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REVIEW ARTICLE

Multi-linear variable separation approach to nonlinear systems

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The multi-linear variable separation approach is reviewed in this article. The method has been recently established and successfully solved a large number of nonlinear systems. One of the most exciting findings is that the basic multi-linear variable separation solution can be expressed by a universal formula including two (1+1)-dimensional functions, and at least one is arbitrary for integrable systems. Furthermore, the method has been extended in two different ways so as to enroll more low dimensional functions in the solution.

Keywords multi-linear variable separation approach (MLVSA), multi-linear variable separation solution (MLVSS), universal formula

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Contents

1 Introduction

Nonlinear phenomena in almost every branch of natural science have been studied extensively and intensively, where it is unavoidable to deal with nonlinear models instead of linear ones. With the development of nonlinear science, many useful analytical and numerical methods have been proposed and established. In the middle of the 19th century, the Fourier transformation method, one of the most important methods for solving linear problems, was successfully extended to nonlinear systems, namely, the inverse scattering transformation (IST) method [1, 2], which hence raised the climax of researches in nonlinear science. Recently, the IST method has received some further progress [3 – 6]. At the same time, people were longing to extend the other powerful method – variable separation approach – to nonlinear science. However, the extension of the variable separation approach is much more difficult, and thus there had been no substantial progress for quite a long period. Recently, some types of nonlinear variable separation approaches have been constructed such as the nonlinearization method [7, 8] (or the formal variable separation approach [9]), the functional variable separation approach [10 – 12], the derivative dependent variable separation approach [13, 14] and the multi-linear variable separation approach [15, 16].

In this article, we will review the multi-linear variable separation approach (MLVSA). The original idea of the MLVSA was first proposed early in 1996 by Lou and his colleague for the Davey-Stewartson (DS) equation [15]. However, no further progress had been achieved ever since until six years later when Lou and his student applied this idea to the DS equation again [16] and succeeded in finding a way to solve many other (2+1)-dimensional integrable systems. One of the important results is that the basic multi-linear variable separation solution (MLVSS) can be expressed by a universal formula:

$$U \equiv \frac{-2\Delta q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}$$
$$\Delta \equiv a_0 a_3 - a_1 a_2 \tag{1}$$

which is valid for suitable fields or potentials of a nonlinear model. In Eq. (1), $\{x,\ y\}$ are space variables, the subscripts denote derivatives, $p \equiv p(x,t)$ is an arbitrary function of the indicated variables, $q \equiv q(y,t)$ may be either an arbitrary function for some models such as the DS system, the Nizhnik-Novikov-Veselov (NNV) system and the two-dimensional sine-Gordon (2DsG) system, or an arbitrary solution of a Riccati equation (or heat conduction equation) for some other systems, a_0, a_1, a_2 and a_3 are constants (which can also be functions of time t). However, for nonintegrable systems, neither p nor q is arbitrary [17].

Just after the derivation of the universal formula (1), one question was put forward immediately that whether it was possible to include more variable separation functions in an exact solution. In order to answer this question, two ways were then proposed which are the extensions of the basic MLVSA. One is achieved by extending the variable separation assumption for the expansion function. The other is fulfilled by expanding the solution form about two functions instead of one to obtain a couple of multi-linear equations. It is remarkable that the extended variable separation solution obtained in this way can be treated as a conditional addition of two basic MLVSSs.

It is also worth mentioning another important and interesting finding that the universal formula (1) and its extended forms (see below) can work as "soliton factories" which can produce a variety of nonlinear excitations, such as dromion, solitoff, curved line and surface soliton, soliton lattice, ring soliton, peakon, compacton, foldon, chaotic soliton, fractal soliton, bubble soliton, tire soliton, ghost soliton, and so on, and various soliton interaction modes, such as soliton fission and fusion, soliton reconnection and reflection, and so on [16, 18, 19].

This article is organized as follows. In Section 2, we first briefly introduce the basic procedure of the MLVSA, and then apply it to the (2+1)-dimensional dispersive long-wave system (2DLWS) as an illustration. In Section 3, we discuss the first type of the generalized MLVSA which is also applied to the 2DLWS as an example. In Section 4, the second type of the generalized MLVSA is presented and then illustrated for a modified NNV system. Section 5 contains a short summary and discussion.

2 The basic multi-linear variable separation approach and its applications

For a general (2+1)-dimensional nonlinear system

$$F(u, u_t, u_x, u_y, ...) = 0 \tag{2}$$

where u is a function of $\{x,\ y,\ t\}$, F is a polynomial function of u and its derivatives with respect to space and time variables, the basic MLVSA can be carried out through the following five steps:

(1) Multi-linearize the nonlinear system

The Painlevé analysis of Eq. (2) discovers that its solution can be written as:

$$u = u'(f, f_x, ...) + u_0 \tag{3}$$

where $u' \equiv u'(f, f_x, ...)$ is constituted by the expansion function $f \equiv f(x, y, t)$ and its derivatives with respect to x, y, t, and u_0 is a seed solution remaining as arbitrary as possible. It should be mentioned that the function u' is a homogeneous function which has the property $u'(\lambda f) = u'(f)$ for arbitrary constant λ [see Eq. (11) for example]. The substitution of Eq. (3) into Eq. (2) leads to a multi-linearized equation with respect to f:

$$F'(f, f_t, f_x, f_y, ...) = 0 \tag{4}$$

due to u' is homogeneous with respect to f. The function F' should possess the property,

$$F'(\lambda f, \lambda f_t, \lambda f_x, \lambda f_y, ...) = \lambda^n F'(f, f_t, f_x, f_y, ...) \tag{5}$$

with n being an integer for all arbitrary constant λ. We call Eq. (5) as bilinear, trilinear and k-linear if $n = 2,\ 3$ and k respectively.

(2) Make a variable separation assumption

The assumption for the expansion function f can be obtained by Darboux transformation, or by generalizing the usual two-soliton solution. The basic form reads

$$f = a_0 + a_1 p + a_2 q + a_3 pq \tag{6}$$

where p and q are functions of $\{x, t\}$ and $\{y, t\}$, respectively, a_i $(i = 0, 1, 2, 3)$ are constants (which can also be functions of time t). It is remarkable that when p and q are both set as exponential functions, the assumption (6) just reduces to the usual two-soliton solution expression for the KdV type systems. The substitution of the assumption (6) into Eq. (5) gives

$$G(p, p_t, p_x, q, q_t, q_y, ...) = 0 \tag{7}$$

(3) Separate variables in Eq. (7)

Now we need to separate functions p and q into two separate equations:

$$G_1(p, p_t, p_x, ...) = 0, \quad G_2(q, q_t, q_y, ...) = 0 \tag{8}$$

If a function (the seed solution u_0 or any arbitrary integration function) is of variables $\{x, t\}$, then it must be in G_1, otherwise in G_2.

(4) Solve the variable separated equations G_1 and G_2

It is still very difficult to solve the variable separated equations for any given seed solution. Fortunately, the intrusion of the seed solution as general as possible makes it easy to solve the complicated equation of p (or q) alternatively. Sometimes it even becomes an easy algebra problem. That is, to view p (or q) as arbitrary and then to solve the seed solution u_0 instead.

(5) Obtain the universal formula

After the above four steps, it is found that almost all the multi-linear variable separable models have a universal formula (1) in common, where at least one of p and q is a two-dimensional arbitrary function for integrable systems.

Here, we take the (2+1)-dimensional dispersive long-wave system (2DLWS) which reads

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0 \tag{9}$$

$$v_t + (uv + u + u_{xy})_x = 0 \tag{10}$$

as an illustrative example [20].

It is not difficult to find the following truncated expansion form of the solution of Eqs. (9) and (10):

$$u = 2(\ln f)_x + u_0(x,t), \quad v = 2(\ln f)_{xy} - 1 \tag{11}$$

through the Painlevé truncation expansion, where $u = u_0(x,t)$, an arbitrary function of the indicated variables, and $v = -1$ is a seed solution of Eqs. (9) and (10). The substitution of Eq. (11) into Eqs. (9) and (10) results in a degenerated tri-linearized equation:

$$\begin{aligned}&(f_{xyt} + u_{0x}f_{xy} + u_0 f_{xxy} + f_{xxxy})f^2\\ &-[(f_{xxy} + 2u_0 f_{xy} + f_{yt} + u_{0x}f_y)f_x\\ &+(u_0 f_{xx} + f_{xt} + f_{xxx})f_y + (f_{xx} + f_t)f_{xy}]f\\ &+2f_x f_y(u_0 f_x + f_{xx} + f_t) = 0\end{aligned} \tag{12}$$

Now, we substitute the basic assumption (6) into Eq. (12), and then separate the obtained equation into two parts:

$$p_t = -p_{xx} - u_0 p_x + (a_1a_2 - a_0a_3)(c_2p^2 + c_1p + c_0) \tag{13}$$

and

$$\begin{aligned}q_t = &-c_2(a_0 + a_2q)^2 + c_1(a_1 + a_3q)(a_0 + a_2q)\\ &-c_0(a_1 + a_3q)^2\end{aligned} \tag{14}$$

It is clear that Eq. (13) depends on the variables x and t, while Eq. (14) is only of the variables y and t. Now we are in the position to solve these two equations. As indicated in the fourth step, alternatively, we can solve u_0 from Eq. (13) rather than p. The result is

$$u_0 = -p_x^{-1}[p_t + p_{xx} - (a_1a_2 - a_0a_3)(c_2p^2 + c_1p + c_0)] \tag{15}$$

Since Eq. (14) is a Riccati equation, its solution can be easily obtained.

Finally, the substitution of the assumption (6) into Eq. (11) gives the multi-linear variable separation solution [20]:

$$u = \frac{2(a_1 + a_3q)p_x}{a_0 + a_1p + a_2q + a_3pq} + u_0 \tag{16}$$

$$v = \frac{2(a_0a_3 - a_1a_2)p_xq_y}{(a_0 + a_1p + a_2q + a_3pq)^2} - 1 \tag{17}$$

with p being an arbitrary function of $\{x,t\}$, u_0 determined by Eq. (15) and q by Eq. (14). It is seen that the quantity $\eta \equiv -(v+1)$ given by Eq. (17) is rightly the universal formula (1).

By using this method, many (2+1)-dimensional mathematical physical models have been solved, such as the NNV equation, the asymmetric NNV (ANNV) equation, the asymmetric DS (ADS) equation, the dispersive long wave equation (DLWE), the Broer–Kaup–Kupershmidt (BKK) system, the higher order BKK system, the non-integrable (2+1)-dimensional KdV equation, the long wave-short wave interaction model (LWSWIM), the Maccari system, the Burgers equation, the 2DsG system, the general $(N+M)$-component AKNS system, and so on [16, 19, 21–28] (and references therein). In addition, the method has also been applied to several (1+1)-dimensional systems such as the negative KdV hierarchy, the Ito system, the shallow water wave equations, the long-wave-short-wave resonant interaction equation [29], etc., (3+1)-dimensional systems such as the Burgers equation [30], the JM equation [31], etc., and differential difference systems such as a (2+1)-dimensional special Toda equation [32], a (2+1)-dimensional differential-difference asymmetric NNV equation [33], a (1+1)-dimensional differential-difference Toda-like equation [34], etc. It is discovered that the MLVSS for differential-difference systems share a similar form (1) with the only difference that p (or q) being a difference function.

3 The first type of generalized multi-linear variable separation approach

The basic MLVSA has been extended in two different ways which could include more low dimensional functions and give more general variable separation solutions. In this section, we review the first type of general MLVSA realized by extending the basic variable separation assumption (6). The method has been applied to the DLWE, the BKK system, the higher order BKK system and the Burgers equation, and so on [35−37].

As a comparison, we still use the 2DLWS (9) and (10) as an illustration. Now, to solve the trilinear equation

(12), we extend the basic assumption (6) to be

$$f = q_0 + \sum_{i=1}^{N} p_i q_i \tag{18}$$

where $\{q_i,\ i = 0, 1, 2, ..., N\}$, and $\{p_i,\ i = 1, 2, ..., N\}$ are functions of $\{y, t\}$ and $\{x, t\}$, respectively. It is easy to find that the extended variable separation ansatz (18) will return back to the basic one (6), when $N = 3,\ q_0 = a_0,\ q_1 = a_1,\ q_2 = a_2 q,\ q_3 = a_3 q,\ p_1 = p_3 = p$, and $p_2 = 1$. Substituting the ansatz (18) into the trilinear equation (12), and then separating the obtained equation gives

$$q_{it} = \sum_{j=0}^{N} (c_{ij} + q_i C_j) q_j, \quad i = 0, 1, ..., N \tag{19}$$

$$p_{it} = (c_{00} - u_0 \partial_x - \partial_x^2) p_i - c_{0i} + \sum_{j=1}^{N} (c_{j0} p_i - c_{ji}) p_j, \quad i = 1, 2, ..., N \tag{20}$$

where $\{c_{ij}, C_j,\ i, j = 0, 1, 2, ..., N\}$ are arbitrary functions of t.

Finally, we obtain the general variable separation solution:

$$u = \pm \frac{2 \sum_{i=1}^{N} p_{ix} q_i}{q_0 + \sum_{i=1}^{N} p_i q_i} + u_0 \tag{21}$$

and

$$v (\equiv -\eta - 1) = \frac{-2 \sum_{i=1}^{N} p_{ix} q_{iy}}{q_0 + \sum_{i=1}^{N} p_i q_i} + \frac{2 \sum_{i=1}^{N} p_{ix} q_i \left(q_{0y} + \sum_{j=1}^{N} p_j q_{jy} \right)}{\left(q_0 + \sum_{i=1}^{N} p_i q_i \right)^2} \equiv U_E \tag{22}$$

It is seen that in addition to one (1+1)-dimensional arbitrary function of $\{x, t\}$ (one of u_0 and p_i), $(N + 1)(N + 2) - 1$ arbitrary functions of t, c_{ij}, and C_j have been included in the general solution (21) and (22). Furthermore, various arbitrary functions of y and $\{x, t\}$ will appear in the solution (22) after the coupled systems of Eqs. (19) and (20) are solved. However, it is rather difficult to solve the coupled system. Fortunately, one can obtain some results for some simple cases. For instance, the following simplest nontrivial case has been utilized to construct abundant nonlinear coherent structures [35]. That is, fixing $N = 1$, $c_{ij} = C_i = 0$, $p_1 = p$, and $q_0 \to a_0 + q_0$, the formula (22) is simplified to

$$v = \frac{2 p_x \left(q_1 q_{0y} - (a_0 + q_0) q_{1y} \right)}{(a_0 + q_0 + p q_1)^2} \equiv V \tag{23}$$

with q_0 and q_1 being arbitrary functions of y, and p being an arbitrary function of $\{x,\ t\}$. It has been proven that the simplified quantity expressed by Eq. (23) does work for many other known models which allow the universal formula (1).

4 The second type of generalized multi-linear variable separation approach

The second type of general MLVSA has been achieved by expanding the solution by about two functions with each assumed to have the basic variable separation assumption form (6). The method has been applied to nonlinear systems such as the modified NNV (MNNV) system and the 2DsG system [35].

As an illustration, we take the MNNV system

$$u_t + u_{xxx} + u_{yyy} + \sigma^2 u_x^3 + \sigma^2 u_y^3 + 3 u_x v_{xx} + 3 u_y v_{yy} = 0 \tag{24}$$

$$v_{xy} + \sigma^2 u_x u_y = 0 \tag{25}$$

which is a main member of the MNNV hierarchy associated with the generalized Lamé system [38]. By means of Painlevé analysis, it is found that the MNNV system has the following Bäcklund transformation:

$$u = \pm \sigma \ln \frac{f}{g} \tag{26}$$

$$v = -\ln(fg) + v_1 + v_2 \tag{27}$$

where $f,\ g$ are two expansion functions of variables $\{x, y, t\}$, $v_1 \equiv v_1(x, t)$, $v_2 \equiv v_2(y, t)$, and $\{0, v_1 + v_2\}$ is a seed solution of the MNNV system. The substitution of Eqs. (26) and (27) into Eqs. (24) and (25) leads to a system of two bilinear equations:

$$(D_t + D_x^3 + D_y^3 + 3 v_{1xx} D_x + 3 v_{2yy} D_y) f \cdot g = 0 \tag{28}$$

$$\sigma D_x D_y f \cdot g = 0 \tag{29}$$

where $D_x,\ D_y,\ D_t$ are the Hirota's bilinear operators [39].

Now, we take two expansion functions f and g both similar to the basic assumption (6), i.e.,

$$f = a_0 + a_1 p + a_2 q + a_3 pq \tag{30}$$

$$g = b_0 + b_1 p + b_2 q + b_3 pq \tag{31}$$

with $p \equiv p(x, t)$, $q \equiv q(y, t)$, and constants a_i, b_i $(i = 0, ..., 3)$. Then substituting Eqs. (30) and (31) into Eqs. (28) and (29) can yield

$$\frac{q_t + q_{yyy} + 3 v_{2yy} q_y}{(a_3 b_2 - a_2 b_3) q^2 + 2(a_3 b_0 - a_2 b_1) q + a_1 b_0 - a_0 b_1}$$

$$= -\frac{p_t + p_{xxx} + 3v_{1xx}p_x}{(a_3b_1 - a_1b_3)p^2 + 2(a_3b_0 - a_1b_2)p + a_2b_0 - a_0b_2} \quad (32)$$

under the condition

$$b_3a_0 + a_3b_0 - a_1b_2 - a_2b_1 = 0 \quad (33)$$

Obviously, from Eq. (32), we can obtain two variable separated equations

$$p_t + p_{xxx} + 3v_{1xx}p_x = -c(t)[(a_1b_3 - a_3b_1)p^2 + 2(a_1b_2 - a_3b_0)p + a_0b_2 - a_2b_0] \quad (34)$$

$$q_t + q_{yyy} + 3v_{2yy}q_y = c(t)[(a_3b_2 - a_2b_3)q^2 + 2(a_3b_0 - a_2b_1)q + a_1b_0 - a_0b_1] \quad (35)$$

where $c(t)$ is an arbitrary function of t. Similarly, it is still very difficult to solve Eqs. (34) and (35) for any fixed v_1 and v_2. Alternatively, one can solve v_1 and v_2 from Eqs. (34) and (35) in terms of p and q, respectively,

$$v_{1xx} = -\frac{1}{3p_x}\{p_t + p_{xxx} + c(t)[(a_3b_1 - a_1b_3)p^2 + 2(a_3b_0 - a_1b_2)p + a_2b_0 - a_0b_2]\} \quad (36)$$

$$v_{2yy} = -\frac{1}{3q_y}\{q_t + q_{yyy} - c(t)[(a_3b_2 - a_2b_3)q^2 + 2(a_3b_0 - a_2b_1)q + a_1b_0 - a_0b_1]\} \quad (37)$$

Finally, the variable separation solution of the MNNV system is obtained

$$u = \pm\sigma \ln \frac{a_0 + a_1p + a_2q + a_3pq}{b_0 + b_1p + b_2q + b_3pq} \quad (38)$$

$$v = -\ln[(a_0 + a_1p + a_2q + a_3pq)(b_0 + b_1p + b_2q + b_3pq)] + v_1 + v_2 \quad (39)$$

where the functions v_1 and v_2 are determined by arbitrary functions p and q via Eqs. (36) and (37), respectively, under the constant relation (33).

Remarkably, it is interesting to consider the potentials $F(\equiv -2u_{xy}/\sigma)$ and $G(\equiv -2v_{xy})$,

$$F = \pm\left\{\frac{2(a_1a_2 - a_0a_3)p_xp_y}{(a_0 + a_1p + a_2q + a_3pq)^2} - \frac{2(b_1b_2 - b_0b_3)p_xp_y}{(b_0 + b_1p + b_2q + b_3pq)^2}\right\} \equiv \pm(U_a - U_b) \quad (40)$$

$$G = \frac{2(a_1a_2 - a_0a_3)p_xp_y}{(a_0 + a_1p + a_2q + a_3pq)^2} + \frac{2(b_1b_2 - b_0b_3)p_xp_y}{(b_0 + b_1p + b_2q + b_3pq)^2} \equiv U_a + U_b \quad (41)$$

where U_a and U_b are just the universal quantity (1) with different parameters. It is revealed from Eqs. (40) and (41) that two universal terms U_a and U_b (each is a solution of the MNNV system) can be combined linearly for the potentials F and G with a necessary condition (33) and the opposite sign of the term U_b.

5 Summary and discussion

In summary, we have reviewed the newly developed multi-linear variable separation approach and its two extended versions. All have been successfully applied to many nonlinear integrable and non-integrable systems in (1+1), (2+1), and (3+1) dimensions, as well as differential-difference equations.

One fundamental finding is that low dimensional functions can exist in the multi-linear variable separation solution and its extended expressions. Furthermore, at least one of them is arbitrary for integrable systems. In this sense, we have proposed a sense of integrability, namely, MLVSA solvability.

Another magnificent finding is that the universal formula and its extended expressions can be used to construct a variety of nonlinear coherent structures even for the multi-valued case, which have been mentioned above. In addition, solitons with chaotic behavior and fractal pattern are also obtained, which indicates that chaos and fractal can enter the integrable systems through characteristic lines or boundaries. Many types of interaction properties for solitary waves and solitons are also discovered.

It is noted that the equivalent variable separation solution has also been obtained by other methods, such as the Darboux transformation method for the NNV [40], ANNV [41], and MNNV [42] systems, the entangled mapping approach and/or the modified extended tanh-function method for (1+1)-dimensional systems like the coupled integrable dispersionless and shallow water wave equations, (1+1)-dimensional ones like the generalized NNV, DS, BKK, etc., and (3+1)-dimensional Burgers system, and so on [43, 44].

As an extension of the linear variable separation method in the nonlinear case, it is meaningful to see whether the method can be used to solve more systems, and to see whether the method can be further extended to obtain more new solutions.

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References

1. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett., 1967, 19: 1095

2. M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge: Cambridge University Press, 1991
3. A. Boutet de Monvel, A. S. Fokas, and D. Shepelsky, Lett. Math. Phys., 2003, 65: 199
4. A. S. Fokas, Nonlinearity, 2004, 17: 1521
5. A. S. Fokas and A. R. Its, J. Phys. A: Math Gen., 2004, 37: 6091
6. A. Boutet de Monvel, A. S. Fokas, and D. Shepelsky, Comm. Math. Phys., 2006, 263: 133
7. C. W. Cao, Sci. China Ser. A, 1990, 33: 528
8. Y. Cheng and Y. S. Li, Phys. Lett. A, 1991, 175: 22
9. S. Y. Lou and L. L. Chen, J. Math. Phys., 1999, 40: 6491
10. K. S. Chou and C. Z. Qu, J. Phys. A: Math. Gen., 1999, 32: 6271
11. C. Z. Qu, S. L. Zhang, and R. C Liu., Physica D, 2000, 144: 97
12. P. G. Estevez, C. Z. Qu, and S. L. Zhang, J. Math. Anal. Appl., 2002, 275: 44
13. S. L. Zhang, S. Y. Lou, and C. Z. Qu, J. Phys. A: Math. Gen., 2003, 36: 12223
14. S. L. Zhang and S. Y. Lou, Physica A, 2004, 335: 430
15. S. Y. Lou and J. Z. Lu, J. Phys. A: Math. Gen., 1996, 29: 4209
16. X. Y Tang., S. Y. Lou, and Y. Zhang, Phys. Rev. E, 2002, 66: 046601
17. X. Y. Tang and S. Y. Lou, Commun. Theor. Phys., 2002, 38: 1
18. X. Y. Tang, J. M. Li, and S. Y. Lou, Phys. Scr., 2007, 75: 201
19. S. Y. Lou and X. Y. Tang, Nonlinear Mathematical Physics Methods, Beijing: Science Press, 2006
20. X. Y. Tang, C. L. Chen, and S. Y. Lou, J. Phys. A: Math. Gen., 2002, 35: L293
21. X. Y. Tang, K. W. Chow, and S. Y. Lou, J. Phys. A: Math. Gen., 2005, 38: 10361
22. X. Y. Tang, Phys. Lett. A, 2003, 314: 286
23. S. Y. Lou, C. L. Chen, and X. Y. Tang, J. Math. Phys., 2002, 43: 4078
24. A. Maccaria, Phys. Lett. A, 2005, 336: 117
25. A. Maccaria, Chaos, Solitons and Fractals, 2006, 27: 363
26. A. Maccaria, J. Math. Phys., 2008, 49: 022702
27. B. B. Thomas, K. K. Victor, and K. T. Crepin, J. Phys. A: Math. Theor., 2008, 41: 135208
28. A. M. Wazwaz, Appl. Math. Comput., 2008, 204: 817
29. J. F. Zhang, C. Q. Dai, C. Z. Xu, J. P. Meng, and X. J. Lai, Phys. Lett. A, 2006, 352: 511
30. J. P. Ying and S. Y. Lou, Chin. Phys. Lett., 2003, 20: 1448
31. X. Y. Tang and Z. F. Liang, Phys. Lett. A, 2006, 351: 398
32. X. M. Qian, S. Y. Lou, and X. B. Hu, J. Phys. A: Gen. Math., 2004, 37: 2401
33. X. M. Qian, S. Y. Lou, and X. B Hu., Z. Naturforsch., 2004, 59a: 645
34. S. F. Shen, Phys. Lett. A, 2007, 365: 210
35. X. Y. Tang and S. Y. Lou, J. Math. Phys., 2003, 44: 4000
36. X. Y. Tang and S. Y. Lou, Commun. Theor. Phys., 2003, 40: 62
37. X. Y. Tang and S. Y. Lou, Chin. Phys. Lett., 2003, 3: 335
38. W. K. Schief, Proc. R. Soc. London Ser. A, 1997, 453: 1671
39. R. Hirota, Phys. Rev. Lett., 1971, 27: 1192
40. H. C. Hu, S. Y. Lou, and Q. P. Liu, Chin. Phys. Lett., 2003, 20: 1413
41. H. C. Hu, X. Y. Tang, S. Y. Lou, and Q. P. Liu, Chaos, Soltions and Fractals, 2004, 22: 327
42. H. C. Hu and S. Y. Lou, Chin. Phys. Lett., 2004, 21: 2073
43. C. Q. Dai, C. J. Yan, and J. F. Zhang, Commun. Theor. Phys., 2006, 46: 389
44. C. Q. Dai and J. F. Zhang, J. Math. Phys., 2006, 47: 043501