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RESEARCH ARTICLE

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The thermodynamics in a dynamical black hole

Abstract Considering the back-reaction of emitting particles to the black hole, a "new" horizon is suggested where thermodynamics can be built in the dynamical black hole. It, at least, means that the thermodynamics of a dynamical black hole should not be constructed at the original event horizon any more. The temperature, "new" horizon position and radiating particles' energy will be consistent again under the theory of equilibrium thermodynamical system.

Keywords Hawking radiation, black hole, information loss paradox, back-reaction, event horizon, thermodynamics

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1 Introduction

In 1970s, Hawking discovered, by using the basic principles of quantum field theory, the striking fact that a classical black hole could radiate thermal spectrum of the particles [1, 2]. The corresponding temperature is $T = \dfrac{\kappa}{2\pi}$, where κ is the surface gravity of the black hole. This discovery connects the black hole with thermodynamics, which states that a black hole's entropy depends on the surface area of its horizon by $S_{\rm BH} = \dfrac{A}{4}$ [3−5]. Then, we can use all the laws of thermodynamics to describe the black hole system. However, it gave rise to a disturbing problem about information conservation during black hole evaporation. Considering the quantum effect, Hawking radiation takes nothing out from the black hole because of its black body spectrum [6, 7].

In 2000, Parikh and Wilczek proposed a direct semi-classical approach to calculate the emission rate when particles tunnel across the event horizon [8]. Their result is considered to be consistent with the information conservation. In their method, energy conservation and WKB approximation $\Gamma = {\rm e}^{-2ImS}$ are used. What is more, a corrected spectrum, which is approximated to the first order, is given. The crucial point is that the barrier is created by the emission particle itself while considering the energy conservation [9, 10]. As a particle emits, the black hole loses energy and the space-time background also changes in the meantime [11]. This means that the black hole will be "dynamical" and shrink as particles radiate, which also could be called the emitting particle's back-reaction. Following this method, many works have been finished for static and stationary black holes [12−17], and even non-stationary black hole [18]. Following the idea of the tunneling by Parikh and Wilczek, Refs. [19−21] use Damour-Ruffini's method [22] to get the corrected thermal radiation of black hole too. This method is a more general one because it can be used to study not only both of scalar field and Dirac field, but also massive and massless particle radiation.

However, we can easily find that it is difficult to define the temperature of the black hole because of the corrected thermal spectrum. Therefore, the thermodynamics of black hole could be difficult to construct in this case.

In this paper, we intend to construct the thermodynamics in a dynamical black hole. The basic idea is that the thermodynamics may not be constructed on the original event horizon of a dynamical black hole. This problem has been discussed in some papers [23, 24] in which they think the thermodynamics of a dynamical black hole should be constructed at the apparent hori-

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zon. So, in Section 2, we use Damour-Ruffini's method to get the relation between the emission rate of the outgoing particles and the horizon $\Gamma = \mathrm{e}^{-4\pi r_h \omega}$. In Section 3, we show that a new horizon must be defined if we want to get the purely thermal spectrum by considering the emission particles' back-reaction. We can easily get the temperature at this new horizon, and can connect thermodynamics with this dynamical black hole.

2 Damour-Ruffini's method and the particles' emission rate from a Schwarzschild black hole

The line element in a Schwarzschild space-time is

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{1}$$

its event horizon is $r_h = 2M$, where M is the mass of the black hole.

The Klein-Gordon equation in a curved space-time is given by

$$\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) - \mu^2\right]\Phi = 0 \tag{2}$$

so we can get the Klein-Gordon equation in Schwarzschild space-time as

$$\left[-\left(1-\frac{2M}{r}\right)^{-1} r^2\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r}\left(r^2 - 2Mr\right)\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - r^2\mu^2\right]\Phi = 0$$

By separating the variables as $\Phi = \frac{1}{r}R_\omega(r,t)Y_{lm}(\theta,\phi)$, we can get the radial equation as

$$\left[-\left(1-\frac{2M}{r}\right)^{-1} r^2\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r}\left(r^2-2Mr\right)\frac{\partial}{\partial r} - r^2\mu^2\right]\frac{R_\omega}{r} = -\frac{l\,(l+1)\,R_\omega}{r} \tag{3}$$

where μ, l are respectively the static mass and the quantum number of angular momentum of the particle.

The tortoise coordinate transformation is

$$r_* = r + \frac{1}{2\kappa}\ln\frac{r-r_h}{r_h}$$

$$\mathrm{d}r_* = \left[1 + \frac{1}{2\kappa\,(r-r_h)}\right]\mathrm{d}r$$

$$\frac{\mathrm{d}}{\mathrm{d}r} = \left[1 + \frac{1}{2\kappa\,(r-r_h)}\right]\frac{\mathrm{d}}{\mathrm{d}r_*}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} = -\frac{1}{2\kappa\,(r-r_h)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}r_*} + \left[1+\frac{1}{2\kappa\,(r-r_h)}\right]^2\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}$$

Substituted by these expressions, Eq. (3) can be rewritten as

$$-\frac{\partial^2 R_\omega}{\partial t^2} + \left(\frac{r - r_h + \frac{1}{2\kappa}}{r}\right)^2\frac{\partial^2 R_\omega}{\partial r_*^2} + \left[\frac{1}{2\kappa r^2} - \frac{r_h\left(r - r_h + \frac{1}{2\kappa}\right)}{r^3}\right]\frac{\partial R_\omega}{\partial r_*} - \left[r^2\mu^2 + \frac{r_h}{r} - l\,(l+1)\right]\frac{r-r_h}{r^3}R_\omega = 0 \tag{4}$$

when $r \to r_h$, Eq. (4) can be changed into

$$-\frac{\partial^2 R_\omega}{\partial t^2} + \frac{1}{(2\kappa r_h)^2}\frac{\partial^2 R_\omega}{\partial r_*^2} = 0 \tag{5}$$

its solution is

$$R_\omega = \mathrm{e}^{-\mathrm{i}\omega t \pm \mathrm{i}2\kappa r_h \omega r_*} \tag{6}$$

Letting $r'_* = 2\kappa r_h r_*$, the solution can also be written as

$$R_\omega = \mathrm{e}^{-\mathrm{i}\omega t \pm \mathrm{i}\omega r'_*} \tag{7}$$

Therefore, the ingoing and outgoing solution are respectively written as following:

$$R_\omega^{\mathrm{in}} = \mathrm{e}^{-\mathrm{i}\omega\left(t + r'_*\right)} = \mathrm{e}^{-\mathrm{i}\omega v} \tag{8}$$

$$R_\omega^{\mathrm{out}} = \mathrm{e}^{-\mathrm{i}\omega\left(t - r'_*\right)} = \mathrm{e}^{-\mathrm{i}\omega v}\mathrm{e}^{\mathrm{i}4\kappa r_h \omega r}\left(\frac{r-r_h}{r_h}\right)^{\mathrm{i}2\omega r_h} \tag{9}$$

where v is the advanced Eddington coordinate.

From Eq. (8) and Eq. (9), we can find that R_ω^{in} has good behavior at the horizon. However, R_ω^{out} is not analytic at the horizon and it can only describe the particles outside the horizon. Using $(r - r_h) \to |r - r_h|\,\mathrm{e}^{-\mathrm{i}\pi}$ to extend the outgoing solution into the horizon, the outgoing solution R_ω^{out} inside the horizon can be written as follows

$$\begin{aligned} R_\omega^{\mathrm{out}}\,(r < r_h) &= \mathrm{e}^{-\mathrm{i}\omega v}\mathrm{e}^{\mathrm{i}4\kappa r_h \omega r}\left(\frac{r_h - r}{r_h}\right)^{\mathrm{i}2\omega r_h}\mathrm{e}^{2r_h\pi\omega} \\ &= \mathrm{e}^{-\mathrm{i}\omega v}\mathrm{e}^{\mathrm{i}4\kappa r_h\omega(r + \frac{1}{2\kappa}\ln\frac{r_h - r}{r_h})}\mathrm{e}^{2r_h\pi\omega} \\ &= \mathrm{e}^{-\mathrm{i}\omega v}\mathrm{e}^{\mathrm{i}2w}r'_*\mathrm{e}^{2r_h\pi\omega} \end{aligned} \tag{10}$$

where $r_* = r + \frac{1}{2\kappa}\ln\frac{r_h - r}{r_h}$, this is the tortoise coordinate transformation inside the horizon.

Therefore, we get the emission rate of the outgoing wave function at the horizon

$$\Gamma = \left|\frac{R_\omega^{\mathrm{out}}\,(r > r_h)}{R_\omega^{\mathrm{out}}\,(r < r_h)}\right|^2 = \mathrm{e}^{-4\pi r_h\omega} = \mathrm{e}^{-\beta\omega} \tag{11}$$

where $\beta = \dfrac{1}{T} = 4\pi r_h = 8\pi M$, and $T = \dfrac{1}{4\pi r_h} = \dfrac{1}{8\pi M}$ is the temperature of Schwarzschild black hole at the event horizon.

3 New horizon and its thermodynamics

Considering the particles' self-interaction: when a particle with energy ω_i emits through the horizon, due to the conservation of energy, the mass of the black hole M will be reduced to $(M - \omega_i)$, the event horizon will change from $r_h^i = 2M$ to $r_h^f = 2(M - \omega_i)$. So emission rate can be written as

$$\begin{aligned}\Gamma &= \prod_i \Gamma_i = \prod_i \mathrm{e}^{-8\pi(M-\omega_i)\omega_i} = \mathrm{e}^{\sum_i -8\pi(M-\omega_i)\omega_i} \\ &= \mathrm{e}^{\int_0^\omega -8\pi\left(M-\omega'\right)\mathrm{d}\omega'} = \mathrm{e}^{-4\pi(2M-\omega)\omega} \\ &= \mathrm{e}^{-8\pi M\omega\left(1-\frac{\omega}{2M}\right)} \end{aligned} \tag{12}$$

where considering the semiclassical approximation, we have used integral to take place of the sum.

Eq. (12) tells us that the emission spectrum has a modified term and the Hawking radiation is not a purely thermal spectrum any more at the event horizon in a dynamical black hole. So it is very difficult to construct the thermodynamics because the temperature is not well-defined.

From Eq. (12), we find that we can definite a new horizon r_n if we want to obtain the purely thermal emission spectrum as $\Gamma = \mathrm{e}^{-4\pi r_n\omega} = \mathrm{e}^{-\beta\omega}$ to construct the thermodynamics when we consider the tunneling particle's back-reaction to the black hole.

By comparing Eq. (11) with Eq. (12) ,we can get the new horizon is

$$r_n = 2M - \omega \tag{13}$$

Therefore, the emission rate is

$$\Gamma = \mathrm{e}^{-4\pi(2M-\omega)\omega} = \mathrm{e}^{-\beta\omega} \tag{14}$$

where $\beta = 4\pi r_n = 4\pi(2M - \omega)$, and the Hawking temperature of the black hole is

$$T = \frac{1}{\beta} = \frac{1}{4\pi r_n} = \frac{1}{4\pi(2M-\omega)} \tag{15}$$

From Eq. (15), we can easily obtain the relation between the new horizon and the event horizons as follows:

$$r_n = 2M - \omega = \frac{r_h^i + r_h^f}{2} \tag{16}$$

Eq. (16) shows that r_n is the average position of the event horizon if we think the event horizon shrinks continuously when a particle tunnels across the horizon. We can construct thermodynamics at the "new" horizon in a dynamical black hole. At least, it shows that the thermodynamics of a dynamical black hole could not be constructed at its original event horizon any more.

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