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RESEARCH ARTICLE

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Nonlinear property of slightly compressible media permeated with air-filled bubbles

Abstract Based on the nonlinear oscillation of an air-filled bubble in weakly compressible media at prestressed state, the effective medium method is used to study the nonlinear property of the slightly compressible media permeated with air bubbles. It is this nonlinear oscillation of air bubbles that results in the nonlinear property of the porous media. Numerical results have confirmed that the nonlinearity of the porous media is usually high, though the optimal porosity is very small. Moreover, the nonlinear property is greatly affected by the prestressed state, porosity, and shear modulus of the matrix media.

Keywords slightly compressible media, porous media, effective medium method, nonlinear parameter

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1 Introduction

Because the elastic and viscoelastic materials permeated with air-filled bubbles are used in a large variety of physical situations, it is very important to determine the acoustic properties of porous media. The effective medium theory (EMT) based on resonance scattering theory can be used to obtain the linear acoustic property of porous media [1, 2]. However, the nonlinear property of the porous media cannot be obtained by the EMT. The nonlinear acoustic property is very important for the rubber-like porous media. Furthermore, it is an important way to strongly enhance the nonlinearity of rubber-like media by introducing bubbles into it [3, 4]. The biomedical applications also promote the research of the nonlinear property of porous media in recent years because bubbles are often encapsulated in thin elastic shells, manufactured for use as contrast agents and considered as a drug or gene delivery vehicle. Therefore, it is very important to obtain the nonlinear property of slightly compressible porous media.

As a unit of porous media, the oscillation of a single bubble is the basis of analyzing the nonlinear property of porous media. The oscillation of an air-filled bubble in an elastic medium is a classical problem in the theory of elasticity. Some works, such as the oscillation of bubbles in an elastic medium [5, 6], viscoelastic medium [7, 8], and the oscillation of bubbles with a shell, have been studied [9−11]. The nonlinear oscillation of the bubble in slightly compressible media at prestressed stated has also been studied recently [11, 12]. One point should be noted that the assumption of slight compressibility corresponds to $\lambda \gg \mu$, where λ and μ are the Lame constants of an elastic medium [12, 13].

Based on the nonlinear oscillation of air-filled bubbles, the effective medium method (EMM) is developed since the porous media should have the same stress and strain with the effective media [13]. Both the linear and nonlinear acoustic properties of the porous media in undeformed and prestressed state can be obtained by EMM. In this paper, the EMM is used to analyze the nonlinear property of slightly compressible elastic media permeated with air bubbles for low frequency longitudinal wave. For simplicity, the damp of the matrix is not taken into account in this paper.

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2 Formulation and solution of the problem

This section gives brief discussions of the nonlinear oscillation of bubbles in slightly compressible elastic media at prestressed state. The method of obtaining the nonlinear property of porous media is also discussed.

The nonlinear oscillation of the bubble at prestressed state is discussed first because it is the basis for analyzing the nonlinear property of porous media. The radial pulsation of a single bubble in slightly compressible elastic medium is obtained from Lagrange's equation [12], that is

$$R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{1}{\rho_s}\left[P_g\left(R_1/R\right)^{3\gamma}-P_\infty-P_e\left(R\right)\right] \tag{1}$$

where R is the actual radius of the bubble, R_1 the equilibrium bubble radius, P_g the equilibrium pressure in the bubble, P_∞ the pressure at the outer boundary of the elastic medium (which is also the surrounding pressure), $P_e\left(R\right)$ the effective pressure resulting from the shear stress, ρ_s the density of the slightly compressible elastic media, and γ the polytropic index ($\gamma = 1.4$ for air). The equilibrium radius R_1 can be obtained by setting $R = R_1$ and $\dot{R} = \ddot{R} = 0$ in Eq. (1), that is

$$P_e\left(R_1\right)=P_g-P_\infty \tag{2}$$

There is $P_e\left(R_1\right) = 0$ if the equilibrium pressure in bubble P_g equals the surrounding pressure P_∞. This state is also called the initial undeformed state. Under the initial undeformed stated, the equilibrium bubble radius R_1 is designated R_0 and the surrounding pressure, which is also called as the initial pressure, is designated $P_{\infty 0}$. If $P_\infty \neq P_{\infty 0}$, the pressure difference between P_g and P_∞ will compel the equilibrium bubble radius R_1 to change and the shear stress will be generated to balance the pressure difference, that is $P_e\left(R_1\right) \neq 0$. Because the effective pressure $P_e\left(R\right)$ has the form of

$$P_e\left(\zeta\right)=\mu\int_0^1\left[(1+\beta)\left(\frac{y}{x^2}-\frac{y^7}{x^8}\right)+(1-\beta)\left(\frac{1}{y}-\frac{y^5}{x^6}\right)\right]\mathrm{d}x \tag{3}$$

where $\zeta = R/R_0$, $y = (x^{-3} + \zeta^3 - 1)^{-1/3}$, x is the variable, β the fitting parameter referred to by Mooney as the coefficient of asymmetry, which must lie in the range $-1 \leqslant \beta \leqslant 1$ [12, 14], the ratio of equilibrium bubble radius $\zeta_1 = R_1/R_0$ varying with the surrounding pressure can be obtained from Eqs. (2) and (3) through the interpolation algorithm [13]. Therefore, the parameter ζ_1 is also a parameter used to estimate the prestressed state of the bubble.

In order to use the EMM, the bubble's dynamic equation is expressed in terms of the volume deviation $U(R_1)$, where $U(R_1) = 4\pi(R^3 - R_1^3)/3$. If the radiation losses is taken into account, the dynamic equation of the bubble can be written as [13]

$$\ddot{U}+\omega_1^2U-\frac{R_1}{c_l}U=GU^2+q\left(\dot{U}^2+2U\ddot{U}\right)-\frac{4\pi R_1}{\rho_s}P_\infty^{\mathrm{inc}} \tag{4}$$

where $G = \dfrac{9\gamma\left(\gamma+1\right)P_g+2P_e'\left(\zeta_1\right)\zeta_1-P_e''\left(\zeta_1\right)\zeta_1^2}{8\pi R_1^5\rho_s}$, $\omega_1^2=\omega_e^2+\omega_g^2$, $\omega_g^2=\dfrac{3\gamma P_g}{\rho_s R_1^2}$, $\omega_e^2=\dfrac{P_e'\left(\zeta_1\right)\zeta_1}{\rho_s R_1^2}$, $q=\dfrac{1}{8\pi R_1^3}$, $\zeta_1 = R_1/R_0$, P_∞^{inc} is the effective surrounding pressure generated by the incidence longitudinal wave, the primes of $P_e\left(\zeta_1\right)$ indicate derivatives with respect to ζ at ζ_1, and c_l is the longitudinal velocity of the elastic medium.

Using the perturbation method, the volume pulsation of Eq. (4) up to quadratic order has the form

$$U(R_1)=-\frac{4\pi R_1P_\infty^{\mathrm{inc}}}{\rho_s(\omega_1^2-\omega^2-\mathrm{i}R_1\omega^3/c_l)}+\frac{(G-3q\omega^2)(P_\infty^{\mathrm{inc}})^2}{\omega_1^2-4\omega^2-8\mathrm{i}R_1\omega^3/c_l}\cdot\left[\frac{4\pi R_1}{\rho_s(\omega_1^2-\omega^2-\mathrm{i}R_1\omega^3/c_l)}\right]^2 \tag{5}$$

for the harmonic longitudinal incidence wave, where ω is the angular frequency of the incident wave. The resonance occurs at the frequency obtained by setting the real part of the first term's denominator in Eq. (5) equal to zero. Therefore, the resonance angular frequency of the bubble ω_1 is determined by ω_e and ω_g because $\omega_1^2 = \omega_e^2 + \omega_g^2$. According to the expression of ω_e and ω_g, one can easily find that ω_e and ω_g are determined by the shear modulus of the elastic medium and the air in the bubble, respectively. If ω_e^2 is much larger than ω_g^2, the effect of the air in the bubble can be ignored. On the contrary, the effect of the shear modulus can be ignored if $\omega_e^2 \ll \omega_g^2$. One can also find from Eq. (5) that the influence of the incidence frequency on the volume pulsation of the bubble can be neglected if the incident frequency is much smaller than the resonance frequency. Thus, the ensuing discussion is restricted to the case of $\omega \ll \omega_1$ for simplicity.

Since there are $\lambda \gg \mu$ and $P_\infty^{\mathrm{inc}} = -\sigma$ for slightly compressible elastic media, whereσ is the stress generated by the incident longitudinal wave [3], Eq. (5) can be simplified as

$$U(R_1)=g_1(R_1)\sigma+g_2(R_1)\sigma^2 \tag{6}$$

where

$$g_1(R_1) = \frac{4\pi R_1^3}{P'_e(\zeta_1)\zeta_1 + 3\gamma P_g} \tag{7}$$

$$g_2(R_1) = \frac{2\pi R_1^3\left[9\gamma(\gamma+1)P_g + 2P'_e(\zeta_1)\zeta_1 - P''_e(\zeta_1)\zeta_1^2\right]}{\left[P'_e(\zeta_1)\zeta_1 + 3\gamma P_g\right]^3} \tag{8}$$

The volume variation produced by bubble oscillation in unit volume can thus be obtained from Eq. (6)

$$V_c = \int U(R_1)n(R_1)\mathrm{d}R_1 = V_{c1}\sigma + V_{c2}\sigma^2 \tag{9}$$

where $n(R_1)\mathrm{d}R_1$ is the number of bubbles with equilibrium radii from R_1 to $R_1 + \mathrm{d}R_1$ in unit volume, V_{c1} and V_{c2} are linear and nonlinear volume perturbation of the bubble, respectively. Substituting Eqs. (7) and (8) into Eq. (9) yields

$$V_{c1} = \frac{3\varphi}{P'_e(\zeta_1)\zeta_1 + 3\gamma P_g} \tag{10}$$

$$V_{c2} = \frac{3\varphi\left[9\gamma(\gamma+1)P_g + 2P'_e(\zeta_1)\zeta_1 - P''_e(\zeta_1)\zeta_1^2\right]}{2\left[P'_e(\zeta_1)\zeta_1 + 3\gamma P_g\right]^3} \tag{11}$$

where φ is porosity which is the total volume of bubbles in unit volume.

Because the effective medium has the same stress and strain with the porous media, the EMM is developed [15]. The effective Lame coefficients $\bar{\lambda}$, $\bar{\mu}$ of the porous media can be expressed as

$$\bar{\mu} = \mu(1-\varphi) \tag{12a}$$

$$\bar{\lambda} + 2\bar{\mu} = \bar{C}_{11}(1 - \bar{C}_{11}^2 V_{c2}\varepsilon) \tag{12b}$$

where $\bar{C}_{11} = \dfrac{(\lambda+2\mu)[P'_e(\zeta_1)\zeta_1 + 3\gamma P_g]}{P'_e(\zeta_1)\zeta_1 + 3\gamma P_g + 3\varphi(\lambda+2\mu)}$, ε is longitudinal strain. Eq. (12) is a nonlinear equation, that is to say the porous media has nonlinear property. Because the nonlinear parameter is used to evaluate the nonlinearity of liquid and solid media, the concept of nonlinear parameter can also be used for evaluating the nonlinearity of the inhomogeneous media [16]. According to the method and definition of Ostrovskii [4], the nonlinear parameter of porous media Γ has the form of

$$\Gamma = V_{c2}\bar{C}_{11}^2 = \frac{3\varphi(\lambda+2\mu)^2\left[9\gamma(\gamma+1)P_g + 2P'_e(\zeta_1)\zeta_1 - P''_e(\zeta_1)\zeta_1^2\right]}{2[P'_e(\zeta_1)\zeta_1 + 3\gamma P_g][P'_e(\zeta_1)\zeta_1 + 3\gamma P_g + 3\varphi(\lambda+2\mu)]^2} \tag{13}$$

From the previous discussions, one can easily conclude that the nonlinear property of porous media resulted from the nonlinear oscillation of bubbles. According to Eq. (13), the nonlinear parameter of porous media is mainly affected by the porosity and the ratio of the equilibrium bubble radius ζ_1. Since ζ_1 varies with the surrounding pressure, the nonlinear property is also affected by the surrounding pressure. Eq. (13) has a maximum at the porosity of

$$\varphi_{\mathrm{op}} = \frac{3\gamma P_g + P'_e(\zeta_1)\zeta_1}{3(\lambda+2\mu)} \tag{14}$$

where φ_{op} is the optimal porosity. The corresponding maximum nonlinear parameter $\Gamma_{\max}$ can be expressed as

$$\Gamma_{\max} = \frac{(\lambda+2\mu)\left[9\gamma(\gamma+1)P_g + 2P'_e(\zeta_1)\zeta_1 - P''_e(\zeta_1)\zeta_1^2\right]}{8\left[P'_e(\zeta_1)\zeta_1 + 3\gamma P_g\right]^2} \tag{15}$$

Because $P'_e(\zeta_1)$ and $P''_e(\zeta_1)$ usually have the same order of magnitude with the shear modulus μ, and P_g is also small compared with $\lambda + 2\mu$, $\Gamma_{\max}$ is thus very high and the corresponding optical porosity is very small.

Because the equilibrium bubble radius varies with the surrounding pressure, the porosity of porous media will vary with the equilibrium bubble radius. If the optimal porosity of porous media in undeformed state is designated φ_{op0}, it can be expressed as

$$\varphi_{\mathrm{op0}} = \frac{\varphi_{\mathrm{op}}}{\zeta_1^3 + \varphi_{\mathrm{op}} - \varphi_{\mathrm{op}}\zeta_1^3} \tag{16}$$

According to Eqs. (14)–(16) the maximum nonlinear parameter of porous media and the optimal porosity is mainly affected by the shear modulus and the ratio of the equilibrium bubble radius ζ_1.

3 Numerical results and discussion

Computations are performed for the plastizole [13] because its shear modulus is very small and $\lambda \gg \mu$. Unless otherwise stated, the parameters of plastizole used in the calculation are $\rho_s = 950$ kg/m^3, $\lambda = 2.01 \times 10^9$ N/m^2, $\mu = 1.01 \times 10^4$ N/m^2. The asymmetry coefficient $\beta = 0$ and the initial pressure $P_{\infty 0} = 1.01 \times 10^5$ Pa is assumed. The nonlinear property of porous plastizole is computed.

In order to obtain the prestress of the porous media, the ratio of the equilibrium bubble radius ζ_1 should be obtained first. Using interpolative algorithm [13], the ζ_1 varying with the surrounding pressure is obtained as shown in Fig. 1. The figure shows that the surrounding pressure strongly affects the equilibrium radius of the bubble in plastizole.

The nonlinear parameter varying with the porosity in undeformed state φ_0 and the ratio of the equilibrium bubble radius ζ_1 is displayed in Fig. 2. The curves in Fig. 2 are the isolines of the nonlinear parameter. The number in the isoline is the value of the nonlinear parameter of the porous media. Figure 2 shows that the same porous media has different nonlinear parameters in different prestressed state. Also, there is a maximum nonlinear parameter for each ζ_1.

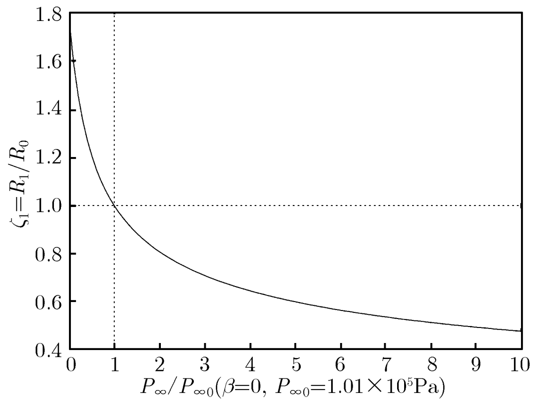

Fig. 1 The ζ_1 varying with the surrounding pressure for bubble in plastizole.

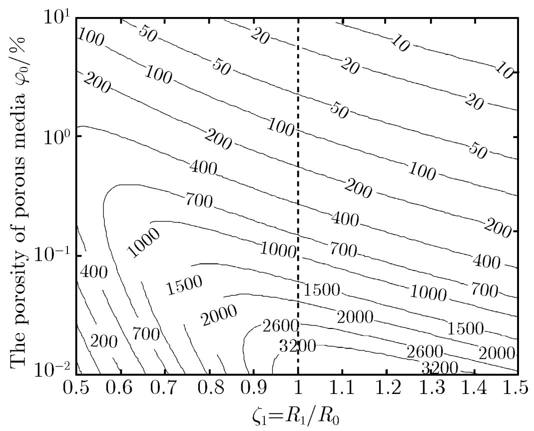

Fig. 2 The nonlinear parameter of porous plastizole Γ varying with the φ_0 and ζ_1.

According to Eq. (15) the maximum value of the nonlinear parameter is mainly determined by the shear modulus and ζ_1. Keeping the same λ as the plastizole, the $\Gamma_{\max}$ varying with the shear modulus and ζ_1 are obtained as shown in Fig. 3. The curves in Fig. 3 are the isolines of the nonlinear parameter and the number in the isoline are the value of $\Gamma_{\max}$. Figure 3 shows that the maximum nonlinear parameter increases quickly with the increase of ζ_1 and the decrease of the shear modulus.

The optimal porosity in undeformed state $\varphi_{\text{op}0}$ can also be obtained from Eqs. (15) and (16). The $\varphi_{\text{op}0}$ varying with the shear modulus and ζ_1 is shown in Fig. 4. The results also show that the $\varphi_{\text{op}0}$ decreases quickly with the decrease of the shear modulus and the increase of ζ_1. Moreover, the $\varphi_{\text{op}0}$ corresponding to $\Gamma_{\max}$ is very small, that is, $\Gamma_{\max}$ is very sensitive to the porosity.

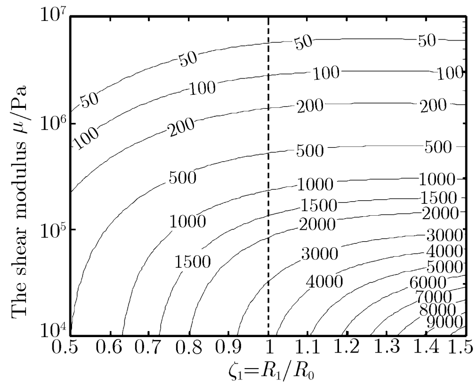

Fig. 3 The $\Gamma_{\max}$ varying with the ζ_1 and the shear modulus of elastic medium.

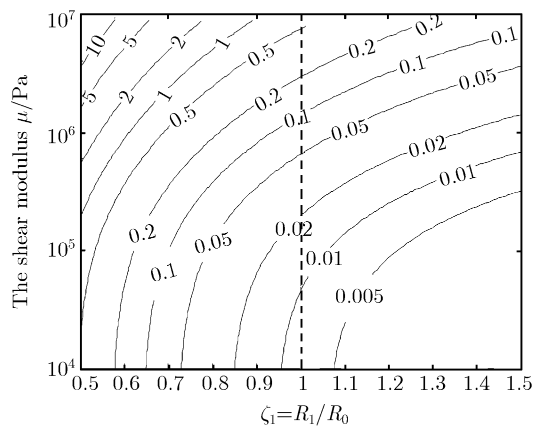

Fig. 4 The $\varphi_{\text{op}0}$ (%) varying with the ζ_1 and the shear modulus of elastic medium.

4 Conclusion

The purpose of this paper is to assess the nonlinear property of slightly compressible elastic media permeated with air-filled bubbles for low-frequency incident wave. As a unit of porous media, the nonlinear oscillation of bubbles at prestressed state is discussed. The nonlinear property of porous media results from the nonlinear oscillation of bubbles. Numerical results confirm that the nonlinear property of porous media is very sensitive to the porosity and the prestressed state. This work is useful to design the porous media with strong nonlinear acoustic property.

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