into a number of breather-like structures that are stable QDBs. Third, when $\lambda = 0.9$, these structures coalesce into one large unstable structure, called a CDB. Since a single large CDB closely approximates a stable breather, the final decay stage can be very slow, thus we can obtain QDBs and CDBs by changing the parameter of the driver and then control the characteristics of the nonlinear dynamic system. This case is given credit to the property of the Duffing equation. In addition, this process can be done in real experiments, so it has great significance for studying the nonlinear dynamic system.

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