

Vitaly N. MELNIKOV

# Models of $G$ time variations in diverse dimensions

© Higher Education Press and Springer-Verlag 2009

**Abstract** A review of different cosmological models in diverse dimensions leading to a relatively small time variation in the effective gravitational constant  $G$  is presented. Among them: the 4-dimensional (4-D) general scalar-tensor model, the multidimensional vacuum model with two curved Einstein spaces, the multidimensional model with the multicomponent anisotropic “perfect fluid”, the  $S$ -brane model with scalar fields and two form fields, etc. It is shown that there exist different possible ways of explaining relatively small time variations of the effective gravitational constant  $G$  compatible with present cosmological data (e.g. acceleration): 4-dimensional scalar-tensor theories or multidimensional cosmological models with different matter sources. The experimental bounds on  $\dot{G}$  may be satisfied either in some restricted interval or for all allowed values of the synchronous time variable.

**Keywords** multidimensional cosmological models, scalar-tensor theories, gravitational constant, acceleration of the Universe, dark energy, variations of  $G$

**PACS numbers** 04.50.-h, 04.80.Cc, 06.20.Jr, 95.36.+x

## Contents

1	Introduction	75
2	Scalar-tensor cosmology and variations of $G$	78
3	$G$ -dot in (1+3+ $N$ )-D cosmology with multicomponent anisotropic fluid	80

3.1	The model	80
3.2	Cosmological parameters	80
3.3	The vacuum model with two Einstein spaces	80
3.4	The model with two Ricci-flat spaces and two-component fluid	82
4	Multidimensional cosmology with anisotropic fluid	82
4.1	The model	82
4.2	Solutions with power-law and exponential dependence of scale factors	83
4.2.1	Solutions with power-law behavior	83
4.2.2	Solutions with exponential behavior	83
4.3	Acceleration and variation of $G$	84
4.3.1	Power-law expansion with acceleration	84
4.3.2	Exponential expansion with acceleration	87
5	$S$ -brane solution with acceleration and small variation of $G$	88
5.1	The model	88
5.2	Solutions with acceleration	89
5.3	Example	90
6	Conclusions	90
	Acknowledgements	91
	References	91

Vitaly N. MELNIKOV<sup>1,2</sup> (✉)

<sup>1</sup> Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya Str., Moscow 119361, Russia

<sup>2</sup> Institute of Gravitation and Cosmology, People’s Friendship University of Russia, 6 Mikhlukho-Maklaya Str., Moscow 117198, Russia  
E-mail: melnikov@phys.msu.ru, melnikov@vniims.ru

Received October 8, 2008; accepted November 2, 2008

## 1 Introduction

In the development of relativistic gravitation and dynamical cosmology after A. Einstein and A. Friedmann, there are three distinct stages recognized: first, investigation of models with matter sources in the form of a perfect fluid, as was originally done by Einstein and Friedmann. Second, studies of models sourced from different physical fields, starting from electromagnetic and scalar ones,

both in classical and quantum cases (see Ref. [1]). And third, which is really topical now, application of ideas and results of unified models for treating fundamental problems of cosmology and black hole physics, especially in high energy regimes and for explanation of the greatest challenge to modern physics — the present acceleration of the Universe, dark matter and dark energy problems. Multidimensional gravitational models play an essential role in the latter approach.

The necessity of studying multidimensional models of gravitation and cosmology [2–6] is motivated by several reasons. First, the main trend of modern physics is the unification of all known fundamental physical interactions: electromagnetic, weak, strong and gravitational ones. In recent decades, there has been a significant progress in unifying weak and electromagnetic interactions, as well as more modest achievements in GUT [7–9], super-symmetric, string and super-string theories.

Now, theories with membranes,  $p$ -branes and the more vague M-theory are being created and studied. Having no definite successful unifying theory now, it is desirable to study common features of these theories and their applications to solving basic problems of gravitation and cosmology. If we really believe in unified theories, the early stages of the Universe's evolution and black hole physics, as unique superhigh energy regions and possibly even low energy stage, where we observe present acceleration, are the most proper and natural arenas for them.

Second, multidimensional gravitational models as well as scalar-tensor theories of gravity are theoretical frameworks for describing possible temporal and range variations of fundamental physical constants (FPC) [1, 10–12]. These ideas originated from papers of E. Milne (1935) and P. Dirac (1937) on relations between the phenomena of micro- and macro-worlds, and up till now they are under thorough study both theoretically and experimentally. The possible discoveries of the fine structure constant and proton to electron mass ratio variations are now at a critical further investigation phase.

Lastly, by applying multidimensional gravitational models to basic problems of modern cosmology and black hole physics, we hope to find answers to such long-standing problems as singular or nonsingular initial states, the creation of the Universe, the creation of matter and its entropy, the cosmological constant, the coincidence problem, the origin of inflation and specific scalar fields which may be necessary for its realization, isotropization and graceful exit problems, the stability and nature of fundamental constants [10, 13–16], the possible number of extra dimensions, their stable compactification, new revolutionary data on present acceleration of the Universe, dark matter and dark energy [15]

etc.

Bearing in mind that multidimensional gravitational models are certain generalizations of general relativity which is tested reliably for weak fields up to 0.0001 and partially in strong fields (binary pulsars), it is quite natural to inquire about their possible observational or experimental windows. From what we already know, among these windows are [13–15]:

- possible deviations from the Newton and Coulomb laws, or new interactions,
- possible variations of the effective gravitational constant with a time rate smaller than that of the Hubble,
- possible existence of monopole modes in gravitational waves,
- different behaviors of strong field objects, such as multidimensional black holes, wormholes and  $p$ -branes,
- standard cosmological tests,
- possible non-conservation of energy in strong field objects and accelerators, that is, if brane-world ideas about gravity in the bulk turn out to be true etc.

Since modern cosmology has already become a unique laboratory for testing standard unified models of physical interactions at energies that are far beyond the level of existing and future man-made accelerators and other installations on the Earth, there exists a possibility of using cosmological and astrophysical data for discriminating between future unified schemes. Data on possible time variations or possible deviations from the Newton law as a new important test should also contribute to the unified theory choice and viable cosmological model choice as well [17, 18].

As no accepted unified model exists, in our approach [2–6, 19] we adopted simple (but general, from the point of view of number of dimensions) models, based on multidimensional Einstein equations with or without sources of different natures:

- cosmological constant,
  - perfect and viscous fluids,
  - scalar and electromagnetic fields,
  - their possible interactions,
  - dilaton and moduli fields with or without potentials,
  - fields of antisymmetric forms (related to  $p$ -branes)
- etc.

Our program's main objective was and is to obtain the exact self-consistent solutions (integrable models) for these models and then to analyze them in cosmological, spherically and axially symmetric cases. In our view, this is a natural and most reliable way to study highly nonlinear systems. It is done mainly within Riemannian geometry. Some simple models in integrable Weyl geometry and with torsion were studied as well. In many cases, we tried to single out models which do not contradict available experimental or observational data on

variations of  $G$ . In some cases, we used our methods also for arbitrary dimensions in studying 4-D models.

As our model [2–6], we use  $n$  Einstein spaces of constant curvature, with the sources being  $(m+1)$ -component perfect fluid, (or fields or form-fields), a cosmological or spherically symmetric metric manifold as a direct product of factor-spaces of arbitrary dimensions. Then, in harmonic time gauge we show that Einstein's multidimensional equations are equivalent to the Lagrange equations with non-diagonal, in general, mini-super-space metric and some exponential potential. After diagonalization of this metric, we performed reduction to sigma-model [20–22] and Toda-like systems [23], further to Liouville, Abel, generalized Emden-Fowler Eqs. etc. and tried to find exact solutions. We suppose that the behavior of extra spaces is as follows: they are constant or dynamically compactified, or like torus, or large, but with barriers, walls etc.

So, we realized and continue to realize the program in arbitrary dimensions (from 1988) [2–6, 15, 19]:

In cosmology we obtained exact general solutions of multidimensional Einstein equations with sources:

- $\Lambda$ ,  $\Lambda$  + scalar field (e.g. nonsingular, dynamically compactified, inflationary) [24];
- perfect fluid, PF + scalar field (e.g. nonsingular, inflationary solutions) [25, 26];
- viscous fluid (e.g. nonsingular, generation of mass and entropy, quintessence and coincidence in 2-component model);
- stochastic behavior near the singularity, billiards in Lobachevsky space,  $D=11$  is critical,  $\varphi$  destroys billiards (1994).

(For all above cases Ricci-flat solutions were obtained for any  $n$ .)

- also, solutions with curvature in one factor-space;
- with curvatures in 2 factor-spaces only for total  $N=10, 11$ ;
- with fields: scalar, dilatons, forms of arbitrary rank [27] (1998) — inflationary,  $\Lambda$  generation by forms ( $p$ -branes) [28];
- first billiards for dilaton-forms ( $p$ -branes) interaction (1999);
- quantum variants (solutions of WDW-equation [24, 26, 29]) for all above cases where classical solutions were obtained;
- dilatonic fields with potentials, their billiard behavior.

For many of these integrable models we also calculated the variation with time of the effective gravitational constant. Comparison with present experimental bounds allowed choosing of particular models or singling out some classes of solutions.

Similar methods were used for obtaining exact solu-

tions in the spherical symmetry case [2–6, 15, 19].

Dirac's Large Numbers Hypothesis (LNH) is the origin of many theoretical studies of time-varying  $G$ . According to LNH, the value of  $\dot{G}/G$  should be approximately the Hubble rate. Although it has become clear in recent decades that the Hubble rate is too high to be compatible with experiments, the enduring legacy of Dirac's bold stroke is the acceptance by modern theories of non-zero values of  $\dot{G}/G$  as being potentially consistent with physical reality.

There are three problems related to  $G$ , whose origin lies mainly in predictions of unified models of physical interactions:

- (1) absolute  $G$  measurements,
- (2) possible time variations of  $G$ ,
- (3) possible range variations of  $G$  – non-Newtonian, or new interactions.

After the original Dirac hypothesis some new ones appeared as well as some generalized theories of gravitation admitting variations of the effective gravitational coupling. We can single out three stages in the development of this field [10]:

- Study of phenomenological theories and hypotheses with variations of FPC, their predictions and confrontation with experiments (1937–1977).
- Creation of theories admitting variations of an effective gravitational constant in a particular system of units, analyses of experimental and observational data within these theories [1] (1977–present).
- Analysis of Fundamental Physical Constants variations within unified models [4, 5] (present).

Different theoretical schemes lead to temporal variations of the effective gravitational constant [30]:

- Empirical models and theories of Dirac type, where  $G$  is replaced by  $G(t)$ .

- Numerous scalar-tensor theories of Jordan-Brans-Dicke type where  $G$  depending on the scalar field  $\phi(t)$  appears.

- Gravitational theories with a conformal scalar field arising in different approaches [31, 1] (they can be treated as special cases of scalar-tensor theories).

- Multidimensional unified theories in which there are dilaton fields and effective scalar fields appearing in our 4-D spacetime from additional dimensions [4, 5]. They may also help in solving the problem of the variable cosmological constant (from Planckian to the present values) and the cosmic coincidence problem.

A striking feature of the present status of theoretical physics is that there is no satisfactory theory unifying all four known interactions; most modern unification theories do not admit unique and universal constant values of physical constants and of the Newtonian gravitational coupling constant  $G$  in particular. Although the bounds

on  $\dot{G}$  and  $G(r)$  are, in some classes of theories, rather wide on purely theoretical grounds, since any theoretical model contains a number of adjustable parameters, we note that observational data concerning other phenomena, in particular cosmological data, may place limits on possible ranges of these adjustable parameters. However, in any case, variations of  $G$  may be an additional test of unified models, generalized theories of gravitation and cosmological models as well [18].

Here, we restrict ourselves to the problem of  $\dot{G}$  (for  $G(r)$  see Refs. [1, 4, 5, 10, 11]). We show that various theories predict the value of  $\dot{G}/G$  to be  $10^{-12}/y$  or less. The significance of this fact for experimental and observational determinations of the value of or upper bound on  $\dot{G}$  is the following: any determination with error bounds significantly below  $10^{-12}/y$  (combined with experimental bounds on other parameters) will typically be compatible with only a small portion of the existing theoretical models and will therefore cast serious doubts on the viability of all other models. In short, a tight bound on  $\dot{G}$ , in conjunction with other astrophysical observations, will be a very effective “theory killer” and/or significantly reduce the class of viable theories. Any step forward in this direction will be of utmost significance [32–34].

Some estimations for  $\dot{G}$  were done long ago in the frames of general scalar-tensor theories using the values of cosmological parameters ( $\Omega$ ,  $H$ ,  $q$ , etc.) known at that time [1, 4, 5, 35–38]. It is easy to show that for modern values, they predict  $\dot{G}/G$  at the level of  $10^{-13}/y$  and less (see also estimations of Miyazaki [39], predicting time variations of  $G$  at the level of  $10^{-13}/y$  for the Machian-type cosmological solution in the Brans-Dicke theory, Fujii (see Ref. [40]), of Mbelek and Lanchieze-Ray [41] on the level of  $10^{-17}/y$  for the simple 5-D KK-theory with an external scalar field and section 2 of the present paper).

The most reliable experimental bounds on  $\dot{G}/G$  (radar ranging of spacecraft and planet dynamics [42, 43] and laser lunar ranging [44, 45]) give the limit less than  $10^{-12}/y$ , so any result at less than this level will be very important for solving the fundamental problem of variations of constants and for discriminating between viable unified theories and cosmological models. So, with further data on lunar laser and radar ranging and realization of MICROSCOPE, ASTROD, LATOR space projects and such multipurpose new generation type space experiments as Satellite Energy Exchange (SEE) for measuring  $\dot{G}$ , the absolute value of  $G$  and Yukawa type forces at meters and Earth radius ranges [46–48] become extremely topical.

In what follows, we shall discuss predictions for  $\dot{G}$  from

generalized scalar-tensor theories and some multidimensional models.

## 2 Scalar-tensor cosmology and variations of $G$

The purpose of this section is to estimate the order of magnitude of the gravitational constant  $G$  variations due to cosmological expansion in the framework of general scalar-tensor theories (STT) of gravity [49].

Consider the general (Bermann-Wagoner-Nordtvedt) class of STT where gravity is characterized by a metric  $g_{\mu\nu}$  and a scalar field  $\phi$ ; the action is

$$S = \int d^4x \sqrt{g} \{ f(\phi) \mathcal{R}[g] + h(\phi) g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2U(\phi) + L_m \} \quad (2.1)$$

Here  $\mathcal{R}[g]$  is the scalar curvature,  $g = |\det g_{\mu\nu}|$ ;  $f$ ,  $h$  and  $U$  are certain functions of  $\phi$ , varying from theory to theory,  $L_m$  is a matter Lagrangian.

This formulation of the theory corresponds to the Jordan conformal frame, in which matter particles move along geodesics and hence the weak equivalence principle is valid and non-gravitational fundamental constants do not change. In other words, this is the frame well describing the existing laboratory, geophysical and cosmological observations.

Among the three functions of  $\phi$  entering into Eq. (2.1), only two are independent since there is a freedom of transformations  $\phi = \phi(\phi_{\text{new}})$  [35]. We use this arbitrariness, choosing  $h(\phi) \equiv 1$ , as is done, e.g., in Ref. [50]. Another standard parametrization is to put  $f(\phi) = \phi$  and  $h(\phi) = \omega(\phi)/\phi$  (the Brans-Dicke parametrization of the general theory (2.1)). In our parametrization  $h \equiv 1$ , the B-D parameter  $\omega(\phi) = f(\phi_\phi)^{-2}$ ; the subscript  $\phi$  denotes a derivative with respect to  $\phi$ . The B-D STT is the particular case  $\omega = \text{const}$ , so that in Eq. (2.1)

$$f(\phi) = \phi^2/(4\omega), \quad h \equiv 1 \quad (2.2)$$

For the conformal scalar field case see Refs. [1, 35].

The field equations that follow from Eq. (2.1) read

$$\square\phi - \frac{1}{2}\mathcal{R}f_\phi + U_\phi = 0 \quad (2.3)$$

$$f(\phi) \left( \mathcal{R}_\mu^\nu - \frac{1}{2}\delta_\mu^\nu \mathcal{R} \right) = -\phi_{,\mu}\phi^{,\nu} + \frac{1}{2}\delta_\mu^\nu \phi^{,\alpha}\phi_{,\alpha} - \delta_\mu^\nu U(\phi) + (\nabla_\mu \nabla^\nu - \delta_\mu^\nu \square)f - T_{\mu(m)}^\nu \quad (2.4)$$

where  $\square$  is the D'Alembert operator, and the last term in Eq. (2.4) is the energy-momentum tensor of matter.

Consider now isotropic cosmological models with the

standard FRW metric

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right] \quad (2.5)$$

where  $a(t)$  is the scale factor of the Universe, and  $k = 1, 0, -1$  for closed, spatially flat and hyperbolic models, respectively. Accordingly, we assume  $\phi = \phi(t)$  and the energy-momentum tensor of matter in the perfect fluid form  $T_{\mu(m)}^\nu = \text{diag}(\rho, -p, -p, -p)$  ( $\rho$  is a density and  $p$  is a pressure).

The field equations in this case can be written as follows:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{3}{a^2}(a\ddot{a} + \dot{a}^2 + k) + U_\phi = 0 \quad (2.6)$$

$$\frac{3f}{a^2}(\dot{a}^2 + k) = \frac{1}{2}\dot{\phi}^2 + U - 3\frac{\dot{a}}{a}\dot{f} + \rho \quad (2.7)$$

$$\frac{f}{a^2}(2a\ddot{a} + \dot{a}^2 + k) = -\frac{1}{2}\dot{\phi}^2 + U - \ddot{f} - 2\frac{\dot{a}}{a}\dot{f} - p \quad (2.8)$$

To connect these equations with observations, let us fix the time  $t$  at the present epoch (i.e., consider the instantaneous values of all quantities) and introduce the standard observables:  $H = \dot{a}/a$  (the Hubble parameter),  $q = -a\ddot{a}/\dot{a}^2$  (the deceleration parameter),  $\Omega_m = \rho/\rho_{\text{cr}}$  (the matter density parameter), where  $\rho_{\text{cr}}$  is the critical density, or, in our model, the r.h.s. of Eq. (2.6) in case  $k = 0$ :  $\rho_{\text{cr}} = 3fH^2$ . This is slightly different from the usual definition  $\rho_{\text{cr}} = 3H^2/(8\pi G)$ . The point is that the locally measured Newtonian constant in STT differs from  $1/(8\pi f)$ ; provided the derivatives  $U_{\phi\phi}$  and  $f_{\phi\phi}$  are sufficiently small, one has [50]

$$8\pi G_{\text{eff}} = \frac{1}{f} \frac{2\omega + 4}{2\omega + 3} \quad (2.9)$$

Since, according to the Solar-system experiments,  $\omega \geq 40\,000$ , for our order-of-magnitude reasoning we can safely put  $8\pi G = 1/f$ , and, in particular, our definition of  $\rho_{\text{cr}}$  now coincides with the standard one.

The time variation of  $G$ , to a good approximation, is

$$\dot{G}/G \approx -\dot{f}/f = gH \quad (2.10)$$

where, for convenience, we have introduced the coefficient  $g$  expressing  $\dot{G}/G$  in terms of  $H$ .

Eqs. (2.6)–(2.8) contain too many arbitrary parameters for making a good estimate of  $g$ . Let us now introduce some restrictions according to the current state of observational cosmology:

(i)  $k = 0$  (a spatially flat cosmological model, so that the total density of matter equals  $\rho_{\text{cr}}$ );

(ii)  $p = 0$  (the pressure of ordinary matter is negligible compared to the energy density);

(iii)  $\rho = 0.3\rho_{\text{cr}}$  (the ordinary matter, including its

dark component, contributes to only 0.3 of the critical density; unusual matter (dark energy), which is here represented by the scalar field, comprises the remaining 70 percent).

Then Eqs. (2.6) and (2.8) can be rewritten in the form

$$\begin{aligned} \frac{1}{2}\dot{\phi}^2 + U - 3H\dot{f} &= 2.1H^2f \\ -\frac{1}{2}\dot{\phi}^2 + U - 2H\dot{f} - \ddot{f} &= (1 - 2q)H^2f \end{aligned} \quad (2.11)$$

Subtracting the second from the first one, we exclude the “cosmological constant”  $U$  and obtain

$$\dot{\phi}^2 - H\dot{f} + \ddot{f} = (1.1 + 2q)H^2f \quad (2.12)$$

The first term in Eq. (2.12) can be represented in the form

$$\dot{\phi}^2 = \dot{f}^2(df/d\phi)^{-2} = \dot{f}^2\omega/f \quad (2.13)$$

and  $\dot{f}/f$  can be replaced with  $-gH$ . The term  $\ddot{f}$  can be neglected for our estimation purposes for an arbitrary function  $f$  and potential  $U(\phi)$ .

Then, Eq. (2.12) divided by  $H^2f$  leads to the quadratic equation with respect to  $g$ :

$$\omega g^2 + g - q' = 0, \quad q' = 1.1 + 2q \quad (2.14)$$

According to present observations, the Universe is expanding with an acceleration, so that the parameter  $q$  is, roughly,  $-0.5 \pm 0.2$ , hence we can take  $|q'| \leq 0.4$ .

In case  $q' = 0$  we simply obtain  $g = -1/\omega$ . Assuming

$$H = h_{100} \cdot 100 \text{ km}/(\text{s} \cdot \text{Mpc}) \approx h_{100} \cdot 10^{-10} \text{ y}^{-1} \quad (2.15)$$

and the present limit  $\omega \geq 40\,000$ , we come to the estimate

$$|\dot{G}/G| \leq 4 \times 10^{-15} h_{100} \text{ y}^{-1} \quad (2.16)$$

where  $h_{100}$  is, by modern views, close to 0.7. So Eq. (2.16) becomes

$$|\dot{G}/G| \leq 4 \times 10^{-15} \text{ y}^{-1} \quad (2.17)$$

For nonzero values of  $q'$ , solving the quadratic equation (2.14) and assuming  $q'\omega \gg 1$ , we arrive at the estimate  $|g| \sim \sqrt{q'/\omega}$ , so that, taking  $q' = 0.4$  and again  $\omega \geq 40\,000$ , we have, instead of Eq. (2.16)

$$|\dot{G}/G| \leq 0.9 \times 10^{-13} h_{100} \text{ y}^{-1} \approx 0.7 \times 10^{-13} \text{ y}^{-1} \quad (2.18)$$

where we have again put  $h_{100} = 0.7$ .

As a result, in the framework of the general STT, present cosmological observations, taking into account the Solar-system data, restrict the possible variation of  $G$  to values less than  $10^{-13}/\text{y}$ . This estimate may be considerably tightened if the matter density parameter  $\Omega_m$  and the (negative) deceleration parameter  $q$  will be determined more precisely.

### 3 $G$ -dot in (1+3+N)-D cosmology with multicomponent anisotropic fluid

We consider here a (4+N)-D cosmology with an isotropic 3-D space and an Einstein internal space [37, 51]. The Einstein equations provide a relation between  $\dot{G}/G$  and other cosmological parameters.

#### 3.1 The model

Let us consider (4+N)-D theory with the gravitational part of the action

$$S_g = \frac{1}{2\kappa^2} \int d^{4+N}x \sqrt{-g} R \quad (3.1)$$

where  $\kappa^2$  is the fundamental gravitational constant. Then the gravitational field equations are

$$R_{P}^M = \kappa^2 \left( T_{P}^M - \delta_{P}^M \frac{T}{N+2} \right) \quad (3.2)$$

where  $T_{P}^M$  is a (4+N)-D energy-momentum tensor,  $T = T_{M}^M$ , and  $M, P = 0, \dots, N+3$ .

For the (4+N)-D manifold we assume the structure

$$M^{4+N} = \mathbb{R}_* \times M_k^3 \times K^N \quad (3.3)$$

where  $\mathbb{R}_*$  is a 1-D time manifold,  $M_k^3$  is a 3-D space of constant curvature,  $M_k^3 = S^3, R^3, L^3$  for  $k = +1, 0, -1$ , respectively, and  $K^N$  is a  $N$ -D Einstein manifold.

The metric is taken in the form

$$ds^2 = g_{MN} dx^M dx^N = -dt^2 + a^2(t) g_{ij}^{(3)}(x) dx^i dx^j + b^2(t) g_{mn}^{(N)}(y) dy^m dy^n \quad (3.4)$$

where  $i, j, k = 1, 2, 3$ ;  $m, n, p = 4, \dots, N+3$ ;  $g_{ij}^{(3)}, g_{mn}^{(N)}$ ,  $a(t)$  and  $b(t)$  are, respectively, the metrics and scale factors for  $M_k^3$  and  $K^N$ .

For  $T_{P}^M$  we adopt the expression of the multicomponent (anisotropic) fluid form

$$(T_{P}^M) = \sum_{\alpha=1}^m \text{diag}(-\rho^\alpha(t), p_3^\alpha(t) \delta_j^i, p_N^\alpha(t) \delta_n^m) \quad (3.5)$$

Under these assumptions the Einstein equations take the form

$$\frac{3\ddot{a}}{a} + \frac{N\ddot{b}}{b} = \frac{\kappa^2}{N+2} \sum_{\alpha=1}^m [-(N+1)\rho^\alpha - 3p_3^\alpha - Np_N^\alpha] \quad (3.6)$$

$$\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{N\dot{a}\dot{b}}{ab} + \frac{2\dot{b}}{b} = \frac{\kappa^2}{N+2} \sum_{\alpha=1}^m [\rho^\alpha + (N-1)p_3^\alpha - Np_N^\alpha] \quad (3.7)$$

$$\frac{\ddot{b}}{b} + (N-1)\frac{\dot{b}^2}{b^2} + \frac{3\dot{a}\dot{b}}{ab} + \frac{\lambda}{b^2} = \frac{\kappa^2}{N+2} \sum_{\alpha=1}^m (\rho^\alpha - 3p_3^\alpha + 2p_N^\alpha) \quad (3.8)$$

Here,

$$R_{mn}[g^{(N)}] = \lambda g_{mn}^{(N)} \quad (3.9)$$

$m, n = 1, \dots, N$ , where  $\lambda$  is constant. The 4-D density is

$$\rho^{\alpha,(4)}(t) = \int_K d^N y \sqrt{g^{(N)}} b^N(t) \rho^\alpha(t) = \rho^\alpha(t) b(t) \quad (3.10)$$

where we have normalized the factor  $b(t)$  by putting

$$\int_K d^N y \sqrt{g^{(N)}} = 1 \quad (3.11)$$

On the other hand, to get the 4-D gravity equations one should put  $8\pi G(t) \rho^{\alpha,(4)}(t) = \kappa^2 \rho^\alpha(t)$ .

Consequently, the effective 4-D gravitational ‘‘constant’’  $G(t)$  is defined by

$$8\pi G(t) = \kappa^2 b^{-N}(t) \quad (3.12)$$

whence its time variation is expressed as

$$\dot{G}/G = -N\dot{b}/b \quad (3.13)$$

#### 3.2 Cosmological parameters

Some inferences concerning the observational cosmological parameters can be extracted just from the equations without solving them [37]. Indeed, let us define the Hubble parameter  $H$ , the density parameters  $\Omega^\alpha$  and the ‘‘deceleration’’ parameter  $q$  by referring to a fixed instant  $t_0$  in the usual way

$$\begin{aligned} H &= \dot{a}/a \\ \Omega^\alpha &= 8\pi G \rho^{\alpha,(4)} / (3H^2) = \kappa^2 \rho^\alpha / 3H^2 \\ q &= -a\ddot{a}/\dot{a}^2 \end{aligned} \quad (3.14)$$

Further, instead of  $G$  let us introduce the dimensionless parameter

$$g = \dot{G}/(GH) = -N\dot{a}\dot{b}/(\dot{a}b) \quad (3.15)$$

The present observational upper bound is  $g < 0.1$ , if we think in accord with [42, 44]

$$\dot{G}/G < 10^{-12} \text{y}^{-1} \quad (3.16)$$

and  $H = (0.7 \pm 0.1) \times 10^{-11} \text{y}^{-1} \approx (70 \pm 10) \text{ km}/(\text{s} \cdot \text{Mpc})$ .

#### 3.3 The vacuum model with two Einstein spaces

Here, we consider the vacuum case when  $T_{P}^M = 0$ . Let us suppose that  $t_0$  is an extremum point of the function  $b(t)$ , i.e.  $\dot{b}(t_0) = 0$ . In this point we get  $\dot{G}(t_0) = 0$ . From

Eqs. (3.6)–(3.8) we get that for  $t = t_0$

$$\frac{3\ddot{a}}{a} + \frac{N\ddot{b}}{b} = 0 \tag{3.17}$$

$$\frac{\ddot{a}}{a} = -\frac{2\dot{a}^2}{a^2} - \frac{2k}{a^2} \tag{3.18}$$

$$\frac{\ddot{b}}{b} = -\frac{\lambda}{b^2} \tag{3.19}$$

Let us suppose that we “live” near the point  $t_0$ , then according to modern observations on acceleration of expansion of the Universe (Refs. [52, 53]) we should put  $\dot{a}(t_0) > 0$  and  $\ddot{a}(t_0) > 0$ . This implies  $k < 0$  due to Eq. (3.18) and  $\dot{b}(t_0) < 0$ ,  $\lambda > 0$  due to Eqs. (3.17) and (3.19). Thus, our 3-D space should have negative curvature and the internal  $N$ -D space should have a positive curvature.

From Eqs. (3.17)–(3.19) we obtain using the definitions of cosmological parameters

$$\frac{|2k|}{H_0^2 a_0^2} = 2 + |q_0| \tag{3.20}$$

$$\frac{d_2|\lambda|}{H_0^2 b_0^2} = 3|q_0| \tag{3.21}$$

Here,  $a_0 = a(t_0)$  and  $b_0 = b(t_0)$ .

Since we suppose that we now “live” near the point  $t_0$ , then we get

$$\frac{\dot{b}}{b} \approx \frac{\ddot{b}_0(t - t_0)}{b_0} \tag{3.22}$$

and due to Eqs. (3.13) and (3.17) we find

$$\frac{\dot{G}}{G} = -N\dot{b}/b \approx -N\frac{\ddot{b}_0}{b_0}(t - t_0) = 3\frac{\ddot{a}_0(t - t_0)}{a_0} \tag{3.23}$$

The subscript “0” refers to  $t_0$ . Using Eq. (3.14) we obtain in our approximation

$$\frac{\dot{G}}{G} \approx -3q_0 H_0^2 (t - t_0) \tag{3.24}$$

Remember that  $q_0 < 0$ , hence,  $\dot{G}/G > 0$  for  $t > t_0$  and  $\dot{G}/G < 0$  for  $t < t_0$ ; in our approximation  $\dot{G}/G$  does not depend upon the dimension of internal space  $N = \dim K$ .

**Exact 1 + 3 + 6 solution**

Now we consider the exact solution from Ref. [54] defined on the manifold

$$M = \mathbb{R}_* \times M^{(3)} \times M^{(6)} \tag{3.25}$$

with the metric

$$ds^2 = (f_1 f_2)^{-\frac{1}{2}} [-2f_1^{-2} (d\tau)^2 + |\lambda_3| g_{ij}^{(3)}(x) dx^i dx^j + f_2 |\lambda_6| g_{mn}^{(N)}(y) dy^m dy^n] \tag{3.26}$$

where  $(M^{(3)}, g^{(3)})$  and  $(M^{(6)}, g^{(6)})$  are Einstein spaces:

$$\text{Ric}[g^{(i)}] = \lambda_i g^{(i)} \tag{3.27}$$

$i = 3, 6$ . Here, we use the notations:  $\lambda_3 = 2k$  and  $\lambda_6 = \lambda$ .

In Eq. (3.26),

$$f_1 = |\tau^2 + \varepsilon_3| \tag{3.28}$$

$$f_2 = -3\varepsilon_6 (\tau^2 + \varepsilon_3) [1 + \tau (h(\tau, \varepsilon_3) + C_1)] + \varepsilon_3 \varepsilon_6 > 0 \tag{3.29}$$

$C_1$  is constant and  $\varepsilon_i = \text{sgn } \lambda_i$ ,  $i = 3, 6$ , and

$$h(\tau, \varepsilon_3) = \frac{1}{2} \ln \left| \frac{\tau - 1}{\tau + 1} \right|, \quad \varepsilon_3 = -1 \tag{3.30}$$

$$h(\tau, \varepsilon_3) = \arctan \tau, \quad \varepsilon_3 = 1 \tag{3.31}$$

As was mentioned above we should restrict our consideration to the case when our 3-D space has negative curvature and the 6-D internal space has positive curvature, i.e.,

$$\varepsilon_3 = -1, \quad \varepsilon_6 = 1 \tag{3.32}$$

The analysis carried out in Ref. [54] tells us that the scale factor of our 3-D space

$$a_3 = a = (f_1 f_2)^{-1/4} |\lambda_3|^{1/2} \tag{3.33}$$

has the minimum at some point  $\tau_*$  when the branch of the solution with  $\tau \in (\tau_-, \tau_+)$  is considered. Here,  $\tau_-, \tau_+$  are roots of the equation  $f_2(\tau) = 0$  belonging to the interval  $(0, 1)$ . In this case the scale factor of our space  $a_3(\tau)$  is monotonically decreasing in the interval  $(\tau_-, \tau_*)$  and monotonically increasing in the interval  $(\tau_*, \tau_+)$ .

The scale factor of the internal 6-D space

$$a_6 = b = (f_1 f_2)^{-1/4} f_2^{1/2} |\lambda_6|^{1/2} \tag{3.34}$$

has a maximum at some point  $\tau_0$ . It is monotonically increasing in the interval  $(\tau_-, \tau_0)$  and monotonically decreasing in the interval  $(\tau_0, \tau_+)$ . For other branches of the solution with either  $\tau \in (\tau_-, \tau_+\infty)$  or  $\tau \in (-\infty, \tau_+)$ , ( $|\tau_-|, |\tau_+| > 1$ ) we get monotonic behavior of scale factors  $a_3(\tau)$  and  $a_6(\tau)$ .

Let us consider the behavior of our solution in synchronous time

$$ds^2 = -dt^2 + a_3^2(t) g_{ij}^{(3)}(x) dx^i dx^j + a_6^2(t) g_{mn}^{(N)}(y) dy^m dy^n \tag{3.35}$$

where

$$t_s = \sqrt{2} \int_{\tau_-}^{\tau} d\tau' (f_1 f_2)^{-1/4} f_1^{-1} \quad (3.36)$$

The function  $t_s(\tau)$  is monotonically increasing from  $t_s(\tau_-) = 0$  to  $T = t_s(\tau_+)$ .

The scale factor of 3-D space has minimum at  $t_0 = t(\tau_*)$ .  $a_3(t)$  is monotonically decreasing from infinity to finite value in the interval  $(0, t_0)$  and monotonically increasing to infinity in the interval  $(t_0, T)$ .

The scale factor of 6-D space has maximum at  $t_* = t(\tau_*)$ . The function  $a_6(t)$  is monotonically increasing from infinity to a finite value in the interval  $(0, t_*)$  and monotonically decreasing to infinity in the interval  $(t_*, T)$ . Only in the case when  $C_1 > 0$  will we get that  $t_* < t_0$  and hence, in the time epoch near  $t_0$ , we get an accelerating expansion of our 3-D space.

### 3.4 The model with two Ricci-flat spaces and two-component fluid

Here we consider another example when two factor spaces are Ricci-flat.

In this case, excluding  $b$  from Eqs. (3.6) and (3.8), we get

$$\frac{N-1}{3N} g^2 - g + q - \sum_{\alpha=1}^m A^\alpha \Omega^\alpha = 0 \quad (3.37)$$

with

$$A^\alpha = \frac{1}{N+2} [2N+1 + 3(1-N)\nu_3^\alpha + 3N\nu_N^\alpha] \quad (3.38)$$

where

$$\nu_3^\alpha = p_3^\alpha / \rho^\alpha, \quad \nu_N^\alpha = p_N^\alpha / \rho^\alpha, \quad \rho^\alpha > 0 \quad (3.39)$$

When  $g$  is small we get from Eq. (3.37)

$$g \approx q - \sum_{\alpha=1}^m A^\alpha \Omega^\alpha \quad (3.40)$$

Note that Eq. (3.40) for  $N=6, m=1, \nu_3^1 = \nu_6^1 = 0$  (so that  $A^1 = 13/8$ ) coincides with the corresponding relation of Wu and Wang [55] obtained for large times in case  $k = -1$  (see also Ref. [36]).

If  $k = 0$ , then in addition to Eq. (3.40), one can obtain a separate relation between  $g$  and  $\Omega^\alpha$ , namely,

$$\frac{N-1}{6N} g^2 - g + 1 - \sum_{\alpha=1}^m \Omega^\alpha = 0 \quad (3.41)$$

(this follows from the Einstein equation  $R_0^0 - \frac{1}{2}R = \kappa^2 T_0^0$ , which is a linear combination of Eqs. (3.6)–(3.8)).

### Two-component example: dust + $(N-1)$ -brane

Consider now a two component case:  $m = 2$  [51]. Let the first component (matter) be dust, i.e.,

$$\nu_3^1 = \nu_N^1 = 0 \quad (3.42)$$

and the second one (quintessence) be a  $(N-1)$ -brane, i.e.

$$\nu_3^2 = 1, \quad \nu_N^2 = -1 \quad (3.43)$$

We remember that, as it was mentioned in Ref. [56], the multidimensional cosmological model on the product manifold  $\mathbb{R} \times M_1 \times \dots \times M_n$  with fields of forms (for review see Ref. [19]) may be described in terms of a multicomponent “perfect” fluid [25] with the following equations of state for  $\alpha$ -s component:  $p_i^\alpha = -\rho^\alpha$  if  $p$ -brane world-volume contains  $M_i$  and  $p_i^\alpha = \rho^\alpha$  in the opposite case. Thus, the field of form matter leads us either to  $\Lambda$ -term, or to stiff matter equations of state in internal spaces.

In this case we get from Eq. (3.40) for small  $g$

$$g \approx q - \frac{2N+1}{N+2} \Omega^1 + 4 \frac{N-1}{N+2} \Omega^2 \quad (3.44)$$

and for  $k = 0$  and small  $g$  we obtain from Eq. (3.41)

$$1 - g \approx \Omega^1 + \Omega^2 \quad (3.45)$$

Now we illustrate these formulas by example when  $N = 6$  ( $K^6$  may be a Calabi-Yau manifold) and

$$-q = \Omega^1 = \Omega^2 = 0.5 \quad (3.46)$$

We get from Eq. (3.44)

$$g \approx -\frac{1}{16} \approx -0.06 \quad (3.47)$$

in agreement with relations (3.15, 3.16).

In this case, the second fluid component corresponds to magnetic (Euclidean)  $NS5$ -brane (in  $D = 10$  type I, Het or II A string models). Here we considered for simplicity the case of the constant dilaton field.

This example tells us that for small enough temporal variation of  $G$  we may find the estimates on  $G$ -dot without consideration of the exact solutions. However, here, we should select the solutions that give us the accelerated expansion of our world. We may use, for instance, the mechanism suggested above: but instead of curvatures in 2 factor-spaces we should consider two fluid components.

## 4 Multidimensional cosmology with anisotropic fluid [57, 58]

### 4.1 The model

Now we consider a cosmological model describing the

dynamics of  $n$  Ricci-flat spaces in the presence of 1-component “perfect-fluid” matter [25]. Metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=0}^n \exp[2x^i(t)]g^i \quad (4.1)$$

is defined on the manifold

$$M = R \times M_0 \times \dots \times M_n \quad (4.2)$$

where manifold  $M_i$  with a metric  $g^i$  is a Ricci-flat space of dimension  $d_i$ ,  $i = 0, \dots, n$ ;  $n \geq 2$ . The multidimensional Hilbert-Einstein equations have the following form:

$$R_N^M - \frac{1}{2}\delta_N^M R = \kappa^2 T_N^M \quad (4.3)$$

where  $\kappa^2$  is the gravitational constant, and the energy-momentum tensor is adopted as

$$(T_N^M) = \text{diag}(-\rho, p_1\delta_{k_1}^{m_1}, \dots, p_n\delta_{k_n}^{m_n}) \quad (4.4)$$

describing anisotropic fluid in general.

We put pressures of the anisotropic perfect fluid in all spaces to be proportional to the density

$$p_i(t) = \left(1 - \frac{u_i}{d_i}\right)\rho(t) \quad (4.5)$$

where  $u_i = \text{const}$ ,  $i = 0, \dots, n$ . Here we put  $\rho > 0$ .

We also impose the following restriction on vector  $u = (u_i) \in R^n$

$$\langle u, u \rangle_* < 0 \quad (4.6)$$

Here, the bilinear form  $\langle \cdot, \cdot \rangle_* : R^n \times R^n \rightarrow R$  is defined by the relation

$$\langle u, v \rangle_* = G^{ij}u_i v_j \quad (4.7)$$

$u, v \in R^{n+1}$ , where

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D} \quad (4.8)$$

are components of the matrix inverse to a matrix of the minisuperspace metric [59, 29]

$$G_{ij} = d_i\delta_{ij} - d_i d_j \quad (4.9)$$

In Eq. (4.8)  $D = 1 + \sum_{i=0}^n d_i$  is the total dimension of  $M$  (4.2). The restriction (4.6) reads

$$\langle u, u \rangle_* \equiv G^{ij}u_i u_j = \sum_{i=0}^n \frac{(u_i)^2}{d_i} + \frac{1}{2-D} \left(\sum_{i=0}^n u_i\right)^2 < 0 \quad (4.10)$$

## 4.2 Solutions with power-law and exponential dependence of scale factors

Let us consider two special families of solutions [25,

26] with the metric written in synchronous time parametrization

$$g = -dt_s \otimes dt_s + \sum_{i=0}^n a_i^2(t_s)g^i \quad (4.11)$$

These solutions have either power-law or exponential dependence of scale factors (w.r.t.  $t_s$ ).

### 4.2.1 Solutions with power-law behavior

The solutions with power-law behavior of scale factors take place for

$$\langle u^{(\Lambda)} - u, u \rangle_* \neq 0 \quad (4.12)$$

The vector  $u_i^{(\Lambda)} = 2d_i$  corresponds to the  $\Lambda$ -term fluid with  $p_i = -\rho$  (vacuum-like). In this case the solutions are defined by the metric (4.11) with the scale factors  $a_i = a_i(t_s) = A_i t_s^{\nu^i}$ , and the density

$$\kappa^2 \rho = \frac{-2 \langle u, u \rangle_*}{\langle u^{(\Lambda)} - u, u \rangle_*^2 t_s^2} \quad (4.13)$$

$$\nu^i = 2 \frac{u^i}{\langle u^{(\Lambda)} - u, u \rangle_*} \quad (4.14)$$

with  $u^i = G^{ij}u_j$  and  $A_i$  are positive constants,  $i = 0, \dots, n$ .

We will use the following explicit formulas for contravariant components

$$u^i = G^{ij}u_j = \frac{u_i}{d_i} + \frac{1}{2-D} \sum_{j=0}^n u_j \quad (4.15)$$

and the scalar product

$$\begin{aligned} \langle u^{(\Lambda)} - u, u \rangle_* = & -\sum_{i=0}^n \frac{(u_i)^2}{d_i} + \frac{2}{D-2} \sum_{i=0}^n u_i \\ & + \frac{1}{D-2} \left(\sum_{i=0}^n u_i\right)^2 \end{aligned} \quad (4.16)$$

### 4.2.2 Solutions with exponential behavior

The solutions with exponential behavior of scale factors take place for

$$\langle u^{(\Lambda)} - u, u \rangle_* = 0 \quad (4.17)$$

In this case, the solutions are determined by the metric (4.11) with the scale factors  $a_i = a_i(t_s) = A_i \exp(\nu^i t_s)$ , and the density  $\rho = \text{const.}$ ,

$$\nu^i = \varepsilon u^i \sqrt{-\frac{2\kappa^2 \rho}{\langle u, u \rangle_*}} \quad (4.18)$$

where  $\varepsilon = \pm 1$ ,  $u^i = G^{ij}u_j$  and  $A_i$  are positive constants,  $i = 0, \dots, n$ . Here  $\rho > 0$  for  $\langle u, u \rangle_* < 0$  and  $\rho < 0$  for  $\langle u, u \rangle_* > 0$ .

**Remark.** The model under consideration was integrated in Ref. [19] for  $\langle u, u \rangle_* < 0$ . The solutions from [25] were generalized in Ref. [20] to the case when a massless minimally coupled scalar field was added. Families of exceptional solutions with power-law and exponential behavior of scale factors in terms of synchronous time variable were singled out in Ref. [20] and correspond to a constant value of the scalar field:  $\varphi = \text{const}$ . When the scalar field is omitted we are led to the solutions presented above (in Ref. [25], these solutions were originally written in the harmonic time parametrization). It may be verified that the exceptional solutions with power-law dependence of scale factors are also valid when the restriction (4.6) is omitted. Moreover, it may be shown that for  $\langle u, u \rangle_* = 0$  the power-law solutions coincide with our vacuum Kasner-like solution. In this case the matter source vanishes since:  $\rho = 0$  in Eq. (4.13).

4.3 Acceleration and variation of  $G$

For the subspace  $(M_0, g^0)$ ,  $g^0$  to be flat,  $d_0 = 3$ , describes our 3-D space and  $(M_i, g^i)$  internal factor-spaces. We are interested in solutions with accelerated expansion of our space and small enough variations of  $G$  obeying experimental tests at the present moment [51]

$$\left| \frac{\dot{G}}{GH} \right| (t_{s0}) < 0.1 \tag{4.19}$$

Here we suppose that internal spaces are compact. Hence our 4-D constant is (see Ref. [37])

$$G = \text{const.} \cdot \prod_{i=1}^n (a_i^{-d_i}) \tag{4.20}$$

4.3.1 Power-law expansion with acceleration

For solutions with power-law expansion, the accelerated expansion of our space takes place for

$$\nu^0 > 1 \tag{4.21}$$

For  $D = 4$  when internal spaces are absent we get

$$\nu^0 = \frac{2}{6 - u_0} \tag{4.22}$$

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{6}(u_0 - 6)u_0 \neq 0 \tag{4.23}$$

that implies  $u_0 \neq 0$  and  $u_0 \neq 6$  (here  $\langle u, u \rangle_* = -\frac{1}{6}u_0^2 < 0$ ). The condition  $\nu^0 > 1$  is equivalent to  $4 < u_0 < 6$  or,

equivalently,

$$-\rho < p < -\frac{\rho}{3} \tag{4.24}$$

that agrees with the well-known result for  $D = 4$ .

For the power law solutions we get

$$\frac{\dot{G}}{G} = -\frac{\sum_{j=1}^n \nu^j d_j}{t_s}, \quad H = \frac{\dot{a}_0}{a_0} = \frac{\nu^0}{t_s} \tag{4.25}$$

and hence

$$\frac{\dot{G}}{GH} = -\frac{\sum_{j=1}^n \nu^j d_j}{\nu^0} \equiv \delta \tag{4.26}$$

The constant parameter  $\delta$  describes variations of the gravitational constant, according to Eq. (4.19)  $|\delta| < 0.1$ .

It follows from the definition of  $\nu^i$  in Eq. (4.14) that

$$\delta = -\frac{\sum_{i=1}^n u^i d_i}{u^0} \tag{4.27}$$

or, in terms of covariant components [see Eq. (4.15)]

$$\delta = -\frac{(D - 4)u_0 - 2 \sum_{i=1}^n u_i}{\frac{1}{3}(5 - D)u_0 + \sum_{i=1}^n u_i} \tag{4.28}$$

Thus, the relations (4.16), (4.21), (4.28) and the constraint (4.12) define a set of parameters  $u_i$  compatible with the acceleration and tests on  $G$ -dot.

In what follows, we will show that these relations do really define a non-empty set of parameters  $u_i$  describing equations of state.

**The case of constant  $G$**

Consider now the most important case  $\delta = 0$ , i.e. when the variation of  $G$  is absent:  $\dot{G} = 0$ . Indeed, there is a tendency of lowering the upper bound on  $G$ , e.g. according to arguments of Ref. [60],  $\delta < 10^{-4}$ . This constraint just follows from the identity

$$\frac{\dot{G}}{G} = \frac{\dot{\alpha}}{\alpha} \tag{4.29}$$

that may take place in some multidimensional models ( $\alpha$  is the fine structure constant).

**Isotropic case.** First, we consider an isotropic case when pressures in all internal spaces are coinciding. This takes place when

$$u_i = v d_i \tag{4.30}$$

$i = 1, \dots, n$ . For pressures in internal spaces we get from Eq. (4.5)

$$p_i = (1 - v)\rho \tag{4.31}$$

$i = 1, \dots, n$ . In the isotropic case we get from Eqs. (4.10) and (4.16)

$$\langle u, u \rangle_* = \frac{1}{2 - D} \left[ -\frac{1}{3}(d - 1)u_0 + 2du_0v - 2dv^2 \right] \tag{4.32}$$

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{2-D} \left[ 2u_0 + 2dv + \frac{1}{3}(d-1)u_0^2 - 2du_0v + 2dv^2 \right] \quad (4.33)$$

For  $\delta = 0$  we get in the isotropic case  $v = \frac{u_0}{2}$  or in terms of pressures

$$p_i = \frac{1}{2}(3p_0 - \rho), \quad i = 1, \dots, n \quad (4.34)$$

Substituting into Eqs. (4.32) and (4.33) we get

$$\langle u, u \rangle_* = -\frac{1}{6}u_0^2 \quad (4.35)$$

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{6}u_0(u_0 - 6) \quad (4.36)$$

Remarkably, we obtain the same relations as in the  $D = 4$  case (see Remark above). For our solution we should put  $u_0 \neq 0$  and  $u_0 \neq 6$ .

Using Eq. (4.30) we get  $u^0 = -u_0/6$  and  $u^i = 0$  for  $i > 0$ , hence  $\nu_i = 0$  for  $i = 1, \dots, n$ , i.e. all internal spaces are static.

Metric (4.11) reads in this case

$$g = -dt_s \otimes dt_s + A_0^2 t_s^{2\nu^0} g^0 + \sum_{i=1}^n A_i^2 g^i \quad (4.37)$$

where  $A_i$  are positive constants, and

$$\nu^0 = \frac{2}{6 - u_0} \quad (4.38)$$

We see that the power  $\nu^0$  is the same as in the  $D = 4$  case. For the density we get from Eq. (4.13)

$$\kappa^2 \rho = \frac{12}{(u_0 - 6)^2 t_s^2} \quad (4.39)$$

Thus, for equations of state (4.5) with relations (4.30) imposed, we are led to the solution Refs. (4.37)–(4.39) with Ricci-flat (e.g. flat) 3-metric and  $n$  static internal Ricci-flat spaces. For  $4 < u_0 < 6$ , or  $-\rho < p_0 < -\frac{\rho}{3}$ , we get an accelerated expansion of our 3-D Ricci-flat space.

**Non-isotropic case.** Let us consider the anisotropic (w.r.t. internal spaces) case with  $\delta = 0$ , or, equivalently [see Eq. (4.28)],

$$(D - 4)u_0 = 2 \sum_{i=1}^n u_i \quad (4.40)$$

This implies

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{6}u_0(u_0 - 6) - \Delta \quad (4.41)$$

$$\langle u, u \rangle_* = -\frac{1}{6}u_0^2 + \Delta \quad (4.42)$$

where

$$\Delta = \sum_{i=1}^n \frac{u_i^2}{d_i} - \frac{1}{d} \left( \sum_{i=1}^n u_i \right)^2 \geq 0 \quad (4.43)$$

The inequality in Eq. (4.43) could be readily proved using the well-known Cauchy-Schwartz inequality:

$$\left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \geq \left( \sum_{i=1}^n b_i c_i \right)^2 \quad (4.44)$$

Indeed, substituting  $b_i = \sqrt{d_i}$  and  $c_i = u_i/\sqrt{d_i}$  into Eq. (4.44) we get Eq. (4.43). The equality in Eq. (4.44) takes place only when the vectors  $(b_i)$  and  $(c_i)$  are linearly dependent, which for our choice, reads:  $u_i/\sqrt{d_i} = v\sqrt{d_i}$  where  $v$  is constant. Thus,  $\Delta = 0$  only in the isotropic case (4.30). In the non-isotropic case we get  $\Delta > 0$ .

In what follows, we will use the relation

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{6}(u_0 - u_0^+)(u_0 - u_0^-) \quad (4.45)$$

where

$$u_0^\pm = 3 \pm \sqrt{9 + 6\Delta} \quad (4.46)$$

are roots of quadratic polynomial (4.41), obeying  $u_0^- < 0$ ,  $u_0^+ > 6$  for  $\Delta > 0$ . It follows from Eq. (4.40) that  $u^0 = -u_0/6$  and hence

$$\nu^0 = -\frac{2u_0}{u_0(u_0 - 6) - 6\Delta}, \quad u_0 \neq u_0^\pm \quad (4.47)$$

The function  $\nu^0(u_0)$  is monotonically increasing: i) from 0 to  $+\infty$  in the interval  $(-\infty, u_0^-)$ ; ii) from  $-\infty$  to  $+\infty$  in the interval  $(u_0^-, u_0^+)$ ; iii) from  $-\infty$  to 0 in the interval  $(u_0^+, +\infty)$ .

The accelerated expansion of our space takes place when  $\nu^0 > 1$ , or, equivalently, when

$$(A) \quad u_0 \in (u_{0*}^-, u_0^-) \quad (4.48)$$

$$(B) \quad u_0 \in (u_{0*}^+, u_0^+) \quad (4.49)$$

$$u_{0*}^\pm = 2 \pm \sqrt{4 + 6\Delta} \quad (4.50)$$

In terms of  $w_0$ -parameter:

$$p_0 = w_0 \rho, \quad w_0 = 1 - \frac{u_0}{3} \quad (4.51)$$

these two branches read:

$$(A) \quad w_0^- = \sqrt{1 + \frac{2}{3}\Delta} < w_0 < \frac{1}{3} + \frac{2}{3}\sqrt{1 + \frac{3}{2}\Delta} = w_{0*}^- \quad (4.52)$$

$$(B) \quad w_0^+ = -\sqrt{1 + \frac{2}{3}\Delta} < w_0 < \frac{1}{3} - \frac{2}{3}\sqrt{1 + \frac{3}{2}\Delta} = w_{0*}^+ \quad (4.53)$$

The first branch (A) describes a superstiff matter ( $w_0 > 1$ ) with negative density. Indeed, the relation  $\rho < 0$  follows from Eq. (4.13) and  $\langle u, u \rangle_* > 0$ , see Eq. (4.40).

The second branch (B) corresponds to matter with a broken weak energy condition (since  $w_0 < -\frac{1}{3}$ ) and positive density (since  $\langle u, u \rangle_* < 0$ ). This matter is a phantom one (i.e.  $w_0 < -1$ ) when  $\Delta \geq 2$ . For  $\Delta < 2$  the interval  $(w_0^+, w_{0*}^+)$  contains both phantom points ( $w_0 < -1$ ) and non-phantom ones ( $w_0 > -1$ ).

**The case of varying  $G$**

Now we consider another important case  $\delta \neq 0$ , i.e. when the variation of  $G$  is non-zero:  $\dot{G} \neq 0$ . We use the bound  $|\delta| < 0.1$ , stating the smallness of  $\delta$ . Using Eq. (4.28) we get

$$\sum_{i=1}^n u_i = \frac{1}{2} dbu_0 \tag{4.54}$$

where  $d = D - 4$  and

$$b = b(\delta) = \frac{1 + \frac{1-d}{3d}\delta}{1 - \frac{\delta}{2}} \tag{4.55}$$

For the scalar product we get from Eq. (4.54)

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{A}{6} u_0^2 - Bu_0 - \Delta \tag{4.56}$$

$$\langle u, u \rangle_* = -\frac{A}{6} u_0^2 + \Delta \tag{4.57}$$

where  $\Delta$  was defined in Eq. (4.43) [see Eqs. (4.10) and (4.16)], and

$$A = A(\delta) = 1 - \frac{(d+2)\delta^2}{12d(1 - \frac{\delta}{2})^2} \tag{4.58}$$

$$B = B(\delta) = \frac{1 - \frac{\delta}{3}}{1 - \frac{\delta}{2}} \tag{4.59}$$

It should be noted that due to  $|\delta| < 0.1$   $A$  is positive  $A > 0$  and close to 1:  $|A - 1| < \frac{1}{3} \times 10^{-2}$ .

For the contravariant component  $u^0$ , we get from Eqs. (4.15) and (4.54)

$$u^0 = -\frac{C}{6} u_0 \tag{4.60}$$

$$C = C(\delta) = 3B - 2 = \frac{1}{1 - \frac{\delta}{2}} \tag{4.61}$$

It follows from Eqs. (4.56) and (4.60) that [see Eq. (4.14)]

$$\nu^0 = -\frac{2Cu_0}{\frac{A}{6}u_0^2 - Bu_0 - \Delta} \tag{4.62}$$

Here  $u_0 \neq u_0^\pm$ , where

$$u_0^\pm = u_0^\pm(\delta) = \frac{1}{A}(3B \pm \sqrt{9B^2 + 6A\Delta}) \tag{4.63}$$

are roots of quadratic polynomial (4.56).

In what follows, we will use the identity

$$\nu^0 - 1 = -\frac{Au_0^2 - 4u_0 - 6\Delta}{Au_0^2 - 6Bu_0 - 6\Delta} \tag{4.64}$$

**Isotropic case.** Let us consider an isotropic case (4.30). In this case we obtain from Eq. (4.54)

$$v = \frac{1}{2} dbu_0 \tag{4.65}$$

or, in terms of pressures

$$p_i = \frac{1}{2}[3bp_0 + (2 - 3b)\rho], \quad i = 1, \dots, n \tag{4.66}$$

For scalar products we get

$$\langle u, u \rangle_* = -\frac{A}{6} u_0^2 \tag{4.67}$$

$$\langle u^{(\Lambda)} - u, u \rangle_* = \frac{A}{6} u_0(u_0 - 6B) \tag{4.68}$$

For our solution we should put  $u_0 \neq 0$  and  $u_0 \neq 6B/A$ .

The metric (4.11) reads in our case

$$g = -dt_s \otimes dt_s + A_0^2 t_s^{2\nu^0} g^0 + t_s^{2\nu} \sum_{i=1}^n A_i^2 g^i \tag{4.69}$$

where  $A_i$  are positive constants,

$$\nu^0 = -\frac{2C}{Au_0 - 6B} \tag{4.70}$$

$$\nu = \nu^i = \frac{2\delta}{d(1 - \frac{\delta}{2})(Au_0 - 6B)} \tag{4.71}$$

$i = 1, \dots, n$ . The last formula follows from Eq. (4.14) and

$$u^i = \frac{u_0}{6d(1 - \frac{\delta}{2})} \tag{4.72}$$

We see that the power  $\nu^0$  does not coincide for  $\delta \neq 0$  with that from  $D = 4$  case.

For the density, since  $A > 0$ , we get from Eq. (4.13)

$$\kappa^2 \rho = \frac{12A}{(u_0 - 6B)^2 t_s^2} > 0 \tag{4.73}$$

The condition of accelerated expansion of the 3-D space  $\nu^0 > 1$  reads as

$$\frac{4}{A(\delta)} < u_0 < \frac{6B(\delta)}{A(\delta)} \tag{4.74}$$

or, equivalently, in terms of the  $w_0$ -parameter ( $p_0 = w_0 \rho$ ) [Eq. (4.51)]

$$w_0^+(\delta) = 1 - \frac{2B(\delta)}{A(\delta)} < w_0 < 1 - \frac{4}{3A(\delta)} = w_{0*}^+(\delta) \tag{4.75}$$

For  $\delta > 0$  we get an isotropic contraction of total internal space  $M_1 \times \dots \times M_n$ . In this case  $w_0^+(\delta) < -1$ , and hence phantom matter may occur with the equation of state close to the vacuum one, since

$$w_0^+(\delta) + 1 = -\frac{\delta(1 + \frac{\delta}{d})}{3[1 - \delta + \frac{(d-1)\delta^2}{6d}]} \quad (4.76)$$

For small  $\delta$  we have  $w_0^+(\delta) = 1 - \frac{\delta}{3} + O(\delta^2)$ .

For  $\delta < 0$  we get an isotropic expansion of the total internal space. In this case  $w_0^+(\delta) > -1$  and the phantom matter does not occur. In both cases  $w_{0*}^+(\delta) < -\frac{1}{3}$  and  $w_{0*}^+(\delta) + \frac{1}{3} = O(\delta^2)$ .

**Non-isotropic case.** Now we consider a non isotropic case  $\Delta > 0$  when  $\delta \neq 0$ .

Using relation (4.64) we obtain

$$\nu^0 - 1 = -\frac{(u_0 - u_{0*}^+)(u_0 - u_{0*}^-)}{(u_0 - u_0^+)(u_0 - u_0^-)} \quad (4.77)$$

where  $u_0^\pm = u_{0*}^\pm(\delta)$  were defined in Eq. (4.43) and

$$u_{0*}^\pm = u_{0*}^\pm(\delta) = 2 \pm \sqrt{4 + 6A\Delta} \quad (4.78)$$

The accelerated expansion of the 3-D space takes place when  $\nu^0 > 1$ , i.e.,

- (A)  $u_0 \in (u_{0*}^-(\delta), u_0^-(\delta))$
- (B)  $u_0 \in (u_{0*}^+(\delta), u_0^+(\delta))$

In terms of  $w_0$ -parameter  $p_0 = w_0\rho$ ,  $w_0 = 1 - \frac{u_0}{3}$ , these two branches read:

- (A)  $w_0^-(\delta) < w_0 < w_{0*}^-(\delta)$
- (B)  $w_0^+(\delta) < w_0 < w_{0*}^+(\delta)$

where

$$w_0^\pm(\delta) = 1 - \frac{u_0^\pm(\delta)}{3}, \quad w_{0*}^\pm(\delta) = 1 - \frac{u_{0*}^\pm(\delta)}{3} \quad (4.80)$$

For small  $\delta$  we have

$$w_0^\pm(\delta) = w_0^\pm(0) - \frac{\delta}{6} \left( 1 \pm \frac{3}{\sqrt{9 + 6\Delta}} \right) + O(\delta^2) \quad (4.81)$$

$$w_{0*}^\pm(\delta) = w_{0*}^\pm(0) + O(\delta^2) \quad (4.82)$$

Thus, for small  $\delta$  the lower and upper bounds on  $w_0$  have a small deviation from those obtained in the case  $\delta = 0$ . For small  $\delta$  the upper bounds shift only on a  $O(\delta^2)$  term while the lower bounds shift on a  $O(\delta)$  term.

The first branch (A) describes a superstiff matter  $w_0 > 1$ , since  $w_0^-(\delta) > 1$  due to  $u_0^-(\delta) < 0$ . It may be shown that the density is negative in this case since  $\langle u, u \rangle_* > 0$ .

For the second branch (B), we get for the upper bound  $w_{0*}^+(\delta) < -1/3$  due to  $u_{0*}^+(\delta) > 4$ . For the lower bound we find that  $w_0^+(\delta) < -1$  only if

$$\Delta > 6(A(\delta) - B(\delta)) = -\frac{\delta}{(1 - \frac{\delta}{2})^2} \quad (4.83)$$

This is the condition on appearance of the phantom matter. For  $\delta > 0$  this inequality is valid, but for  $\delta < 0$  it is satisfied only for a big enough  $\Delta$ .

#### 4.3.2 Exponential expansion with acceleration

For solutions from Section 4.2.2, an accelerated expansion of our space takes place for  $\nu^0 > 0$ . For  $D = 4$ , when internal spaces are absent, we get  $u^0 = -u_0/6$  and  $\langle u, u \rangle_* = -\frac{1}{6}u_0^2$ ,  $\langle u^{(\Lambda)} - u, u \rangle_* = \frac{1}{6}(u_0 - 6)u_0 = 0$ , which implies  $u_0 = 6$ , or, equivalently,  $p = -\rho$ .

We get

$$\nu^0 = -\varepsilon \sqrt{\frac{\kappa^2 \rho}{3}} \quad (4.84)$$

which agrees with the well-known result for the  $D = 4$  de-Sitter solution with cosmological constant  $\Lambda = \kappa^2 \rho > 0$ . The condition  $\nu^0 > 0$  is equivalent to  $\varepsilon = -1$ .

For our exponential solutions we get

$$\frac{\dot{G}}{G} = -\sum_{j=1}^n \nu^j d_j, \quad H = \frac{\dot{a}_0}{a_0} = \nu^0 \quad (4.85)$$

and hence

$$\dot{G}/(GH) = -\frac{1}{\nu^0} \sum_{j=1}^n \nu^j d_j \equiv \delta \quad (4.86)$$

i.e. we get the same relation as in Eq. (4.26).

The constant parameter  $\delta$  describing the variation of the gravitational constant obeys the restriction  $|\delta| < 0.1$ .

Due to Eq. (4.86) we get the same relations (4.27) and (4.28) for  $\delta$  as in the power-law case.

#### The case of constant $G$

**Isotropic case.** Consider the important case  $\delta = 0$ , i.e., when the variation of  $G$  is absent:  $\dot{G} = 0$ .

First, we consider the isotropic case when pressures coincide in all internal spaces, see Eq. (4.30). Here, we obtain the same relations as in  $D = 4$  case. For our solution, we should put  $u_0 \neq 0$  and hence, due to Eq. (4.17),  $u_0 = 6$ , i.e.  $p_0 = -\rho$ .

Using Eq. (4.30), we get  $u^0 = -u_0/6 = -1$  and  $u^i = 0$  for  $i > 0$ , hence  $\nu_i = 0$  for  $i = 1, \dots, n$ , i.e., all internal spaces are static.

The metric (4.11) reads in this case as

$$g = -dt_s \otimes dt_s + A_0^2 \exp(2\nu^0 t_s) g^0 + \sum_{i=1}^n A_i^2 g^i \quad (4.87)$$

where  $A_i$  are positive constants, and

$$\nu^0 = -\varepsilon \sqrt{\frac{\kappa^2 \rho}{3}} \quad (4.88)$$

For accelerated expansion we get  $\varepsilon = -1$ . We see that the power  $\nu^0$  is the same as in  $D = 4$  case.

**Anisotropic case.** Consider now the anisotropic (w.r.t. internal spaces) case with  $\delta = 0$ , or, equivalently when Eq. (4.40) is satisfied. It follows from Eq. (4.40) that  $u^0 = -u_0/6$  and hence

$$\nu^0 = -\varepsilon \frac{u_0}{6} \sqrt{\frac{12\kappa^2\rho}{u_0^2 - 6\Delta}}, \quad u_0 = u_0^\pm \quad (4.89)$$

The accelerated expansion of our space takes place when  $\nu^0 > 0$ , or, equivalently, when either

$$\begin{aligned} \text{(A)} \quad & u_0 = u_0^-, \quad \varepsilon = 1 \quad \text{or} \\ \text{(B)} \quad & u_0 = u_0^+, \quad \varepsilon = -1 \end{aligned} \quad (4.90)$$

In terms of the parameter  $w_0$ , see Eq. (4.51), these two branches read:

$$\text{(A)} \quad w_0 = w_0^- = \sqrt{1 + \frac{2}{3}\Delta} \quad (4.91)$$

$$\text{(B)} \quad w_0 = w_0^+ = -\sqrt{1 + \frac{2}{3}\Delta} \quad (4.92)$$

The first branch (A) describes the super-stiff matter ( $w_0 > 1$ ) with negative density  $\rho < 0$ .

The second branch (B) corresponds to matter with positive density (since  $\langle u, u \rangle_* < 0$ ). This matter is the phantom one (i.e.,  $w_0 < -1$ ) when  $\Delta > 0$ .

#### The case of varying $G$

Now, we consider the case  $\delta \neq 0$ , i.e., when  $\dot{G} \neq 0$ . We take the observational bound  $|\delta| < 0.1$ . Using Eq. (4.28), we get relations (4.54) and (4.55). It follows from Eqs. (4.56) and (4.60) that

$$\nu^0 = -\varepsilon \frac{Cu_0}{6} \sqrt{\frac{12\kappa^2\rho}{Au_0^2 - 6\Delta}} \quad (4.93)$$

Here  $u_0 = u_0^\pm(\delta)$  are defined in Eq. (4.63).

**Isotropic case.** Let us consider the isotropic case (4.30). We should put  $u_0 \neq 0$  and hence  $u_0 = 6B/A > 0$ . The metric (4.11) reads

$$g = -dt_s \otimes dt_s + A_0^2 \exp(2\nu^0 t_s) g^0 + \exp(2\nu t_s) \sum_{i=1}^n A_i^2 g^i \quad (4.94)$$

where  $A_i$  are positive constants,

$$\nu^0 = -\varepsilon \frac{Cu_0}{6} \sqrt{\frac{12\kappa^2\rho}{Au_0^2}}, \quad \text{and} \quad (4.95)$$

$$\nu = \nu^i = \varepsilon \frac{\delta u_0}{6d(1-\delta/2)} \sqrt{\frac{12\kappa^2\rho}{Au_0^2}}, \quad i = 1, \dots, n \quad (4.96)$$

We see that the power  $\nu^0$  does not coincide for  $\delta \neq 0$  with that in the  $D = 4$  case.

The accelerated expansion condition for 3-D space,  $\nu^0 > 0$ , reads as

$$u_0 = \frac{6B(\delta)}{A(\delta)}, \quad \varepsilon = -1 \quad (4.97)$$

or, equivalently, in terms of  $w_0$  (4.51) ( $p_0 = w_0\rho$ )

$$w_0 = w_0^+(\delta) = 1 - \frac{2B(\delta)}{A(\delta)} \quad (4.98)$$

For  $\delta > 0$ , we get an isotropic contraction of the whole internal space  $M_1 \times \dots \times M_n$ . In this case  $w_0^+(\delta) < -1$  and hence phantom matter occurs with the equation of state close to the vacuum one since

$$w_0^+(\delta) + 1 = -\frac{\delta(1+\delta/d)}{3[1-\delta+(d-1)\delta^2/(6d)]} \quad (4.99)$$

For small  $\delta$  we have  $w_0^+(\delta) = -1 - \delta/3 + O(\delta^2)$ .

For  $\delta < 0$  we get an isotropic expansion of the whole internal space. Then,  $w_0^+(\delta) > -1$ , and phantom matter does not occur.

**Anisotropic case.** Consider the anisotropic case  $\Delta > 0$  when  $\delta \neq 0$ . Here  $u_0 = u_0^\pm(\delta)$ , see Eq. (4.63).

Accelerated expansion of 3-dimensional space takes place when  $\nu^0 > 0$ , or, equivalently, when either

$$\begin{aligned} \text{(A)} \quad & u_0 = u_0^-(\delta), \quad \varepsilon = 1; \quad \text{or} \\ \text{(B)} \quad & u_0 = u_0^+(\delta), \quad \varepsilon = -1 \end{aligned} \quad (4.100)$$

In terms of the parameter  $w_0$  ( $p_0 = w_0\rho$ ,  $w_0 = 1 - \frac{u_0}{3}$ ) these two branches read:

$$\text{(A)} \quad w_0 = w_0^-(\delta), \quad \text{(B)} \quad w_0 = w_0^+(\delta) \quad (4.101)$$

where  $w_0^\pm(\delta) = 1 - u_0^\pm(\delta)/3$ .

For small  $\delta$  [see Eq. (4.81)] the parameter  $w_0^\pm(\delta)$  has a small deviation from that obtained for  $\delta = 0$ .

The branch (A) describes super-stiff matter  $w_0 > 1$  since  $w_0^-(\delta) > 1$  due to  $u_0^-(\delta) < 0$ . It may be shown that the density is negative in this case since  $\langle u, u \rangle_* > 0$ .

For branch (B) we find that  $w_0^+(\delta) < -1$  only if (4.83) is satisfied.

This is the condition on appearance of the phantom matter. For  $\delta > 0$  this inequality is valid, but for  $\delta < 0$  it is satisfied only for a big enough value of the anisotropy parameter  $\Delta$ , see Eq. (4.83).

## 5 S-brane solution with acceleration and small variation of $G$

### 5.1 The model

In this section we deal with  $S$ -brane solutions describing

two electric branes and a set of  $l$  scalar fields [61].

The model is governed by the action

$$S = \int d^D x \sqrt{|g|} \{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a=1,2} \frac{1}{N_a!} \exp[2\lambda_a(\varphi)] (F^a)^2 \} \quad (5.1)$$

Here  $g = g_{MN}(x) dx^M \otimes dx^N$  is a metric of pseudo-Euclidean signature  $(-, +, \dots, +)$ ,  $F^a = dA^a$  is a form of rank  $N_a$ ,  $(h_{\alpha\beta})$  is a non-degenerate symmetric matrix,  $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$  is a vector of  $l$  scalar fields,  $\lambda_a(\varphi) = \lambda_{a\alpha} \varphi^\alpha$ , is a linear function. Here  $a = 1, 2$  and  $\alpha, \beta = 1, \dots, l$ . In Eq. (5.1)  $|g| = |\det g_{MN}|$ .

We consider as an example the manifold

$$M = (0, +\infty) \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \quad (5.2)$$

where  $M_i$  are oriented Riemannian Ricci-flat spaces of dimension  $d_i$ ,  $i = 1, \dots, 5$ , and  $d_1 = 1$ .

Let two electric branes be defined by sets  $I_1 = \{1, 2, 3\}$  and  $I_2 = \{1, 2, 4\}$ . They intersect on  $M_1 \times M_2$ . The first brane also covers  $M_3$  and the second one covers  $M_4$ . The first brane corresponds to the form  $F^1$ , and the second one corresponds to the form  $F^2$ .

For world-volume dimensions of branes we get  $d(I_s) = N_s - 1 = 1 + d_2 + d_{2+s}$ ,  $s = 1, 2$  and  $d(I_1 \cap I_2) = 1 + d_2$  is the dimension of the branes intersection.

Consider now the  $S$ -brane solution governed by the function  $\hat{H} = 1 + P\rho^2$ , where  $\rho$  is a time variable,  $P = KQ^2/8$ ,  $s = 1, 2$ .

$$K = K_s = d(I_s) \left[ 1 + \frac{d(I_s)}{2-D} \right] + \lambda_{s\alpha} \lambda_{s\beta} h^{\alpha\beta} \quad (5.3)$$

$s = 1, 2$  is supposed to be non zero. Thus,  $K_1 = K_2 = K$ . Here  $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$ .

The branes intersection rule is the following one

$$d(I_1 \cap I_2) = \frac{d(I_1)d(I_2)}{D-2} + \lambda_{1\alpha} \lambda_{2\beta} h^{\alpha\beta} - \frac{1}{2}K \quad (5.4)$$

This relation corresponds to Lie algebra  $A_2$  [62, 19]. Remember that  $K_s = (U^s, U^s)$ ,  $s = 1, 2$ , where ‘‘electric’’  $U^s$  vectors and scalar products were defined in Refs. [27, 20] [see also Refs. [21, 62]]. Relations  $K_1 = K_2$  and (5.4) just follow from the formula  $(A_{ss'}) = (2(U^s, U^{s'}) / (U^{s'}, U^{s'}))$ , where  $(A_{ss'})$  is the Cartan matrix for  $A_2$  (with  $A_{12} = A_{21} = -1$ ).

We consider the following exact solution

$$g = \hat{H}^{2A} [-d\rho \otimes d\rho + \hat{H}^{-4B} (\rho^2 g^1 + g^2) + \hat{H}^{-2B} g^3 + \hat{H}^{-2B} g^4 + g^5] \quad (5.5)$$

$$\exp(\varphi^\alpha) = \hat{H}^{B\lambda_1^\alpha + B\lambda_2^\alpha} \quad (5.6)$$

$$F^1 = -Q\hat{H}^{-2}\rho d\rho \wedge \tau_1 \wedge \tau_2 \wedge \tau_3 \quad (5.7)$$

$$F^2 = -Q\hat{H}^{-2}\rho d\rho \wedge \tau_1 \wedge \tau_2 \wedge \tau_4 \quad (5.8)$$

where

$$A = 2K^{-1} \sum_{s=1,2} \frac{d(I_s)}{D-2} \quad (5.9)$$

$$B = 2K^{-1} \quad (5.10)$$

$s = 1, 2$ . Here  $g_1 = dx \otimes dx$ ,  $\tau_1 = dx$  and  $\tau_i$  denotes a volume form on  $M_i$ . Remember that all Ricci-flat metrics  $g^2, \dots, g^5$  have Euclidean signatures.

This solution is a special case of a more general solution from Ref. [63] corresponding to Lie algebra  $A_2$ .

## 5.2 Solutions with acceleration

Let us introduce a ‘‘synchronous’’ time variable  $\tau = \tau(\rho)$  from the following relation:

$$\tau = \int_0^\rho d\bar{\rho} [\hat{H}(\bar{\rho})]^A \quad (5.11)$$

We put  $P < 0$  and hence  $K < 0$ , which implies  $A < 0$ . Consider two intervals of the parameter  $A$ : (i)  $A < -1$  and (ii)  $-1 < A < 0$ .

For the case (i) the function  $\tau = \tau(\rho)$  is monotonically increasing from 0 to  $+\infty$ , for  $\rho \in (0, \rho_1)$ , where  $\rho_1 = |P|^{-1/2}$ , while for the case (ii) it is monotonically increasing from 0 to the finite value  $\tau_1 = \tau(\rho_1)$ .

Let space  $M_5$  be our 3-D space with a scale factor  $a_5 = \hat{H}^A$ .

For the branch (i) we get an asymptotical relation  $a_5 \sim \text{const } \tau^\nu$ , for  $\tau \rightarrow +\infty$ , where

$$\nu = \frac{A}{A+1} \quad (5.12)$$

and  $\nu > 1$ . For (ii) we obtain  $a_5 \sim \text{const} \cdot (\tau_1 - \tau)^\nu$ , for  $\tau \rightarrow \tau_1 - 0$ , where  $\nu < 0$  due to Eq. (5.12). Thus, we get an asymptotical accelerated expansion of 3-D factor space  $M_5$  in both cases (i) and (ii) and  $a_5 \rightarrow +\infty$ .

Moreover, it may be readily verified that the accelerated expansion takes place for all  $\tau > 0$ , i.e.

$\dot{a}_5 > 0$ ,  $\ddot{a}_5 > 0$ . Here, and in what follows, we denote  $\dot{f} = df/d\tau$ .

Indeed, using the relation  $d\tau/d\rho = \hat{H}^A$  [see Eq. (5.11)] we get

$$\dot{a}_5 = \frac{d\rho}{d\tau} \frac{da_5}{d\rho} = \frac{2|A||P|\rho}{\hat{H}} \quad (5.13)$$

and

$$\ddot{a}_5 = \frac{d\rho}{d\tau} \frac{d}{d\rho} \frac{da_5}{d\rho} = \frac{2|A||P|}{\hat{H}^{2+A}} (1 + |P|\rho^2) \quad (5.14)$$

that certainly implies inequalities for derivatives of scale-factor  $a_5$ .

Now we consider the variation of  $G$ . For our model the 4-D gravitational coupling (in Jordan frame) is

$$G = \text{const.} \cdot \prod_{i=1}^4 (a_i^{-d_i}) = \hat{H}^{2A} \rho^{-1} \quad (5.15)$$

where

$$a_1 = \hat{H}^{A-2B} \rho, \quad a_2 = \hat{H}^{A-2B}, \quad a_3 = a_4 = \hat{H}^{A-B} \quad (5.16)$$

are scale factors of “internal” spaces  $M_1, \dots, M_4$ , respectively.

The function  $G(\tau)$  has minimum at the point  $\tau_0$  corresponding to

$$\rho_0 = \frac{|P|^{-1}}{1+4|A|} \quad (5.17)$$

At this point the variation of  $G$  is zero. This follows from the explicit relation for the dimensionless variation of  $G$

$$\delta = \dot{G}/(GH) = 2 + \frac{1-|P|\rho^2}{2A|P|\rho^2} \quad (5.18)$$

where  $H = \frac{\dot{a}_5}{a_5}$  is the Hubble parameter. The function  $G(\tau)$  is monotonically decreasing from  $+\infty$  to  $G_0 = G(\tau_0)$  for  $\tau \in (0, \tau_0)$  and monotonically increasing from  $G_0 = G(\tau_0)$  to  $+\infty$  for  $\tau \in (\tau_0, \bar{\tau}_1)$ . Here  $\bar{\tau}_1 = +\infty$  for the case (i) and  $\bar{\tau}_1 = \tau_1$  for the case (ii). The scale factors  $a_2(\tau), a_3(\tau), a_4(\tau)$  are monotonically increasing from 1 to 0 for  $\tau \in (0, \bar{\tau}_1)$ , since the powers  $A-B$  and  $A-2B$  are positive and  $P < 0$ . The scale factor  $a_1(\tau)$ , is monotonically decreasing from zero to  $a_1(\tau_2)$  for  $\tau \in (0, \tau_2)$  and monotonically increasing from  $a_1(\tau_2)$  to zero for  $\tau \in (\tau_2, \bar{\tau}_1)$ , where  $\tau_2$  is the point of maximum.

We should treat only solutions with accelerated expansion of our space and small enough variations of the gravitational constant obeying the present experimental constraint

$$|\delta| < 0.1 \quad (5.19)$$

Here, like in the case of the model with two curvatures [64],  $\tau$  is restricted by the interval containing  $\tau_0$ . It follows from Eq. (5.18) that in the asymptotical regions  $\delta \rightarrow 2$ , which does not agree with experimental bounds (5.19). This restriction is satisfied for the interval containing the point  $\tau_0$  where  $\delta = 0$ .

The calculation of  $G$ -dot in the linear approximation near  $\tau_0$ , gives the following approximate relation for dimensionless parameter of reciprocal variation of  $G$  [61]

$$\delta \approx (8 + 2|A|^{-1})H_0(\tau - \tau_0) \quad (5.20)$$

where  $H_0 = H(\tau_0)$  (compare with the analogous relation in Ref. [64]). This relation gives approximate bounds on values of the time variable  $\tau$  allowed by the restriction on  $G$ -dot.

It should be stressed that the solution under consideration with  $P < 0$ ,  $d_1 = 1$  and  $d_5 = 3$  takes place when the configuration of branes, the matrix  $(h_{\alpha\beta})$  and dilatonic coupling vectors  $\lambda_a$ , obey the relations (5.3), (5.4) with  $K < 0$ . This is not possible when  $(h_{\alpha\beta})$  is positive definite, since in this case  $K > 0$ . In the next subsection we give an example of a setup obeying (5.3) and (5.4), by introducing “phantom” fields.

### 5.3 Example

Let us consider the following particular example:  $N_1 = N_2$ , i.e. the ranks of forms are equal, and  $l = 2$ ,  $(h_{\alpha\beta}) = -(\delta_{\alpha\beta})$ , i.e. there are two “phantom” scalar fields. We also put  $d_3 = d_4$ .

Then relations (5.3) and (5.4) read as

$$\lambda_1^2 = \lambda_2^2 = d(I) \left[ 1 + \frac{d(I)}{2-D} \right] - K \quad (5.21)$$

and

$$\lambda_1 \lambda_2 = d_\cap + \frac{[d(I)]^2}{2-D} + \frac{1}{2}K \quad (5.22)$$

where  $d(I) = d(I_1) = d(I_2) = 1 + d_2 + d_3$ ,  $d_\cap = d(I_1 \cap I_2) = 1 + d_2$  and  $K < 0$ . Relations (5.21) and (5.22) are compatible since it may be verified that they imply

$$\frac{\lambda_1 \lambda_2}{|\lambda_1| |\lambda_2|} \in (-1, +1) \quad (5.23)$$

i.e. vectors  $\lambda_1, \lambda_2$  belonging to Euclidean space  $\mathbb{R}^2$  and obeying relations (5.21), (5.22) do exist. The left side of Eq. (5.23) gives  $\cos \theta$ , where  $\theta$  is the angle between these two vectors.

We get in this special case

$$A = \frac{4d(I)}{K(D-2)} \quad (5.24)$$

For  $K \rightarrow -\infty$  the allowed time interval  $(\tau_-, \tau_+)$  of accelerated expansion obeying  $G$ -dot restriction (5.19) vanishes, i.e.  $\tau_+ - \tau_- \rightarrow 0$  [see Eq. (5.20)].

## 6 Conclusions

In this paper we considered different cosmological models in diverse dimensions leading to a relatively small time variation of the effective gravitational constant  $G$ .

We estimated the possible variations of the gravita-

tional constant  $G$  in the framework of a generalized (Bergmann-Wagoner-Nordtvedt) scalar-tensor theory of gravity on the basis of the field equations without using their special solutions. Specific estimates were essentially related to the values of other cosmological parameters (Hubble and acceleration parameters, dark matter density, etc.), but the values of  $\dot{G}/G$  compatible with modern observations did not exceed  $10^{-12}/y$ .

We also considered the multidimensional cosmological model with an  $m$ -component anisotropic (“perfect”) fluid. The multidimensional Hilbert-Einstein equations led to relations between  $\dot{G}$  and cosmological parameters.

In case of two factor spaces with non-zero curvatures without matter, we suggested a mechanism for predicting small  $\dot{G}$ . When the 3-D-space has a negative curvature and the internal space has a positive curvature, we obtained at some time interval an accelerating expansion of our 3-dimensional space and a small value of  $\dot{G}/G$ . We have shown that this result is compatible with the exact  $1+3+6$  solution from Ref. [54]. (Recall that only three exact solutions are known for a vacuum cosmological model with a product of two Einstein spaces, see Ref. [54].)

We also presented another example where two factor spaces are Ricci-flat and for a two-component example (dust + 5-brane) we obtained a small enough variation of  $G$ .

Further, we considered multidimensional cosmological models describing the dynamics of  $n+1$  Ricci-flat factor spaces  $M_i$  in the presence of a one-component anisotropic (perfect) fluid with pressures in all spaces proportional to the density:  $p_i = w_i \rho$ ,  $i = 0, \dots, n$ . Solutions with accelerated expansion of our 3-D space  $M_0$  and small enough variation of the gravitational constant  $G$  were found. These solutions have either exponential or power-law behavior of scale factors w.r.t. a synchronous time variable. In both cases they exist for two branches of parameter  $w_0$ . The first branch describes superstiff matter with  $w_0 > 1$  (and negative energy density), while the second one may contain phantom matter with  $w_0 < -1$  (and positive energy density). Here, contrary to the two-curvature model, the experimental bounds on  $\dot{G}$  are satisfied for all allowed values of the synchronous time variable.

We considered an  $S$ -brane solution with two non-composite electric branes and a set of  $l$  scalar fields as well. The solution, corresponding to Lie algebra  $A_2$ , contains five factor spaces, and the fifth one,  $M_5$ , is interpreted as our 3-D space. As in the model with two non-zero curvatures, we found that there exists a time interval where accelerated expansion of our 3-D space co-exists with a small enough value of  $\dot{G}/G$  obeying the experi-

mental bounds. Similar results for other rank 2 algebras were obtained in Ref. [65].

Thus, here we have shown that there exist different possible ways of explaining relatively small time variations of the effective gravitational constant  $G$  compatible with modern cosmological data (e.g. acceleration): we may consider either 4-D scalar-tensor theories or multidimensional cosmological models with different matter sources. The experimental bounds on  $\dot{G}$  may be satisfied either in some time interval or for all allowed values of the synchronous time variable [from  $(0, +\infty)$  for the power-law case or from  $(-\infty, +\infty)$  for the exponential case].

We also recently considered [66] the multidimensional gravity with a Lagrangian containing the Ricci tensor squared and the Kretschmann invariant. In a Kaluza-Klein approach with a single compact extra space of arbitrary dimension, with the aid of a slow-change approximation (as compared with the Planck scale), we built a class of spatially flat cosmological models in which both the observed scale factor  $a(\tau)$  and the extra-dimensional one,  $b(\tau)$ , grow exponentially at large times, but  $b(\tau)$  grows slowly enough to admit variations of the effective gravitational constant  $G$  within observational limits.

**Acknowledgements** The work was supported in part by the RFBR grant 07-02-13614-ofi-ts.

---

## References

1. K. P. Staniukovich and V. N. Melnikov, *Hydrodynamics, Fields and Constants in the Theory of Gravitation*, Energoatomizdat, Moscow, 1983 (in Russian); English updated translation of first 5 sections in: V. N. Melnikov, *Fields and Constraints in the Theory of Gravitation*, CBPF MO-02/02, Rio de Janeiro, 2002
2. V. N. Melnikov, *Multidimensional Classical and Quantum Cosmology and Gravitation, Exact Solutions and Variations of Constants*, CBPF-NF-051/93, Rio de Janeiro, 1993
3. V. N. Melnikov, in: *Cosmology and Gravitation*, Edited by M. Novello, Editions Frontiers, Singapore, 1994: 147
4. V. N. Melnikov, *Multidimensional Cosmology and Gravitation*, CBPF-MO-002/95, Rio de Janeiro, 1995
5. V. N. Melnikov, in: *Cosmology and Gravitation II*, Edited by M. Novello, Editions Frontiers, Singapore, 1996: 465
6. V. N. Melnikov, *Exact Solutions in Multidimensional Gravity and Cosmology III*, CBPF-MO-03/02, Rio de Janeiro, 2002: 297
7. S. A. Kononogov, V. N. Melnikov, and V. V. Khrushchov, *Izmeritel'naya Tekhnika*, 2007, 3: 3
8. S. A. Kononogov, V. N. Melnikov, and V. V. Khrushchov, *Izmeritel'naya Tekhnika*, 2008, 8: 3
9. S. A. Kononogov, V. N. Melnikov, and V. V. Khrushchov,

- Izmeritel'naya Tekhnika, 2008, 10: 3
10. V. N. Melnikov, *Int. J. Theor. Phys.*, 1994, 33: 1569
  11. V. de Sabbata, V. N. Melnikov, and P. I. Pronin, *Prog. Theor. Phys.*, 1992, 88: 623
  12. V. N. Melnikov, in: *Gravitational Measurements, Fundamental Metrology and Constants*, Edited by V. de Sabbata and V. N. Melnikov, Kluwer Academic Publ., Dordrecht, 1988: 283
  13. V. N. Melnikov, Gravity as a key problem of the millennium, in: *Proceedings of 2000 NASA/JPL Conference on Fundamental Physics in Microgravity*, Solvang, CA, USA, 2000; NASA Document D-21522, 2001: 4.1–4.17
  14. V. N. Melnikov, arXiv: gr-qc/0007067
  15. V. N. Melnikov, Gravity and cosmology as key problems of the millennium, in: *Albert Einstein Century International Conference*, Paris, June 2005, AIP Conf. Proc., NY, 2006 861: 109
  16. S. A. Kononogov and V. N. Melnikov, Fundamental physical constants, gravitational coupling and space SEE Project, *Izmer. Tekhnika*, 2005, 6: 3
  17. V. N. Melnikov, Problems of Gravitation, Cosmology and Fundamental Constants, in: *Physical Interpretation of Relativity Theory*, Edited by M. C. Duffy et al., Moscow, BM-STU Publ., 2007: 17
  18. V. N. Melnikov, *Grav. Cosmol.*, 2007, 13, No. 2(50): 81
  19. V. D. Ivashchuk and V. N. Melnikov, *Class. Quantum Grav.*, 2001, 18: R82; arXiv: hep-th/0110274
  20. V. D. Ivashchuk and V. N. Melnikov, *Phys. Lett. B*, 1997, 403: 23
  21. V. D. Ivashchuk and V. N. Melnikov, *Class. Quantum Grav.*, 1997, 14: 3001; arXiv: hep-th/9705036
  22. V. D. Ivashchuk and V. N. Melnikov, *Class. Quantum Grav.*, 1998, 15: 3941
  23. V. R. Gavrilov, V. D. Ivashchuk, and V. N. Melnikov, *J. Math. Phys.*, 1995, 36: 5829; arXiv: gr-qc/9407019
  24. U. Bleyer, V. D. Ivashchuk, V. N. Melnikov, and A. I. Zhuk, *Nucl. Phys. B*, 1994, 429: 177
  25. V. D. Ivashchuk and V. N. Melnikov, *Int. J. Mod. Phys. D*, 1994, 3(4): 795; arXiv: gr-qc/9403063
  26. V. D. Ivashchuk and V. N. Melnikov, *Grav. Cosmol.*, 1995, 1: 133; arXiv: hep-th/9503223
  27. V. D. Ivashchuk and V. N. Melnikov, *Grav. Cosmol.*, 1996, 2, No. 4 (8): 297; arXiv: hep-th/9612089
  28. M. A. Grebeniuk, V. D. Ivashchuk, and V. N. Melnikov, *Grav. Cosmol.*, Supplement, 1999, 5: 1
  29. V. D. Ivashchuk, V. N. Melnikov, and A. I. Zhuk, *Nuovo Cim. B*, 1989, 104: 575
  30. V. N. Melnikov, in: *Problems of Gravitation and Elementary Particle Theories (PGEPT)*, 1976, 7: 190 (in Russian)
  31. K. A. Bronnikov, V. N. Melnikov, and K. P. Staniukovich, *ITP-68-69*, Kiev, 1968
  32. V. N. Melnikov, Variations of  $G$  and SEE Project, in: *Proc. Rencontre de Moriond-99: Gravitational Waves and Experimental Gravity*, Editions Frontiers, 1999
  33. V. N. Melnikov, *Int. J. Mod. Phys. A*, 2002, 17: 4325
  34. V. N. Melnikov, Integrable Cosmological Models in DD and Variations of Fundamental Constants, in: *Proc. VII Asia-Pacific International Conference on Gravitation and Astrophysics, ICGA-7*, Edited by J. M. Nester, C. M. Chen, and J. P. Hsu, Singapore: World Scientific, 2006: 53
  35. N. A. Zaitsev and V. N. Melnikov, in: *Problems of Gravitation Theory and Particles Theory*, 10: 131, Moscow, Atomizdat, 1979 (in Russian); English version in Ref. [3]
  36. V. D. Ivashchuk and V. N. Melnikov, *Nuovo Cimento B*, 1988, 102: 131
  37. K. A. Bronnikov, V. D. Ivashchuk, and V. N. Melnikov, *Nuovo Cimento B*, 1988, 102: 209
  38. V. N. Melnikov, *Gravitation and Cosmology*, 2000, 6: 81; arXiv: gr-qc/0007067
  39. A. Miyazaki, arXiv: gr-qc/0102003
  40. Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, Cambridge: Cambridge University Press, 2003
  41. J. P. Mbelek and M. Lachieze-Ray, arXiv: gr-qc/0205089
  42. R. Hellings, *Phys. Rev. Lett.*, 1983, 51: 1609
  43. E. V. Pitjeva, in: *Dynamics and Astrometry of Natural and Artificial Celestial Bodies*, Netherlands: Kluwer Acad. Publ., 1997: 251
  44. J. O. Dickey, P. L. Bender, J. E. Faller, X X Newhall, R. L. Ricklefs, J. G. Ries, P. J. Shelus, C. Veillet, A. L. Whipple, J. R. Wiant, J. G. Williams, and C. F. Yoder, *Science*, 1994, 265: 482
  45. K. Nordtvedt, in: *Proceedings of the 18th Course of the School on Cosmology and Gravitation: The gravitational Constant. Generalized gravitational theories and experiments*, 30 April–10 May 2003, Erice, Edited by G. T. Gillies, V. N. Melnikov and V. de Sabbata, Kluwer Acad. Publ., 2004: 289; arXiv: gr-qc/0301024
  46. A. Sanders and W. Deeds, *Phys. Rev. D*, 1992, 46: 480
  47. A. J. Sanders, V. N. Melnikov et al., *Class. Quant. Grav.*, 2000, 17: 2331
  48. V. N. Melnikov and A. Sanders, *Ciencia Ergo Sum*, 2001, 8: 357
  49. K. A. Bronnikov, V. N. Melnikov, and M. Novello, *Grav. Cosmol.*, 2002, Suppl. II, 8: 18
  50. B. Boisseau, G. Esposito-Farese, D. Polarski, and A. A. Starobinsky, *Phys. Rev. Lett.*, 2000, 85: 2236
  51. V. N. Melnikov and V. D. Ivashchuk, Problems of  $G$  and multidimensional models, in: *Proc. JGRG11*, Edited by J. Koga et al., Waseda Univ., Tokyo, 2002: 405; arXiv: gr-qc/0208021
  52. A. G. Riess et al., *Astron. J.*, 1998, 116: 1009
  53. S. Perlmutter et al., *Astrophys. J.*, 1999, 517: 565
  54. V. R. Gavrilov, V. D. Ivashchuk, and V. N. Melnikov, *Class. Quantum Grav.*, 1996, 13, No. 11: 3039
  55. Y.-S. Wu and Z. Wang, *Phys. Rev. Lett.*, 1986, 57: 1978
  56. V. D. Ivashchuk and V. N. Melnikov, *J. Math. Phys.*, 2000, 41: 6341; arXiv: hep-th/9904077
  57. J.-M. Alimi, V. D. Ivashchuk, S. A. Kononogov, and V. N. Melnikov, *Grav. Cosmol.*, 2006, 12, Nos. 2–3 (46–47): 173; arXiv: gr-qc/0611015

58. V. D. Ivashchuk, S. A. Kononogov, V. N. Melnikov, and M. Novello, *Grav. Cosmol.*, 2006, 12, No. 4 (48): 273; arXiv: hep-th/0610167
59. V. D. Ivashchuk and V. N. Melnikov, *Phys. Lett. A*, 1989, 135: 465
60. V. Baukh and A. Zhuk, *Phys. Rev. D*, 2006, 73: 104016
61. J.-M. Alimi, V. D. Ivashchuk, and V. N. Melnikov, *Grav. Cosmol.*, 2007, 13, No. 2 (50): 137; arXiv: 0711.3770
62. V. D. Ivashchuk and V. N. Melnikov, *J. Math. Phys.*, 1998, 39: 2866; arXiv: hep-th/9708157
63. I. S. Goncharenko, V. D. Ivashchuk, and V. N. Melnikov, *Grav. Cosmol.*, 2007, 13, No. 4 (52): 262; arXiv: math-ph/0612079
64. H. Dehnen, V. D. Ivashchuk, S. A. Kononogov and V. N. Melnikov, *Grav. Cosmol.*, 2005, 11, No. 4 (44): 340; arXiv: gr-qc/0602113
65. V. D. Ivashchuk, S. A. Kononogov, and V. N. Melnikov, *Grav. Cosmol.*, 2008, 14, No. 3 (55): 235
66. K. A. Bronnikov, S. A. Kononogov, V. N. Melnikov, and S. G. Rubin, *Grav. Cosmol.*, 2008, 14, No. 3(55): 230