Front. Phys. China, 2009, 4(2): 225–230
DOI 10.1007/s11467-009-0011-0
RESEARCH ARTICLE

Finite thickness lens model for self-focusing (defocusing) in Kerr medium

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Received November 1, 2008; accepted December 1, 2008

A "finite thickness lens" model for self-focusing (defocusing) in Kerr medium is presented. An on-axis normalization transmittance formula is presented for arbitrary nonlinear phase shift for the finite thickness Kerr medium by introducing a nonlinear $ABCD$-matrix for the transition of a Gaussian beam from linear to nonlinear medium, without complex calculation for the beam radius at the far field aperture. The variation of the peak and valley transmittance difference is found to enhance linearly as the phase shift at the focus increases by increasing the thickness of the medium. If the ratio of the Rayleigh distance divided by the thickness of the medium (d/z_0) is constant and small enough, the peak and valley transmittance difference stays constant. Finally, a qualitative formula is presented to express the relationship between the system parameters and the on-axis phase shift at the focus.

Keywords finite thickness lens model, $ABCD$ matrix, transmittance

PACS numbers 78.66.-w, 78.67.Hc, 78.90.+t

1 Introduction

Self-focusing in a thin film Kerr medium is a well-known nonlinear optical phenomenon and has been extensively studied experimentally and theoretically [1–13]. Based on the aberration-less theory of self-focusing, in which the beam is assumed to maintain its Gaussian profile during propagation in the Kerr medium, a nonlinear $ABCD$ matrix has been proposed to deal with the Gaussian beam propagation through a nonlinear thin medium. Then, the effect of the Kerr medium can be interpreted as due to a thin lens and propagation over a negative distance [14].

However, the nonlinear sample in most reported experiments is not so thin, that is, the thickness of the sample is finite, not close to thin, mostly from 0.1 mm to 5 mm [15–19].

In this paper we introduce a new "finite thickness lens" model by introducing a nonlinear $ABCD$-matrix for the transition of a Gaussian beam to describe the effects of self-focusing (defocusing) in finite thickness Kerr medium. In comparison with the reported "thin-lens" model, this new model is simpler and more suitable to the practical sample. We consider more factors in calculating the on-axis transmittance function in z-scan such as the following: the medium is not so thin that the complex radius of curvature q at the input surface and the output surface are different from each other; and the thickness d of the sample cannot be neglected in the illation of the focal length F, that is, the elements of the nonlinear matrix are dependent on the thickness of the sample and the beam parameters.

2 Theoretical method

We now consider an optical system made of a Kerr medium of refractive index n and a Gaussian beam with the beam waist w_0 at $z = 0$ and the Rayleigh distance z_0 given by $z_0 = \pi w_0^2/\lambda$, with λ (632.8 nm) the wavelength of the beam, used to implement the z-scan technique. As shown in Fig. 1, the nonlinear sample, with thickness d, is modeled as a lens of focal length F located at a distance z. The photo-detector, with a small aperture, is located at a distance L (see Fig. 1), and that the different elements of the optical set up does not change the Gaussian distribution, thus we can describe the propagation of the beam using the $ABCD$ law. Assuming only that axis intensity is detected and the photo-detector is set at the far field, that is $L \gg z_0$.

In the following, we are going to obtain the form of F

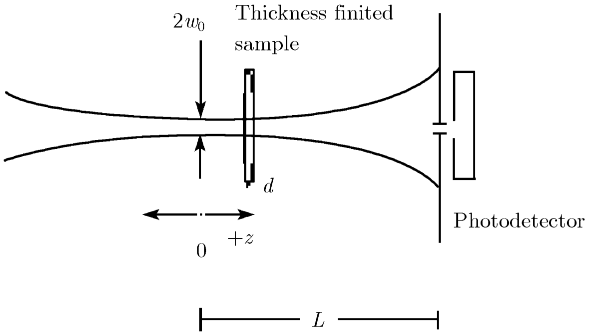

Fig. 1 Optical system of z-scan. The sample is Kerr medium such as semiconductor quantum dots, and the thickness is finite.

for a Kerr medium of thickness d, with refractive index

$$n = n_0 + n_2 I \tag{1}$$

where, n_0 and n_2 are the linear and nonlinear refractive index, respectively, and I is the intensity of the Gaussian beam. Considering that this sample is illuminated by a Gaussian beam, and in the parabolic approximation, this refractive index can be written with a quadratic radial dependence [1]:

$$n \approx n_0 + \frac{2n_2P}{\pi w^2} - \frac{4n_2P}{\pi w^4}r^2 \tag{2}$$

where P is the total power, $w = w(z)$ is the beam radius,$w(z) = w_0[1 + (z/z_0)^2]^{1/2}$, and $r$is the radial coordinate. The Kerr medium with this type of refractive index, in Fig. 2, is known as lens-like and therefore the material has an associated $ABCD$ matrix of the form [20].

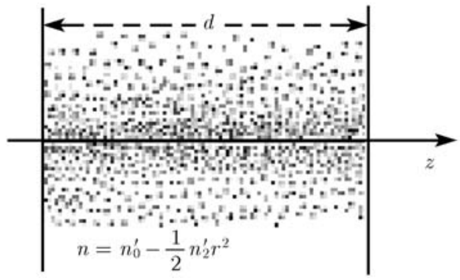

Fig. 2 Kerr medium with the type of refractive index $n = n'_0 - \frac{1}{2}n'_2r^2$, with thickness d.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\gamma d) & \sin(\gamma d)/n'_0\gamma \\ -n'_0\gamma\sin(\gamma d) & \cos(\gamma d) \end{pmatrix} \tag{3}$$

where, $\gamma^2 = n'_2/n'_0, n'_2 = \dfrac{8n_2P}{\pi\omega^4}$ and $n'_0 = n_0 + \dfrac{2n_2P}{\pi\omega^2}$.

From the theory of $ABCD$ matrix, the focal length of the system is given by $F = -1/C$. Thus, for a Kerr sample with thickness d, the focal length is given by

$$F = \frac{1}{\left[\left(n_0+\dfrac{2n_2P}{\pi w^2}\right)\dfrac{8n_2P}{\pi w^4}\right]^{1/2}\sin\left[\left(\dfrac{8n_2P}{\pi w^4\left(n_0+\dfrac{2n_2P}{\pi w^2}\right)}\right)^{1/2}d\right]} \tag{4}$$

In the following, we are going to get the transmittance function T by introducing the focal length into the function.

First of all, assume that the complex radius of curvature of the input surface is q_i. Meanwhile, the elements of the matrix A, B, C and D can be expressed by F. Thus, the nonlinear transformation for the lens-like medium can be written as:

$$\begin{pmatrix} A & B \\ C & C \end{pmatrix} = \begin{pmatrix} \sqrt{1-\dfrac{1}{F^2n'_0n'_2}} & \dfrac{1}{Fn'_0n'_2} \\ -\dfrac{1}{F} & \sqrt{1-\dfrac{1}{F^2n'_0n'_2}} \end{pmatrix} \tag{5}$$

Through the finite thickness lens with focal length F, the output ray then goes through a linear distant L from the output surface to the far field where it meets the aperture. The linear $ABCD$ matrix for this process is

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 1 & L/n_v \\ 0 & 1 \end{pmatrix} \tag{6}$$

where $n_v = 1$ is the refraction index in the free space. At the plane of the aperture the $ABCD$ matrix is

$$\begin{aligned} \begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix} &= \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} A+CL & B+LD \\ C & D \end{pmatrix} \end{aligned} \tag{7}$$

On the one hand, the complex function which describes the radius of curvature $R(z)$ at the aperture is described by [21]

$$q_a(z,F) = \frac{A_Tq_i + B_T}{C_Tq_i + D_T} \tag{8}$$

By all appearances, q_i can be written as $q_i = (z - d/2) + z_0\mathrm{i}$, as $z_0\mathrm{i}$ is the complex radius of curvature at the waist of the Gaussian beam.

On the other hand, function $q_a(z, F)$ is given by [21]

$$\frac{1}{q_a} = \frac{1}{R_a} - \frac{\lambda\mathrm{i}}{\pi n_v\omega_a^2} \tag{9}$$

where R_a and ω_a are the radius of curvature and the beam radius at the aperture. Note that, as we consider the thickness of the lens, the input surface locates at $z - d/2$, not z[22].

The inverted Eq. (8) can be directly compared to Eq. (9). Then, we substitute Eq. (7) into Eq. (8), rationalize the imaginary expression of Eq. (8) using $(x+\mathrm{i}y)^{-1} = (x-\mathrm{i}y)(x^2+y^2)^{-1}$. Comparing the imaginary part of the resulting expression with that of Eq. (9), we can write the imaginary part as:

$$-\frac{\lambda \mathrm{i}}{\pi n_v \omega_a^2} = \mathrm{Im}\left(\frac{1}{q_a}\right)_{(z,F)} = \frac{\left[(A+CL)\left(z-\frac{d}{2}\right)+(B+DL)\right]Cz_0 - \left[C\left(z-\frac{d}{2}\right)+D\right](A+CL)z_0}{\left[(A+CL)\left(z-\frac{d}{2}\right)+(B+DL)\right]^2+(A+CL)^2 z_0^2} \tag{10}$$

3 Results and discussion

3.1 Transmittance function

In this section we will use the imaginary part of the complex radius of curvature calculated above to calculate the final expression for the on-axis transmittance coefficient T.

For the on axis condition, we can easily obtain the final expression for the transmittance coefficient T determined as the ratio of the output light intensity to that in the similar configuration without nonlinear sample (or with the sample with negligible low optical nonlinearity, i.e., with $F \to \infty$) [23]:

$$T = \frac{1/\omega_a^2}{1/\omega_{a(F\Rightarrow\infty)}^2} \tag{11}$$

Substitute Eq. (10) into Eq. (11), we get

$$T = \frac{\mathrm{Im}\dfrac{1}{q_a}}{\mathrm{Im}\dfrac{1}{q_{a(F\Rightarrow\infty)}}} = \frac{\dfrac{\left[(A+CL)\left(z-\frac{d}{2}\right)+(B+DL)\right]Cz_0 - \left[C\left(z-\frac{d}{2}\right)+D\right](A+CL)z_0}{\left[(A+CL)\left(z-\frac{d}{2}\right)+(B+DL)\right]^2+(A+CL)^2 z_0^2}}{\dfrac{-z_0}{\left(z-\frac{d}{2}+L\right)^2+z_0^2}} \tag{12}$$

where A, B, C and D are the elements of the combined "$ABCD$" matrix in Eq. (5). Eq. (12) is the normalization transmittance formula we calculated above for finite thickness medium.

First, let us extend the formula to the thin medium, i.e., $d \to 0$. We get the expression of the photo-induced focal length F as:

$$F = \frac{\pi w^4}{8 n_2 d P} \tag{13}$$

which is the same to Eq. (9) in Ref. [1]. By substituting Eq. (13) into Eq. (5), and then substituting Eq. (5) into Eq. (12), we get the expression of the transmittance function of thin ($d \to 0$) medium as:

$$T = \frac{(z+L)^2+z_0^2}{(z+L)^2+z_0^2-\dfrac{2L(zL+z^2+z_0^2)}{F}} \tag{14}$$

This can be easily reduced to a more simple equation in the far field approximation ($L \gg z, z_0$):

$$T = \frac{1}{1-2z/F} \tag{15}$$

For weak nonlinearity approximation $F \gg z_0$, when $|z| \ll z_0$, Eq. (15) takes the form

$$T = 1 + \frac{2z}{F} \tag{16}$$

Eq. (16) is the same as Eq. (12) in Ref. [1], which means Eq. (12) is also fit for the condition of thin medium. As the nonlinear phase shift is not restricted during our calculation above, we can apply Eq. (12) to both small and large nonlinear phase shift, shown in Fig. 3 and Fig. 4, respectively. In this paper, we define $x = z/z_0$. From Fig. 3, we can see that, for small nonlinear phase shift, the peak and valley occur at the same distance with respect to focus, which is much the same as result of the Gaussian decomposition method given by Weaire *et al.* [6, 24]. For large nonlinear phase shift, this symmetry no

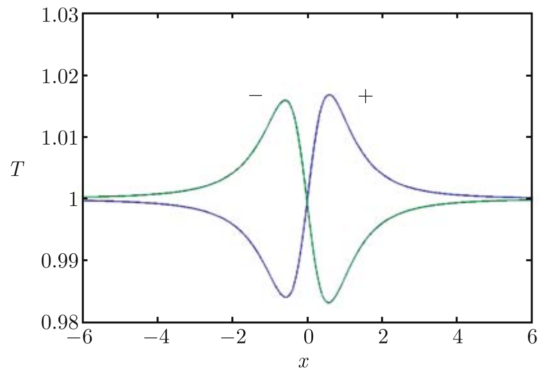

Fig. 3 Calculated closed-aperture z-scan transmittance curve for purely nonlinear refractions, using the finite thickness lens model presented in this paper, with small thickness d = 20 μm, P = 10 mW, z_0 = 5 cm, and a little on-axis phase shift at the focus ($\Delta\Phi_0 = \pm 0.255$).

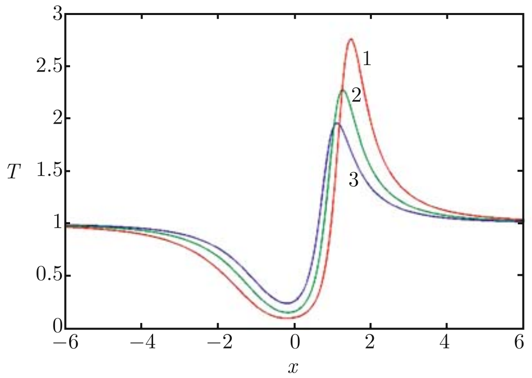

Fig. 4 Calculated closed-aperture z-scan transmittance curve for purely nonlinear refractions, using the finite thickness lens model presented in this paper, with small thickness d = 2 μm, P = 10 mW. Curve 1, 2, and 3, correspond to z_0 = 3 μm, 5 μm and 7 μm respectively, and on-axis phase shift at the focus $\Delta\Phi_0 = 0.87$, 2.26, and 5.15 respectively.

longer holds and with increasing nonlinear phase shifts, the valley of the transmittance is severely suppressed (nearly to zero of the transmittance) and the peak is greatly enhanced, as shown in Fig. 4, which is similar to Fig. 1 (b) in Ref. [10] which applied the large-phase-shift z-scan theory proposed by Kwak *et al.* [10].

In summary, we confirm that the computational methods provide an adequate description of the "finite thickness lens" model, which lends us confidence to carry out the following analysis. In summary, we confirm that the computational methods provide an adequate description of the "finite thickness lens" model, which lends us confidence to carry out the following analysis.

3.2 Finite thickness media

Here we define the range of finite thickness d from 0.1 mm to 10 mm. A mass of calculated curves show that, for small $\Delta\Phi_0$, the peak and valley occur at the same distance with respect to focus. With larger phase distortions ($\Delta\Phi_0 > 1$), numerical calculation (see in Fig. 5) shows that this symmetry no longer holds and peak

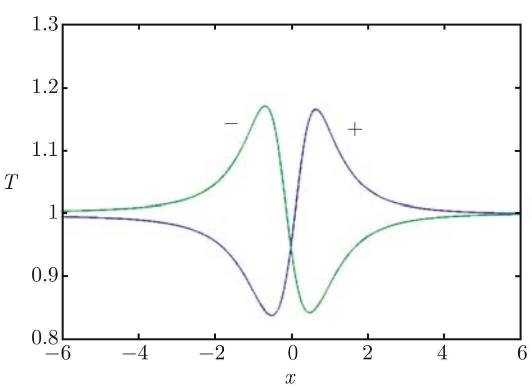

Fig. 5 Calculated closed-aperture z-scan transmittance curve for purely nonlinear refractions, using the finite thickness lens model presented in this paper, with P = 10 mW, z_0 = 5 cm, finite thickness d = 2 mm and a larger on-axis phase shift at the focus ($\Delta\Phi_0 = \pm 2.55$).

and valley both move toward $\pm z$ for the corresponding sign of non-linearity ($\pm\Delta\Phi_0$) such that their separation remains nearly constant.

Now, let us look into the relationship between the system parameters and the peak and valley transmittance difference $\Delta T_{p\text{-}V}$ in the finite thickness lens model.

Let us start from the relationship between the phase shift at the focus $\Delta\Phi_0$ and the peak and valley transmittance difference $\Delta T_{p\text{-}V}$ by increasing the thickness of the medium from 0.1 mm to 4.8 mm, shown in Fig. 6. It exhibits some useful features. First, for a given order of nonlinearity, it can be considered universal. In other words, they are independent of the laser wavelength and the sign of nonlinearity, as long as the far field condition is met. Second, for on-axis condition, the variation of $\Delta T_{p\text{-}V}$ is found to be almost linearly dependent on $|\Delta\Phi_0|$.

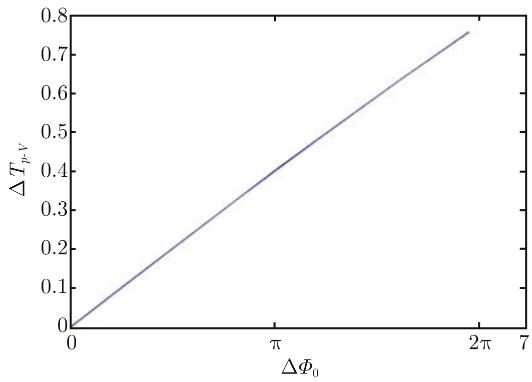

Fig. 6 Calculated $\Delta T_{p\text{-}V}$ as a function of the phase shift at the focus $\Delta\Phi_0$, with constant Rayleigh distance z_0 = 4.96 cm, $w_0 = 10^{-4}$ m, $n_0 = 1.3$, $n_2 = 10^{-11}$, P = 10 mW, L = 2 m and d from 0.1 mm to 4.8 mm.

Second, it is apparent from relations derived so far that a way to obtain larger z-scan signals ($\Delta T_{p\text{-}V}$) is to increase $\Delta\Phi_0$ through either thicker samples (larger d) or stronger focusing (shorter z_0), shown in Fig. 7 and Fig. 8, respectively.

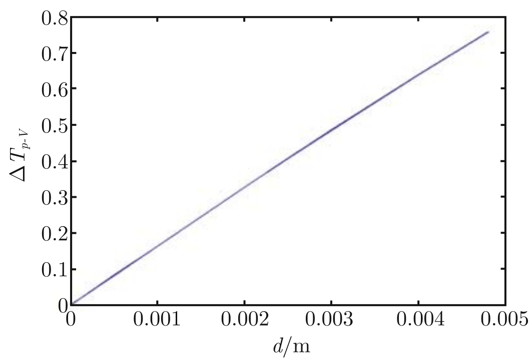

Fig. 7 Calculated $\Delta T_{p\text{-}V}$ as a function of thickness d of the medium, with constant Rayleigh distance z_0 = 4.96 cm. The sensitivity, as indicated by the slope of the curves, decrease slowly for thicker medium.

Finally, let us keep the ratio d/z_0 constant and small enough, shown in Fig. 9. The peak and valley transmittance difference stays constant as the thickness d increases. Thus, it is reasonable for us to define the effective length of the medium as $L_{\rm eff}$, $L_{\rm eff} = \frac{d}{z_0}$. That is, for finite thickness medium, Eq. (12) can be applied by increasing the Rayleigh distance appropriately. Numerical fitting indicates that the following qualitative formula can be used to express the relationship between the system parameters and the on-axis phase shift at the focus:

$$\Delta \Phi_0 = \frac{2n_2 Pdk}{\pi w_0^2 z_0} \tag{17}$$

Eq. (17) is equivalent to Eq. (6) in Ref. [6] in the condition of the aberration-less approximation.

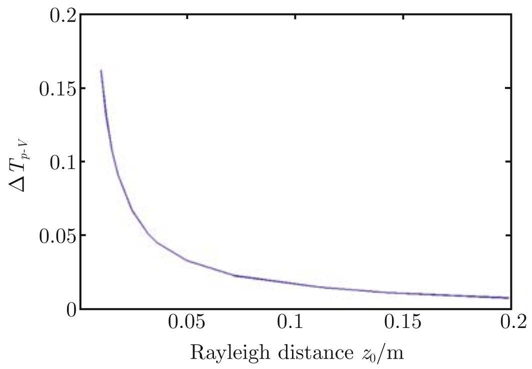

Fig. 8 Calculated $\Delta T_{p\text{-}V}$ as a function of the Rayleigh distance z_0, which increases by increasing w_0, and, with constant thickness $d = 0.2$ mm.

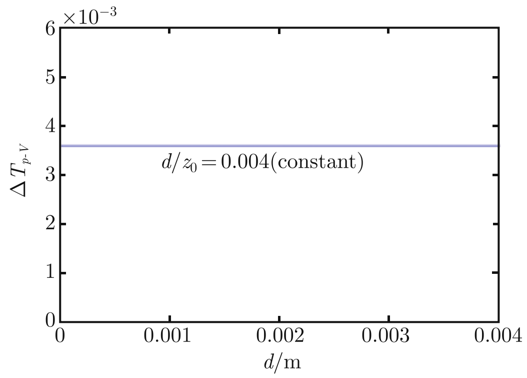

Fig. 9 Calculated $\Delta T_{p\text{-}V}$ as a function of thickness of the medium d, with d/z_0=constant (0.004).

4 Conclusion

A "finite thickness lens" model for self-focusing (defocusing) in finite thickness Kerr medium is presented. This model provides a simple calculation of the on-axis transmittance function in z-scan which is very complicated to be calculated by other models. An on-axis normalization transmittance formula is presented for arbitrary nonlinear phase shift for finite thickness Kerr medium by introducing nonlinear $ABCD$-matrix for the transition of a Gaussian beam from linear to nonlinear medium. This formula meets the analysis solution in the condition of weak nonlinearity approximation for thin medium in Ref. [1] when we extend the thickness d to zero.

By analyzing the function T, we found that the variation of the peak and valley transmittance difference enhances linearly as the phase shift at the focus increases by increasing the thickness of the medium. If the ratio of the Rayleigh divided by the thickness of the medium (d/z_0) is constant and small enough, the peak and valley transmittance difference stays constant, which means that we can treat finite thickness medium as thin medium as long as we keep d/z_0 to be an appropriate small value. Finally, a qualitative formula is presented to express the relationship between the system parameters and the on-axis phase shift at the focus.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 10574011).

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