

Xiao-feng PANG

# Features and states of microscopic particles in nonlinear quantum-mechanics systems

© Higher Education Press and Springer-Verlag 2008

**Abstract** In this paper, we present the elementary principles of nonlinear quantum mechanics (NLQM), which is based on some problems in quantum mechanics. We investigate in detail the motion laws and some main properties of microscopic particles in nonlinear quantum systems using these elementary principles. Concretely speaking, we study in this paper the wave-particle duality of the solution of the nonlinear Schrödinger equation, the stability of microscopic particles described by NLQM, invariances and conservation laws of motion of particles, the Hamiltonian principle of particle motion and corresponding Lagrangian and Hamilton equations, the classical rule of microscopic particle motion, the mechanism and rules of particle collision, the features of reflection and the transmission of particles at interfaces, and the uncertainty relation of particle motion as well as the eigenvalue and eigenequations of particles, and so on. We obtained the invariance and conservation laws of mass, energy and momentum and angular momentum for the microscopic particles, which are also some elementary and universal laws of matter in the NLQM and give further the methods and ways of solving the above questions. We also find that the laws of motion of microscopic particles in such a case are completely different from that in the linear quantum mechanics (LQM). They have a lot of new properties; for example, the particles possess the real wave-corpucle duality, obey the classical rule of motion and conservation laws of energy, momentum and mass, satisfy minimum uncertainty re-

lation, can be localized due to the nonlinear interaction, and its position and momentum can also be determined, etc. From these studies, we see clearly that rules and features of microscopic particle motion in NLQM is different from that in LQM. Therefore, the NLQM is a new physical theory, and a necessary result of the development of quantum mechanics and has a correct representation of describing microscopic particles in nonlinear systems, which can solve problems disputed for about a century by scientists in the LQM field. Hence, the NLQM built is very necessary and correct. The NLQM established can promote the development of physics and can enhance and raise the knowledge and recognition levels to the essences of microscopic matter. We can predict that nonlinear quantum mechanics has extensive applications in physics, chemistry, biology and polymers, etc.

**Keywords** microscopic particle, nonlinear systems, nonlinear quantum mechanics, basic principle, motion rule, dynamic property, nonlinear Schrödinger equation

**PACS numbers** 02 .60.Cb, 03.65.-w

## 1 Introduction

As is known, the quantum mechanics established by several great scientists such as Bohr, Born, Schrödinger and Heisenberg, etc., in the early 1900s [1–6] is the science describing the properties and rules of motion of microscopic particles (MIP). It is a foundation of modern science, in which the state of microscopic particles is described by the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r}, t) \psi \quad (1)$$

where  $\hbar^2 \nabla^2 / 2m$  is the kinetic energy operator,  $V(\mathbf{r}, t)$  is

Xiao-feng PANG<sup>1,2</sup> (✉)

<sup>1</sup> Institute of Life Science and Technology, University of Electronic Science and Technology of China, Chengdu 610054, China

<sup>2</sup> International Centre for Materials Physics, The Chinese Academy of Sciences, Shenyang 110015, China  
E-mail: pangxf@mail.sc.cninfo.net

Received November 10, 2007; accepted January 2, 2008

the externally applied potential operator,  $m$  is the mass of particles,  $\psi(\mathbf{r}, t)$  is a wave function describing the states of particles,  $\mathbf{r}$  is the coordinate or position of the particle, and  $t$  is the time. Eq. (1) is a wave equation, and if only the externally applied potential is known, we can find the solutions of the equation. However, for all externally applied potentials, the solutions of the equation are always a linear or dispersive wave, for example, at  $V(\mathbf{r}, t)=0$ , its solution is a plane wave as follows:

$$\psi(\mathbf{r}, t) = A' \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \quad (2)$$

where  $k$  is the wavevector of the wave,  $\omega$  is its frequency, and  $A'$  is its amplitude. When  $V(\mathbf{r}, t) \neq 0$ , its solutions are a de Broglie wave or a Bloch wave, and so on. Therefore, the MIP is represented by a wave in quantum mechanics. It always disperses in total space and cannot be localized. In other words, the solutions of Eq. (1) in the case of any potential possessing only a wave feature and not a particulate nature or corpuscle-wave duality, can only use the  $|\psi(\mathbf{r}, t)|^2$  representing the probability occurred particle at position  $\mathbf{r}$ . As is known, the wave feature and probability concept of MIP is incompatible with the traditional concept of stability and the determinant size of particles [7–9]. Meanwhile, the wave function  $\psi(\mathbf{r}, t)$ , expressing the MIP, is a function of position  $\mathbf{r}$ . Thus, a contradictory problem occurs in quantum mechanics, namely that if MIP is represented by a wave, then it should not have the determinant position  $\mathbf{r}$ . If we could use a vector  $\mathbf{r}$  to represent the position of the MIP, then its position is determinant, thus there is no uncertainty relation of position and momentum of the MIP in quantum mechanics. On the contrary, the probability concept, uncertainty relation, and statistical average values of mechanical quantities of MIP are then the elementary concepts and principle of quantum mechanics and some intrinsic features of MIP, but are all contradictory with regard to particles. Thus, we have reasons to improve and develop quantum mechanics [7–9].

However, why does quantum mechanics have these questions? This is worth studying deeply and in detail. As is known, Eq. (1) describes the motion of a particle; the corresponding Hamiltonian operator of the system is

$$\hat{H}(t) = \hbar^2 \nabla^2 / 2m + V(\mathbf{r}, t) \quad (3)$$

Obviously, it consists only of kinetic and potential operator of particles; the potential operator is only determined by an externally applied field, and not related to the state or wavefunction of the particle, thus the potential can only change the states of MIP, and cannot change its nature and essence. Therefore, the natures and features of MIP are only determined by the kinetic term. There is no force or energy to obstruct and suppress the dis-

persing effect of kinetic energy in the system, thus the MIP disperses and propagates in total space, and cannot be localized at all. This is the main reason why MIP has only wave feature in quantum mechanics. Meanwhile, the Hamiltonian in Eq. (3) does not represent practical essences and features of MIP. In real physics, the energy operator of the systems and number operator of particles are always associated with the states of particles, i.e., they are related to the wavefunction of MIP. On the other hand, Eq. (2) or (3) can describe only the states and feature of a single particle, and cannot describe the states of many particles. However, a system composed of one particle does not exist in nature. The simplest system in nature is the hydrogen atom, but it consists of two particles. In such a case, when we study the states of particles in realistic systems composed of many particles and many bodies using quantum mechanics, we have to use a simple and uniform average-potential unassociated with the states of particles to replace the complicated and nonlinear interaction among these particles [10, 11]. Thus, these complicated effects and nonlinear interactions determining essences and natures of particles are ignored completely. Therefore, the state and properties of particles determined by the average potential is not real and correct. Then we can only say that quantum mechanics is an approximate and linear theory and cannot represent completely the properties of motion of MIPs. We here refer to it as linear quantum mechanics (LQM). Meanwhile, a lot of hypotheses or theorems of particles in quantum mechanics also do not agree with conventional understanding, and have excited a long-time debate between scientists. Up to now, there is no unified conclusion. Therefore, it is necessary to improve and develop LQM. However, what is its direction of development? From the above studies we know that a key shortcoming or defect of LQM is its ignoring of the dependence of the Hamiltonian or energy operator of the systems on the states of particles and nonlinear interactions among these particles. As a matter of fact, the nonlinear interactions always exist in any realistic physics systems including the hydrogen atom [10–15]. At the same time, it is also a reasonable assumption that the Hamiltonian or energy operator of the systems depend on the states of particles [12–25]. Therefore, to establish a correct new quantum theory, we must break through the elementary hypotheses of LQM, and use the above reasonable assumptions to include the nonlinear interactions among the particles and the dependences of the Hamiltonian of the systems on the state of particles. Thus, we must establish nonlinear quantum mechanism (NLQM) to study the rules of motion of microscopic particles in realistic systems with nonlinear interactions [12–25].

Pang worked out the NLQM describing the properties of motion of MIPs in nonlinear systems [12 – 27]. The elementary principles, theory, calculated rules and applications of NLQM were described in Pang *et al.*'s books [26, 27]. This paper is essentially composed of three parts. The first presents a review of the LQM, as well as theoretical and experimental fundamentals that establish the nonlinear quantum mechanical theory. The theory itself and its essential features are covered in the second part. In the final part, extensive applications of this theory in physics, biology and polymers are introduced. The whole volume forms a complete system of the NLQM. Concrete contents are as follows. Linear quantum mechanics: its successes and problems; macroscopic quantum effects and properties of motion of quasiparticles; the fundamental principles and theory of nonlinear quantum mechanics; wave-corpucle duality of MIP in the NLQM; nonlinear interaction and localization of the MIP; nonlinear versus linear quantum mechanics; problem solving in the NLQM; properties and states of the MIP in different nonlinear systems; nonlinearly quantum-mechanical properties of the excitons and phonons; properties of nonlinear excitations and motions of protons, polarons and magnons in different systems.

For the development of quantum mechanics from linear range to nonlinear domain, Pang worked at and investigated this problem for about 20 years [12 – 27]. In this investigation, Pang first sought the roots of these problems existing in the LQM. Subsequently, Pang [26 – 30] broke through the restrictions of the elementary hypotheses for the independence of the Hamiltonian of the systems on the states of the particles and the linearity of the theory in the LQM, and proposed and established the elementary principles and theory of the NLQM, based on the relations among the nonlinear interaction and soliton motions and macroscopic quantum effect, and incorporating modern theories of superconductors, superfluids and solitons, according to the features of macroscopic quantum effects and soliton theory. A lot of practices and experiences demonstrate that the NLQM is successful [26 – 30].

## 2 Fundamental principles of nonlinear quantum mechanics

Based on the earlier discussion, the fundamental principles of nonlinear quantum mechanics (NLQM) may be summarized as follows [12 – 30].

(1) Microscopic particles in a nonlinear quantum system are described by the following wave function:

$$\phi(\mathbf{r}, t) = \varphi(\mathbf{r}, t) e^{i\theta(\mathbf{r}, t)} \quad (4)$$

where both the amplitude  $\varphi(\mathbf{r}, t)$  and phase  $\theta(\mathbf{r}, t)$  of the wave function are functions of space and time.

(2) In the nonrelativistic case, the wave function  $\phi(\mathbf{r}, t)$  satisfies the generalized nonlinear Schrödinger equation (NLSE), i.e.,

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi \pm b |\phi|^2 \phi + V(\mathbf{r}, t) \phi + A(\phi) \quad (5)$$

or

$$\mu \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi \pm b |\phi|^2 \phi + V(\mathbf{r}, t) \phi + A(\phi) \quad (6)$$

where  $\mu$  is a complex number,  $V$  is an external potential field,  $A$  is a function of  $\phi(\mathbf{r}, t)$ , and  $b$  is a coefficient indicating the strength of nonlinear interaction.

In the relativistic case, the wave function  $\phi(\mathbf{r}, t)$  satisfies the nonlinear Klein-Gordon equation (NLKGE), including the generalized Sine-Gordon equation (SGE) and the  $\phi^4$ -field equation, i.e.,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_j^2} = \beta \sin \phi + \gamma \frac{\partial \phi}{\partial t} + A(\phi), \quad j = 1, 2, 3 \quad (7)$$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_j^2} \mp \alpha \phi \pm \beta |\phi|^2 \phi = A(\phi), \quad j = 1, 2, 3 \quad (8)$$

where  $\gamma$  represents a dissipative or frictional effects,  $\alpha$  is a constant,  $\beta$  is a coefficient indicating the strength of nonlinear interaction and  $A$  is a function of  $\phi(\mathbf{r}, t)$ .

From the above fundamental principles, we see clearly that the NLQM breaks through the fundamental hypotheses of the LQM in two aspects, namely the linearity of dynamic equations and independence of the Hamiltonian operator with the wave function of the particles. In the NLQM, the dynamic equations are all some nonlinear partial differential equations, in which nonlinear terms of state wave function  $\phi$  are involved. The Hamiltonian operator corresponding to these equations are all related to the state wave function as shown in Eqs. (16) and (17). Hence, so far as this point is concerned, the NLQM is really a break-through or a new development in quantum mechanics.

## 3 The properties of microscopic particles in nonlinear quantum mechanics

3.1 The solutions of nonlinear Schrödinger equation and its wave-particle features

In the one-dimensional case, the solution of Eq. (5) at

$V(x, t) = A(\phi) = 0$  can be found by using some methods, for example, the inverse scattering method, which is of the form [26, 27, 31]:

$$\phi(x, t) = A_0 \operatorname{sech} \left\{ \frac{A_0 \sqrt{b}}{\sqrt{2}\hbar} [\sqrt{2m}(x - x_0) - v_e t] \right\} \cdot e^{i v_e [\sqrt{2m}(x - x_0) - v_c t] / 2\hbar} \quad (9)$$

where  $A_0 = \sqrt{\frac{v_e^2 - 2v_c v_e}{2b}}$ . This solution is completely different from Eq. (2). In fact, it is a bell-type non-topological soliton as shown in Fig. 1. Therefore, the microscopic particle in NLQM is a soliton. Here,  $\varphi(x, t) = A_0 \operatorname{sech} \left\{ A_0 [\sqrt{2m}(x - x_0) - v_e t] / \sqrt{2}\hbar \right\}$  is the envelop of the solution, and  $\exp \left\{ i v_e [\sqrt{2m}(x - x_0) - v_e t] / 2\hbar \right\}$  is its carrier wave. The form of soliton of MIP is shown in Fig. 1. The envelop  $\phi(x, t)$  is a slow varying function and the mass center of the particle, the position of the mass center is just at  $x_0$ ,  $A_0$  is its amplitude, and its width is given by  $W = 2\pi\hbar / (\sqrt{mb}A_0)$ . Thus, the size of the soliton is  $A_0 W = 2\pi\hbar / \sqrt{mb}$ , and a constant. This shows that the particle has determinant form and is localized,  $v_e$  is the group velocity of the particle (soliton),  $v_c$  is the phase speed of the carrier wave. For a certain system,  $v_e$  and  $v_c$  and the size of the particle are determinant and do not change with time. According to the soliton theory [32, 33], the bell-type soliton in Eq. (9) can move freely over macroscopic distances in a uniform

velocity  $v_e$  in space-time retaining its form, energy, momentum and other quasi-particle properties. In this condition, its mass, momentum and energy are some constants, and can be represented by [26, 27]

$$N_s = \int_{-\infty}^{\infty} |\phi|^2 dx' = 2\sqrt{2}A_0$$

$$p = -i \int_{-\infty}^{\infty} (\phi^* \phi_{x'} - \phi \phi_{x'}^*) dx' = 2\sqrt{2}A_0 v_e = N_s v_e = \text{const}$$

$$E = \int_{-\infty}^{\infty} \left[ |\phi_{x'}|^2 - \frac{1}{2} |\phi|^4 \right] dx' = E_0 + \frac{1}{2} M_{\text{sol}} v_e^2 \quad (10)$$

where  $x' = x / \sqrt{\hbar^2 / 2m}$ ,  $t' = t / \hbar$ , and  $M_{\text{sol}} = N_s = 2\sqrt{2}A_0$  is effective mass of MIP, which is a constant. The energy, mass and momentum of the particle are invariant and cannot be dispersed in its motion. Just so, the position vector  $\mathbf{r}$  and position  $x$  (or  $x, y, z$ ) has definitively physical significance, and denotes exactly the positions of MIPs at time  $t$ . Thus, the wave function  $\phi(\mathbf{r}, t)$  or  $\phi(x, t)$  can represent exactly the states of MIP at the position  $\mathbf{r}$  or  $x$  and time  $t$ . This is consistent with the concept of particles or corpuscles. At the same time, in Fig. 1(d), we show the collision property of two soliton solutions of Eq. (9) by numerical simulation technique. From this figure, we see clearly that the two particles can

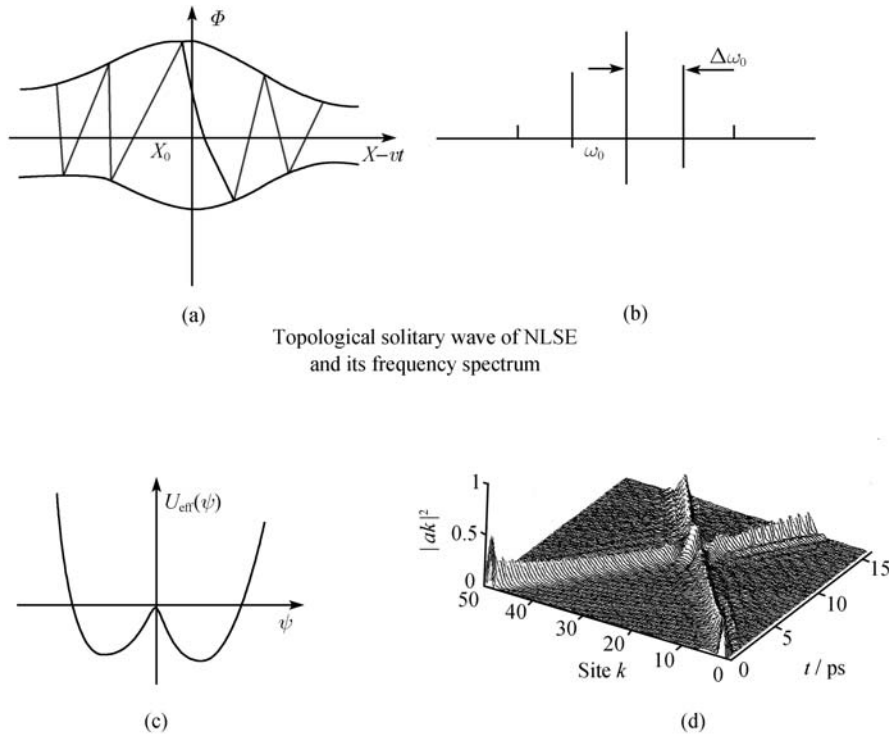


Fig. 1 The solution in Eq. (9) at  $V = A = 0$  in Eq. (5) and its features.

go through each other while retaining their form after the collision, which is the same with that of the classical particles. Therefore, the microscopic particle in NLQM has an obvious particulate feature. However, the envelope of the solution in Eq. (9) is a solitary wave. It has a certain wavevector and frequency as shown in Fig. 1(b), and can propagate in space accompanying the carrier wave, i.e., the carrier wave carries the envelope to propagate in space-time; the feature of propagation depends only on the concrete nature of MIPs. Figure 1(b) shows the width of the frequency spectrum of the envelope  $\varphi(x, t)$ , the frequency spectrum has a localized structure around the carrier frequency  $\omega_0$ . Therefore, the microscopic particle in NLQM has exactly wave-particulate double feature [26, 27]. This consists of Davisson and Germer's experimental result of electron diffraction on double seam in 1927.

However, we must remember that the solution of dynamic equations (5)–(8) in nonlinear quantum mechanics in the limits of weak nonlinear interaction are not exactly the solutions of the dynamic equations in linear quantum mechanics. To see this clearly, we first examine the velocity of the skirt of the soliton given in Eq. (9), which is rewritten as

$$\phi(x', t') = 2\sqrt{2} \operatorname{sech} [\sqrt{2}k(x' - v_e t')] \cdot e^{iv_e(x' - v_e t')/2} \quad (11)$$

for  $b = 1$ ,  $V(\mathbf{r}, t') = A(\phi) = 0$  in Eq. (5). As is known, the nonlinear term in Eq. (5) sharpens the peak, while its dispersion term has the tendency to leave it off. Then, for weak nonlinear interaction and small skirt  $\phi(x', t')$ , it may be approximated by (for  $x > v_e t$ )

$$\phi = 2\sqrt{2}ke^{-\sqrt{2}k(x' - v_e t')}e^{iv_e(x' - v_e t')/2} \quad (12)$$

and the small term  $|\phi|^2 \phi$  in Eq. (5) in such a case can be approximated by

$$i\phi_{t'} + \phi_{x'x''} \approx 0 \quad (13)$$

Substituting Eq. (12) into Eq. (13), we get  $v_e \approx 2\sqrt{2}k$ , which is the group speed of the particle. (Near the top of the peak, we must take both the nonlinear and dispersion terms into account because their contributions are of the same order. The result is the group speed.). Here, we have only checked the formula for the region where  $\phi(x, t)$  is small; that is, when a particle is approximated by Eq. (12), it satisfies the approximate wave equation (13) with  $v_e \approx 2\sqrt{2}k$ .

However, if Eq. (13) is treated as a linear equation, its solution is of the form:

$$\phi'(x, t) = Ae^{i(kx - \omega t)} \quad (14)$$

We now have  $\omega = k^2$ , which gives the phase velocity  $\omega/k$  as  $v_c = k$  and the group speed  $\partial\omega/\partial k = v_{gr} = k$ . Apparently, this is different from  $v_e = 2\sqrt{2}k$ . This is because the solution Eq. (12) is essentially from Eq. (14). Therefore, the solution Eq. (14) is not the solution of nonlinear Schrödinger equation (5) with  $V(x, t) = A(\phi) = 0$  in the case of weak nonlinear interactions. Solution Eq. (13) is a “divergent solution” ( $\phi(x, t) \rightarrow \infty$  at  $x \rightarrow -\infty$ ), which is not an “ordinary plane wave”. The concept of group speed does not apply to a divergent wave. Thus, we can say that the soliton is made from a divergent solution, which is abandoned in the linear waves. The divergence develops by the nonlinear term to yield waves of finite amplitude. When the nonlinear term is very weak, the soliton will diverge; and if we suppress divergence no soliton will result. These circumstances are clearly seen from the following soliton in the case of nonlinear coefficient  $b \neq 1$ :

$$\phi(x, t) = 2\sqrt{\frac{2}{b}} \operatorname{sech} [\sqrt{2}k(x' - v_e t')] e^{iv_e(x' - v_e t')/2} \quad (15)$$

If the nonlinear term approaches zero ( $b \rightarrow 0$ ), the solitary wave diverges ( $\phi(x, t) \rightarrow \infty$ ). If we want to suppress the divergence, then we have to set  $k = 0$ . In such a case, we get Eq. (14) from Eq. (15). This illustrates that the nonlinear Schrödinger equation or nonlinear quantum mechanics can reduce to the linear Schrödinger equation or linear quantum mechanics if and only if the nonlinear interaction and the group speed of the particle are zero. Therefore, we can conclude that the particles (solitons) of nonlinear quantum mechanical equations in the weak nonlinear interaction limit is not the same as that in linear quantum mechanics. If the nonlinear interaction is zero, the nonlinear quantum mechanics reduce to the linear quantum mechanics. However, real physical systems or materials are made up of a great number of MIPs, and nonlinear interactions always exist in the systems or among the particles. The nonlinear interactions arise from interactions among the MIPs or between the MIPs and the environment. The nonlinear quantum mechanics should be the correct and more appropriate theory for real systems. It should be used often and extensively, even in weak nonlinear interaction cases. The linear quantum mechanics, on the other hand, is an approximation to the more general nonlinear quantum theory and can be used to study motions of MIPs in systems in which there exists only very weak and negligible nonlinear interactions [26, 27].

### 3.2 Stability of microscopic particle described by NLQM

As is known, in classical physics the macroscopic particles are certainly stable. Stability is an elementary feature of a particle. However, is the microscopic particle (MIP) described by NLQM or the solution of nonlinear Schrödinger equation (NLSE) in Eq. (5), for instance, Eq. (9), stable? This is also a basic problem in NLQM, and need to be proved further. In the absence of an externally applied field, the stability of the MIPs in NLQM can be demonstrated by means of the initial and structural stabilities. However, how are MIP's behaviors exposed in an externally applied field? If the motion of all the MIPs is located in a finite range where the potential is lowest, we can say that the MIPs are stable according to the minimum theorem of energy. As a matter of fact, when there are a lot of particles in the system, the interactions with one another among the particles are very complicated; it is very difficult to define the behavior of each one individually. Therefore, we cannot adopt again the strategies of initial stability and collision to study their stability. Instead, we take advantage of the following consideration: when a mechanical system is in a state of minimal energy we may say that it is stable, and to change this state, external energy must be supplied. Pang used this minimal energy consideration to demonstrate the stability of the MIPs as follows.

Let  $\phi(x, t)$  represent the field of the particle; meanwhile, we assume that it possess derivatives of all orders, and all integrations for it be convergent and finite. The Lagrange density function corresponding to the NLSE Eq. (5) at  $A(\phi) = 0$  is given as follows:

$$\begin{aligned} \mathcal{L} = & \frac{i\hbar}{2} (\phi^* \phi_t - \phi \phi_t^*) - \frac{\hbar^2}{2m} (\nabla\phi \cdot \nabla\phi^*) \\ & - V(x) \phi^* \phi + b(\phi^* \cdot \phi) \phi \end{aligned} \quad (16)$$

The momentum density of this field is defined as  $P = \partial L' / \partial \phi$ , where  $L' = \mathcal{L}$ . Thus, the Hamiltonian density of the field is as follows:

$$\begin{aligned} \mathcal{H} = & \frac{i\hbar}{2} (\phi^* \partial_t \phi - \phi \partial_t \phi^*) - \mathcal{L} \\ = & \frac{\hbar^2}{2m} (\nabla\phi \cdot \nabla\phi^*) + V(x) \phi \phi^* - b(\phi \phi^*)^2 \end{aligned} \quad (17)$$

From Eqs. (16) and (17), we see clearly that the Lagrange and Hamiltonian operators of the systems corresponding to Eq. (5) are all related to the state wave function of particles and involve all nonlinear interactional energy,  $b(\phi \phi^*)^2$  related to the states of MIP. This is in essence different from Eq. (3) in LQM. Then the natures and features of MIP are simultaneously determined by

the kinetic and nonlinear interaction terms in nonlinear quantum mechanics. Just so, there is a force or energy to obstruct and suppress the dispersing effect of kinetic energy in the system, thus the MIP cannot disperse and propagate in total space, and is localized all the time. This is just the essential reason that the MIP has a particulate nature or corpuscle-wave duality as mentioned above in Section 3.2 in nonlinear quantum mechanics. Therefore, we can say that the above fundamental principles of the NLQM in Eqs. (4)–(8) breaks through the fundamental hypothesis for the independence of Hamiltonian operator with the wave function of the particles in the LQM. This is a new development.

In the general case, the total energy of the particles is a function of  $t'$  and is represented by

$$E(t') = \int_{-\infty}^{\infty} \left[ \left| \frac{\partial \phi}{\partial x'} \right|^2 - \frac{b}{2} |\phi \phi^*|^2 + V(x') |\phi|^2 \right] dx' \quad (18)$$

However, in this case,  $b$  and  $V(x')$  are not functions of  $t'$ . So, the total energy of the systems is a conservative quantity, i.e.,  $E(t') = E = \text{const.}$ , as shown in Eq. (10). We can demonstrate that when  $x' \rightarrow \pm\infty$ , the solutions of Eq. (5) at  $A(\phi) = 0$  and  $\phi(x', t')$  should tend to zero rapidly [26, 27], i.e.,

$$\lim_{|x'| \rightarrow \infty} \phi(x', t') = \lim_{|x'| \rightarrow \infty} \frac{\partial \phi}{\partial x'} = 0$$

Then

$$\int_{-\infty}^{\infty} \phi^* \phi dx' = \text{const. or a function of } t'$$

The average position of the particles  $\phi$  or position of mass centre of MIP can be represented as  $\langle x' \rangle = x'_g = x_0 = \int_{-\infty}^{\infty} \phi'^* \phi dx' / \int_{-\infty}^{\infty} \phi^* \phi dx'$ . Thus, the average velocity of the particles or velocity of mass centre of MIP can be denoted by

$$\begin{aligned} v_g = & \frac{d \langle x' \rangle}{dt'} = \frac{d}{dt'} \left( \frac{\int_{-\infty}^{\infty} \phi'^* x' \phi dx'}{\int_{-\infty}^{\infty} \phi^* \phi dx'} \right) \\ = & -2i \frac{\int_{-\infty}^{\infty} \phi^* \frac{\partial \phi'}{\partial x'} dx'}{\int_{-\infty}^{\infty} \phi^* \phi dx'} \end{aligned} \quad (19)$$

However, for different solutions of the same NLSE (5),  $\int_{-\infty}^{\infty} \phi^* \phi dx'$ ,  $\langle x' \rangle$  and  $d \langle x' \rangle / dt'$  can have different values. Therefore, it is unreasonable to compare the energy between a definite solution and other solutions. We should

compare the energy of one particular solution to that of another solution. The comparison is only meaningful for many MIP systems that have the same values of  $\int_{-\infty}^{\infty} \phi^* \phi dx' = k$ ,  $\langle x' \rangle = u$  and  $d\langle x' \rangle / dt' = \dot{u}$  at the same time  $t'_0$ . Based on these, we can determine the stability of the soliton solutions of Eq. (5), for example, Eq. (9). Thus, we assume that the different solutions of the NLSE (5) at  $A(\phi) = 0$  satisfy the following boundary conditions at definite time  $t'_0$ :

$$\int_{-\infty}^{\infty} \phi^* \phi dx' = k, \quad \langle x' \rangle \Big|_{t'=t'_0} = u(t'_0) \tag{20}$$

$$\frac{d\langle x' \rangle}{dt'} \Big|_{t'=t'_0} = \dot{u}(t'_0)$$

Now we assume the solution of NLSE (5) at  $A(\phi) = 0$  to be of the form:

$$\phi(x', t') = \varphi(x', t') e^{i\theta(x', t')} \tag{21}$$

Substituting Eq. (21) into Eq. (18), we obtain the energy formula:

$$E = \int_{-\infty}^{\infty} \left[ \left( \frac{\partial \varphi}{\partial x'} \right)^2 + \varphi^2 \left( \frac{\partial \theta}{\partial x'} \right)^2 - b\varphi^4 + V(x') \varphi^2 \right] dx' \tag{22}$$

Eq. (20) becomes

$$\int_{-\infty}^{\infty} \varphi^2 dx' = k, \quad \frac{\int_{-\infty}^{\infty} x' \varphi^2 dx'}{\int_{-\infty}^{\infty} \varphi^2 dx'} = u(t'_0) \tag{23}$$

Finding the extreme value of the functional Eq. (22) under the boundary conditions Eq. (23) by means of the Lagrange uncertain factor method, we obtain the following Euler equations:

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \left\{ V(x') + C_1(t'_0) C_2(t'_0) [x' - u(t'_0)] + C_3(t'_0) \left[ 2 \frac{\partial \theta}{\partial t'} - \dot{u}(t'_0) \right] + \left( \frac{\partial \theta}{\partial t'} \right)^2 \right\} \varphi - b\varphi^3 \tag{24}$$

$$= 0$$

$$\frac{\partial^2 \varphi}{\partial (x')^2} \varphi^2 + 2 \frac{\partial \theta}{\partial t'} \varphi \frac{\partial \varphi}{\partial t'} + 2C_3(t'_0) \varphi \frac{\partial \varphi}{\partial t'} = 0 \tag{25}$$

where the Lagrange factors  $C_1$ ,  $C_2$  and  $C_3$  are all func-

tions of  $t'$ . Now, let  $C_3(t'_0) = -\frac{1}{2} \dot{u}(t'_0)$ , if

$$2 \frac{\partial \theta}{\partial x'} - \dot{u}(t'_0) \neq 0$$

we can get from Eq. (25)

$$\frac{2}{\varphi} \frac{\partial \varphi}{\partial x'} = \frac{-\frac{\partial^2 \theta}{\partial x'^2}}{-\frac{\partial \theta}{\partial x'} - \frac{1}{2} \dot{u}(t'_0)}$$

Integration of the above equation yields

$$\varphi^2 = \frac{g(t')}{\frac{\partial \theta}{\partial x'} - \frac{1}{2} \dot{u}(t'_0)} \quad \text{or} \quad \frac{\partial \theta}{\partial x'} \Big|_{t'=t'_0} = \frac{g(t'_0)}{\varphi^2} + \frac{\dot{u}(t'_0)}{2} \tag{26}$$

where  $g(t'_0)$  is an integral constant. Thus,

$$\theta(x', t') = g(t'_0) \int_0^x \frac{dx'}{\varphi^2} + \frac{\dot{u}(t'_0)}{2} x' + M(t'_0) \tag{27}$$

Here,  $M(t'_0)$  is also an integral constant. Again, let

$$C_2(t'_0) = \frac{1}{2} \ddot{u}(t'_0) \tag{28}$$

Substituting Eqs. (26)–(28) into Eq. (24), we obtain

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \left\{ V(x') + \frac{\ddot{u}(t'_0)}{2} x' + \left[ C_1(t'_0) - \frac{\ddot{u}(t'_0)}{2} u(t'_0) + \frac{u^2(t'_0)}{4} \right] \right\} \varphi - b\varphi^3 + \frac{g^2(t'_0)}{\varphi^3} \tag{29}$$

Letting

$$C_1(t'_0) = \frac{u(t'_0) \ddot{u}(t'_0)}{2} - \frac{\dot{u}^2(t'_0)}{2} + M(t'_0) + \beta' \tag{30}$$

where  $\beta'$  is an undetermined constant, which is a function of  $t'$ -independent, and assuming  $Z = x' - u(t'_0)$ , then

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \frac{\partial^2 \varphi}{\partial Z^2}$$

is only a function of  $Z$ . To make the right-hand side of Eq. (30) be also a function of  $Z$ , the coefficients of  $\varphi$ ,  $\varphi^3$  and  $1/\varphi^3$  must also be functions of  $Z$ , thus,  $g(t'_0) = g_0 = \text{const.}$ , and

$$V(x') + \frac{\ddot{u}(t'_0)}{2} x' + M(t'_0) - \frac{u^2(t'_0)}{4} = \tilde{V}_0(Z)$$

Then, Eq. (29) becomes

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \left\{ \tilde{V} [x' - u(t'_0)] + \beta' \right\} \varphi - b\varphi^3 + \frac{g^2(t'_0)}{\varphi^3} \tag{31}$$

Since  $\tilde{V}(Z) = \tilde{V}_0 [x' - u(t'_0)] = 0$  in the present case. Hence, Eq. (31) becomes

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \beta' \varphi - b\varphi^3 + \frac{g^2(t'_0)}{\varphi^3} \quad (32)$$

Therefore,  $\varphi$  is the solution of Eq. (32) for the parameters  $\beta' = \text{constant}$  and  $g(t'_0) = \text{const.}$  For sufficiently large  $|Z|$  we may assume that  $|\varphi| \leq \tilde{\beta}'/|Z|^{1+\Delta}$ , where  $\Delta$  is a small constant. However, in Eq. (32) we can only retain the solution  $\varphi(Z)$  corresponding to  $g(t'_0)$  to ensure that  $\lim_{|\epsilon| \rightarrow \infty} d^2\varphi/dZ^2 = 0$ , thus, Eq. (32) becomes

$$\frac{\partial^2 \varphi}{\partial (x')^2} = \beta' \varphi - b\varphi^3 \quad (33)$$

As a matter of fact, if  $\partial\theta/\partial t' = \dot{u}/2$ , and considering Eqs. (30) and (31) we can verify that the solution in Eq. (9) can satisfy Eq. (33). In such a case, it is not difficult to show that the energy corresponding to the solution Eq. (9) of Eq. (33) has a minimal value under the boundary conditions of Eq. (33) [26, 27]. Thus, we can conclude that the soliton solution of NLSE (5), or the MIP in NLQM is stable in such a case.

### 3.3 The invariance and conservation laws of mass, energy and momentum of particles

It is known from classical physics that the invariance and conservation laws of mass, energy and momentum and angular momentum are some elementary and universal laws of matter including classical particles in nature. We demonstrate here also that the microscopic particles described by the nonlinear Schrödinger equation in nonlinear quantum mechanics also have such properties. They satisfy the conventional conservation laws of mass, momentum and energy. This shows that the microscopic particles in the nonlinear quantum mechanics also have a corpuscle feature. Therefore, the proposed nonlinear quantum mechanical theory reflects the common rules of motions of matter in nature. To solve this problem, we first should give [26, 27] the Lagrangian  $L = \int \mathcal{L} dx$ , where  $\mathcal{L}$  is denoted by Eq. (16) and Hamiltonian  $H = \int \mathcal{H} dx$ , where  $\mathcal{H}$  is denoted by Eq. (17), for the systems corresponding to Eq. (5), respectively. Thus, the number density, the number current, the densities of momentum and energy for the particle can be defined by

$$\begin{aligned} \rho &= |\phi|^2, \quad p = -i\hbar(\phi^* \phi_x - \phi \phi_x^*) \\ J &= i\hbar(\phi^* \phi_x - \phi \phi_x^*) \\ \epsilon &= \frac{\hbar^2}{2m} |\phi_x|^2 - \frac{b}{2} |\phi \phi^*|^2 + V(x) |\phi|^2 \end{aligned} \quad (34)$$

where  $\phi_x = \frac{\partial}{\partial x} \phi(x, t)$ ,  $\phi_t = \frac{\partial}{\partial t} \phi(x, t)$ . From Eq. (5) and its conjugate equation as well as Eqs. (16), (17) and (34) we can obtain

$$\begin{aligned} \frac{\partial p}{\partial t'} &= \frac{\partial}{\partial x'} \left[ 2 \left( \frac{\partial \phi}{\partial x'} \right)^2 + (b|\phi \phi^*|^2 - 2V|\phi|^2) \right. \\ &\quad \left. - \left( \phi^* \frac{\partial^2}{\partial x'^2} \phi + \phi \frac{\partial^2}{\partial x'^2} \phi^* \right) + 2iV \left( \phi^* \frac{\partial \phi}{\partial x'} \right) \right] \\ \frac{\partial \rho}{\partial t'} &= \frac{\partial J}{\partial x'} \\ \frac{\partial \epsilon}{\partial x'} &= \frac{\partial}{\partial x'} \left[ \rho p + i \left( \frac{\partial \phi^*}{\partial x'} \frac{\partial^2 \phi}{\partial x'^2} - \frac{\partial \phi}{\partial x'} \frac{\partial^2 \phi^*}{\partial x'^2} \right) \right. \\ &\quad \left. - iV \left( \phi^* \frac{\partial \phi}{\partial x'} - \phi \frac{\partial \phi^*}{\partial x'} \right) \right] \end{aligned} \quad (35)$$

Thus, we get the following forms for the integral of motion:

$$\begin{aligned} \frac{\partial}{\partial t'} M &= \frac{\partial}{\partial t'} \int \rho dx' = 0, \quad \frac{\partial}{\partial t'} P = \frac{\partial}{\partial t'} \int p dx' = 0 \\ \frac{\partial E}{\partial t'} &= \frac{\partial}{\partial t'} \int \epsilon dx' = 0 \end{aligned} \quad (36)$$

These formulae represent just the conservation of mass, momentum and energy in such a case. This shows that the mass, momentum and energy of the particles (solitons) in the nonlinear quantum mechanical systems still obey general rules of conservation of matter in physics. In the case of  $V(x, t) = \text{const.}$ , we can find out easily the values of mass, momentum and energy of the particles of Eq. (5) [26, 27], as are shown in Eq. (10).

### 3.4 The invariance and general conservation laws of motion of particles described by nonlinear Schrödinger equation

We have learned from Eqs. (31)–(34) that some conservation laws for microscopic particles described by the nonlinear Schrödinger equation (5) in nonlinear quantum mechanics are always related to the invariance of the action relative to several groups of transformations through the Noether theorem in light of Gelfand and Fomin's (1963) and Bulman and its Kermel's (1989) ideas (see Sulem and Sulem *et al.*'s book and references therein [26, 27, 34]). Therefore, we first give the Noether theorem for nonlinear Schrödinger equation.

To simplify the equation, we introduce the following notations:

$$\begin{aligned} \bar{\xi} &= (t, x) = (\xi_0, \xi_1, \dots, \xi_a) \partial_0 = \partial_t \\ \partial &= (\partial_0, \partial_1, \dots, \partial_d) \text{ and } \Phi = (\Phi_1, \Phi_2) = (\phi, \phi^*) \end{aligned}$$

According to the Lagrangian Eq. (16) of the nonlinear Schrödinger equation, the action of the system

$$S\{\phi\} = \int_{t_0}^{t_1} \int L'(\phi, \nabla \phi, \phi_t, \phi^*, \nabla \phi^*, \phi_t^*) dx dt$$

where  $L' = \mathcal{L}$  is the Lagrange density function, now becomes

$$S\{\phi\} = \int_D \int_{x'}^\infty L'(\Phi, \partial\Phi) d\bar{\xi} \quad (37)$$

Under the action of a transformation  $T^\varepsilon$  which depends on the parameter  $\varepsilon$ , we have  $\bar{\zeta} \rightarrow \tilde{\zeta}(\bar{\zeta}, \Phi, \varepsilon)$ ,  $\Phi \rightarrow \tilde{\Phi}(\bar{\zeta}, \Phi, \varepsilon)$ , where  $\tilde{\zeta}$  and  $\tilde{\Phi}$  are assumed to be differentiable with respect to  $\varepsilon$ . When  $\varepsilon = 0$ , the transformation reduces to the identity. For infinitesimally small  $\varepsilon$ , we have  $\tilde{\zeta} = \bar{\zeta} + \delta\varepsilon$ ,  $\tilde{\Phi} = \Phi + \delta\Phi$ . At the same time,  $\Phi(\bar{\zeta}) \rightarrow \tilde{\Phi}(\tilde{\zeta})$  by the transformation  $T^\varepsilon$ , and the domain of integration  $D$  is transformed into  $\tilde{D}$ ,

$$S\{\phi\} \rightarrow \tilde{S}\{\tilde{\phi}\} = \int_{\tilde{D}} \int_{x^1}^\infty L'(\tilde{\Phi}, \partial\tilde{\Phi}) d\tilde{\zeta}$$

where  $\partial$  denotes differentiation with respect to  $\tilde{\zeta}$ . The change  $\delta S = \tilde{S}\{\tilde{\phi}\} - S\{\phi\}$  in the limit of  $\varepsilon$  under the above transformation can be expressed as

$$\begin{aligned} \delta S &= \int_D \int_{x^1}^\infty [L'(\tilde{\Phi}, \partial\tilde{\Phi}) - L'(\Phi, \partial\Phi)] d\bar{\xi} \\ &+ \int_D \int_{x^1}^\infty L'(\Phi, \partial\Phi) \sum_{v=0}^d \frac{\partial\delta\xi_v}{\partial\xi_v} d\bar{\xi} \end{aligned} \quad (38)$$

where we used the Jacobian expansion  $\frac{\partial(\tilde{\xi}_0, \dots, \tilde{\xi}_d)}{\partial(\xi_0, \dots, \xi_d)} = 1 + \sum_{v=0}^d \frac{\partial\delta\xi_v}{\partial\xi_v}$ , and  $L'(\tilde{\Phi}, \partial\tilde{\Phi})$ , in the second term on the right-hand side has been replaced by the leading term  $L'(\Phi, \partial\Phi)$  in the expansion. Now define

$$\begin{aligned} \delta\tilde{\Phi}_i &= \tilde{\Phi}_i(\bar{\xi}) - \Phi_i(\bar{\xi}) = \partial_v \Phi_i \delta\xi_v + \delta\Phi_i(\xi) \\ \partial_v \tilde{\Phi}_i(\bar{\xi}) - \partial_v \Phi_i(\bar{\xi}) &= (\partial_v - \partial_v) \tilde{\Phi}_i(\bar{\xi}) \\ &+ \partial_v [\tilde{\Phi}_i(\bar{\xi}) - \Phi_i(\bar{\xi})] \end{aligned} \quad (39)$$

with

$$\partial_v = \frac{\partial\bar{\xi}_\mu}{\partial\xi_v} \partial_\mu = \left( \delta_{v\mu} + \frac{\partial\delta\xi_\mu}{\partial\xi_v} \right) \partial_\mu = \bar{\partial}_v + \frac{\partial\delta\xi_\mu}{\partial\xi_v} \bar{\partial}_\mu$$

We then have

$$\begin{aligned} &L'(\tilde{\Phi}, \partial\tilde{\Phi}) - L'(\Phi, \partial\Phi) \\ &= \frac{\partial L'}{\partial\Phi_i} [\tilde{\Phi}_i(\bar{\xi}) - \Phi_i(\bar{\xi})] \\ &+ \frac{\partial L'}{\partial(\partial_v \Phi_i)} [\partial_v \tilde{\Phi}_i(\bar{\xi}) - \partial_v \Phi_i(\bar{\xi})] \\ &= \frac{\partial L'}{\partial\Phi_i} \partial\Phi_i + \partial_\mu (L' \delta\xi_\mu) - L' \frac{\partial\delta\xi_v}{\partial\xi_v} \\ &+ \partial_v \left[ \frac{\partial L'}{\partial(\partial_v \Phi_i)} \right] \delta\Phi_i - \partial_\mu \left[ \frac{\partial L'}{\partial(\partial_v \Phi_i)} \right] \delta\Phi_i \end{aligned}$$

Eq. (35) can now be replaced by

$$\begin{aligned} \delta S &= \int_D \int_{x'}^\infty \left\{ \frac{\partial L'}{\partial\Phi_i} - \frac{\partial}{\partial\xi_v} \left[ \frac{\partial L'}{\partial(\partial_v \Phi_i)} \right] \right\} \delta\Phi_i d\bar{\xi} \\ &+ \int_D \int_{x'}^\infty \frac{\partial}{\partial\xi_v} \left[ L' \delta\xi_v + \frac{\partial L'}{\partial(\partial_v \Phi_i)} \delta\Phi_i \right] d\bar{\xi} \end{aligned}$$

where we have used

$$\begin{aligned} &\frac{\partial}{\partial\xi_v} (L' \delta\xi_v) \\ &= L' \frac{\partial\delta\xi_v}{\partial\xi_v} + \frac{\partial L'}{\partial\Phi_i} \partial_v \Phi_i \delta\xi_v + \frac{\partial^2 L'}{\partial(\partial_\mu \Phi_i)} \partial_{v\mu}^2 \Phi_i \delta\xi_v \\ &\cdot \frac{\partial L'}{\partial(\partial_v \Phi_i)} \partial_v \int_{x'}^\infty \delta\Phi_i \frac{\partial}{\partial\xi_v} \left[ \frac{\partial L'}{\partial(\partial_v \Phi_i)} \delta\Phi_i \right] \\ &- \frac{\partial}{\partial\xi_v} \left[ \frac{\partial L'}{\partial(\partial_v \Phi_i)} \delta\Phi_i \right] \delta\Phi_i \end{aligned}$$

Using the Euler-Lagrange equation, the first term on the right-hand side in the equation of  $\delta S$  vanishes. We can get the Noether theorem, i.e., (A) if the action Eq. (37) is invariant under the infinitesimal transformation of the dependent and independent variables  $\phi \rightarrow \phi + \delta\phi$ ,  $\bar{\xi} \rightarrow \bar{\xi} + \delta\bar{\xi}$  where  $\bar{\xi} = (t, x_1, \dots, x_d)$ , the following conservation law holds [28, 29]

$$\begin{aligned} &\frac{\partial}{\partial\xi_v} \left[ L' \delta\xi_v - \frac{\partial L'}{\partial(\partial_v \Phi_i)} \delta\Phi_i \right] = 0, \quad \text{or} \\ &\frac{\partial}{\partial\xi_v} \left[ L' \delta\xi_v + \frac{\partial L'}{\partial(\partial_v \Phi_i)} \left( \delta\Phi_i - \frac{\partial\Phi_i}{\partial\xi_\mu} \delta\xi_\mu \right) \right] = 0 \end{aligned} \quad (40)$$

in terms of  $\delta\tilde{\Phi}_i$  defined above.

If the action is invariant under the infinitesimal transformation

$$\begin{aligned} t \rightarrow \bar{t} &= t + \delta t(x, t, \phi), \quad x \rightarrow \bar{x} = x + \delta x(x, t, \phi) \\ \phi(x, t) &\rightarrow \bar{\phi}(\bar{t}, \bar{x}) = \phi(t, x) + \delta\phi(t, x) \end{aligned}$$

then

$$\begin{aligned} &\int \left[ \frac{\partial L'}{\partial\phi_t} (\partial_t \phi \partial_t + \nabla\phi \cdot \delta\mathbf{x} - \delta\phi) \right. \\ &\left. + \frac{\partial L'}{\partial\phi_t^*} (\partial_t \phi^* \partial_t + \nabla\phi^* \cdot \delta\mathbf{x} - \delta\phi^*) - L\delta t \right] dx \end{aligned}$$

is a conserved quantity.

For the nonlinear Schrödinger equation in Eq. (5) with  $A(\phi) = 0$  in the nonlinear quantum mechanics, we have

$$\frac{\partial L'}{\partial\phi_t} = \frac{i}{2}\phi^*, \quad \text{and} \quad \frac{\partial L'}{\partial\phi_t^*} = -\frac{i}{2}\phi$$

where  $L' = \mathcal{L}$  is given in Eq. (16). Several conservation laws and invariance can be obtained from the Noether theorem.

(a) Invariance under time translation and energy con-

servation law

The action Eq. (37) is invariant under the infinitesimal time translation  $t \rightarrow t + \delta t$  with  $\delta x = \delta \phi = \delta \phi^* = 0$ , then Eq. (40) becomes

$$\begin{aligned} \partial_t \left[ \nabla \phi \cdot \nabla \phi^* - \frac{b}{2} (\phi \phi^*)^2 + V(x, t) \phi^* \phi \right] \\ - \nabla \cdot (\phi_t \nabla \phi^* + \phi_t^* \nabla \phi) = 0 \end{aligned}$$

This results in the conservation of energy

$$\begin{aligned} E = \int \left( \nabla \phi \cdot \nabla \phi^* - \frac{b}{2} (\phi^* \phi)^2 + V(x, t) \phi^* \phi \right) dx \\ = \text{const.} \end{aligned} \quad (41)$$

(b) Invariance of the phase shift or gauge invariance and mass conservation law It is very clear that the action related to the nonlinear Schrödinger equation is invariant under the phase shift  $\bar{\phi} = e^{i\theta} \phi$ , which for infinitesimal  $\theta$  gives  $\delta \phi = i\theta \phi$ , with  $\delta t = \delta x = 0$ . In this case, Eq. (18) becomes

$$\partial_t |\phi|^2 + \nabla \cdot \{i(\phi \nabla \phi^* - \phi^* \nabla \phi)\} = 0 \quad (42)$$

This results in the conservation of mass or number of particles.

$$N = \int |\phi|^2 dx = \text{const.}$$

and the continuum equation

$$\frac{\partial N}{\partial t} = \nabla \cdot \mathbf{j}$$

where  $\mathbf{j}$  is the mass current density

$$\mathbf{j} = -i(\phi \nabla \phi^* - \phi^* \nabla \phi)$$

(c) Invariance of space translation and momentum conservation law

If the action is invariant under an infinitesimal space translation  $x \rightarrow x + \delta x$  with  $\delta t = \delta \phi = \delta \phi^* = 0$ , then Eq. (40) becomes

$$\begin{aligned} \partial_t [i(\phi \nabla \phi^* - \phi^* \nabla \phi) \\ + \nabla \cdot \{2(\nabla \phi^* \times \nabla \phi + \nabla \phi \times \nabla \phi^* + L)\}] = 0 \end{aligned}$$

This leads to the conservation of momentum

$$\mathbf{P} = i \int (\phi \nabla \phi^* - \phi^* \nabla \phi) dx = \text{const.} \quad (43)$$

Note that the center of mass of the microscopic particles is defined by

$$\langle x \rangle = \frac{1}{N} \int x |\phi|^2 dx$$

We then have

$$\begin{aligned} N \frac{d\langle x \rangle}{dt} &= \int x \partial_t |\phi|^2 dx \\ &= - \int x \nabla \cdot [i(\phi \nabla \phi^* - \phi^* \nabla \phi)] dx \\ &= \int i(\phi \nabla \phi^* - \phi^* \nabla \phi) dx \\ &= \mathbf{P} = -\mathbf{J} = - \int \mathbf{j} dx \end{aligned} \quad (44)$$

This is the definition of momentum in classical mechanics. It shows clearly that the microscopic particles described by the nonlinear Schrödinger equation have the feature of classical particles.

(d) Invariance under space rotation and angular momentum conservation law

If the action Eq. (37) is invariant under a rotation of angle  $\delta \theta$  around an axis  $\mathbf{I}$  such that  $\delta t = \delta \phi = \delta \phi^* = 0$  and  $\delta \mathbf{x} = \delta \theta \mathbf{I} \times \mathbf{x}$ , this leads to the conservation of the angular momentum

$$\mathbf{M} = i \int \mathbf{x} \times (\phi^* \nabla \phi - \phi \nabla \phi^*) dx$$

Besides the above, Sulem also derived another invariance of the nonlinear Schrödinger equation from the Noether theorem for nonlinear Schrödinger equation.

(e) Galilean Invariance

If the action is invariant under the Galilean transformation

$$x \rightarrow x'' = x - vt, \quad t \rightarrow t'' = t$$

$$\phi(x, t) \rightarrow \phi''(x'', t'')$$

$$= -i \left( \frac{1}{2} vx'' + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} t'' \right) \phi(x, t)$$

which can also retain the nonlinear Schrödinger equation invariance. For an infinitesimal velocity  $v$ ,  $\delta \mathbf{x} = -vt$ ,  $\delta t = 0$  and  $\delta \hat{\phi} = \phi''(x'', t'') - \phi(x, t) = -(i/2) vx\phi(x, t)$ . After integration over the space variables, Eq. (40) leads to the conservation law Eq. (44) which implies that the velocity of the center of mass of the microscopic particles is a constant. It is also the same, even though the particle is in motion. This exhibits clearly that the microscopic particles have the particulate nature.

3.5 The nonlinear quantum mechanics describes Hamiltonian systems, the behavior of which is determined by a set of canonical conjugate variables. The states of particles can be described by Lagrangian and Hamilton equations.

Using the above variables one can determine the Poisson

bracket and write further the equations of motion of microscopic particles in the form of Hamilton's equations. For Eq. (5) with  $V(\mathbf{r}, t) = A(\phi) = 0$ , the variables  $\phi$  and  $\phi^*$  satisfy the Poisson bracket [26, 27, 34]:

$$\{\phi^{(a)}(x), \phi^{(b)}(y)\} = i\delta^{ab}\delta(x-y) \quad (45)$$

where

$$\{A, B\} = i \int_{-\infty}^{\infty} \left( \frac{\delta A \delta B}{\delta \phi \delta \phi^*} - \frac{\delta B \delta A}{\delta \phi \delta \phi^*} \right)$$

The corresponding Lagrangian density  $\mathcal{L}$  in Eq. (16) associated with Eq. (5) with  $A(\phi) = 0$  can be written in terms of  $\phi(x, t)$  and its conjugate  $\phi^*$  viewed as independent variables. The action of the system can be written as

$$S(\phi, \phi^*) = \int_{t_b}^{t_1} \int_D L' dx dt \quad (46)$$

and its variation for infinitesimal  $\delta\phi$  and  $\delta\phi^*$  is of the form

$$\delta S = \int_{t_b}^t \int_D \left[ \frac{\partial L'}{\partial \phi} \delta\phi + \frac{\partial L'}{\partial \nabla \phi} \delta \nabla \phi + \frac{\partial L'}{\partial \phi_t} \delta \phi_t \right] dx dt + \text{c.c.} \quad (47)$$

where  $L' = \mathcal{L}$ ,  $\partial L' / (\nabla \phi)$  denotes the vector with components  $\partial L' / (\partial_i \phi)$  ( $i = 1, 2, 3$ ). After integrating by parts, we get

$$\begin{aligned} \delta S = & \int_{t_b}^t \int_D \left[ \frac{\partial L'}{\partial \phi} - \nabla \cdot \frac{\partial L'}{\partial \nabla \phi} - \partial_t \left( \frac{\partial L'}{\partial \phi_t} \right) \right] \delta \phi dx dt \\ & + \left( \frac{\partial L'}{\partial \phi_t} \delta \phi \right)_{t_0}^{t_1} + \text{c.c.} \end{aligned} \quad (48)$$

A necessary and sufficient condition for a function  $\phi(x, t)$  with known values  $\phi(x, t_0)$  and  $\phi(x, t_1)$  to yield an extremum of the action  $S$  is that it must satisfy the Euler-Lagrange equation:

$$\frac{\partial L'}{\partial \phi} = \nabla \cdot \left( \frac{\partial L'}{\partial \nabla \phi} \right) + \partial_t \left( \frac{\partial L'}{\partial \phi_t} \right) \quad (49)$$

Eq. (49) gives the nonlinear Schrödinger equation (5) if the Lagrangian density Eq. (16) is used. Therefore, the dynamic equation, or the nonlinear Schrödinger equation in nonlinear quantum mechanics can be derived from the Euler-Lagrange equation, if the Lagrangian function of the system is known. This is different from linear quantum mechanics, in which a dynamic equation, or the linear Schrödinger equation, cannot be obtained from the Euler-Lagrange equation. This is a unique property of nonlinear quantum mechanics.

The above derivation of the nonlinear Schrödinger equation based on the variational principle is a foundation for other methods such as the "collective coordinates", the "variational approach", and the "Rayleigh-

Ritz optimization principle", where a solution is assumed to maintain a prescribed approximate profile (often bell-type) [26, 27]. Such methods greatly simplify the problem, reducing it to a system of ordinary differential equations for the evolution of a few characteristics of the systems.

The Hamiltonian density  $\mathcal{H}$  corresponding to Eq. (5) with  $A(\phi) = 0$  is Eq. (17) [26–29]. Introducing the canonical variables,

$$\begin{aligned} q_1 &= \frac{1}{2}(\phi + \phi^*), & p_1 &= \frac{\partial L'}{\partial (\partial_t q_1)} \\ q_2 &= \frac{1}{2i}(\phi - \phi^*), & p_2 &= \frac{\partial L'}{\partial (\partial_t q_2)} \end{aligned}$$

where  $L' = \mathcal{L}$ , the Hamiltonian density takes the form

$$\mathcal{H} = \sum_i p_i \partial_t q_i - \mathcal{L}$$

and the corresponding variation of the Lagrangian density  $L' = \mathcal{L}$  can be written as

$$\delta L' = \sum_i \frac{\delta L'}{\delta q_i} \delta q_i + \frac{\delta L'}{\delta (\nabla q_i)} \delta (\nabla q_i) + \frac{\delta L'}{\delta (\partial_t q_i)} \delta (\partial_t q_i) \quad (50)$$

From Eq. (50), the definition of  $p_i$ , and the Euler-Lagrange equation,

$$\frac{\partial L'}{\partial \phi} = \nabla \cdot \frac{\partial L'}{\partial \nabla \phi} + \partial_t p_i$$

one obtains the variation of the Hamiltonian in the form of

$$\delta H = \sum_i \int (\partial_t q_i \delta p_i - \partial_t p_i \delta q_i) dx$$

Thus, the Hamilton equation can be derived:

$$\frac{\partial q_i}{\partial t} = \delta \mathcal{H} \delta p_i, \quad \frac{\partial p_i}{\partial t} = -\delta \mathcal{H} \delta q_i \quad (51)$$

or in complex form:

$$i \partial_t \phi = \delta \mathcal{H} \delta \phi^*$$

This is also interesting. It shows that the nonlinear Schrödinger equation of dynamics describing microscopic particle can be obtained from the classical Hamilton equation in nonlinear quantum mechanics, if the Hamiltonian of the system is known. Obviously, such methods of finding dynamic equations are impossible in the linear quantum mechanics. As is known, the Euler-Lagrange equation and Hamilton equation are important equations in classical theoretical (analytic) mechanics, and were used to describe laws of motions of classical particles. These equations are now used to depict properties of motions of microscopic particles in nonlinear quantum

mechanics. This shows sufficiently the classical features of microscopic particles in nonlinear quantum mechanics. On the other hand, from this study, we seek new ways of finding the equation of motion of the microscopic particles in nonlinear quantum mechanics, i.e., if the Lagrangian or Hamiltonian of the system is known in the coordinate representation, then we can obtain the equation of motion of MIP from the Euler-Lagrange or Hamilton equations [26, 27, 34].

### 3.6 The motion of particles obeys a classical rule of motion in nonlinear quantum mechanics

Now utilizing Eq. (5) with  $A(\phi) = 0$  and its conjugate equation we can obtain [12, 26, 27, 35]

$$\begin{aligned} \frac{d}{dt'} \int_{-\infty}^{\infty} \phi^* \phi'_x dx' &= \int_{-\infty}^{\infty} \phi_t^* \phi'_x dx' + \int_{-\infty}^{\infty} \phi^* (\phi'_t)_x dx' \\ &= i \int_{-\infty}^{\infty} \left\{ \phi^* \frac{\partial}{\partial x'} [\phi_{x'x'} + b\phi^* \phi^2 - V\phi] \right. \\ &\quad \left. - [\phi_{x'x'}^* - b\phi(\phi^*)^2 - V\phi^*] \phi'_x \right\} dx' \\ &= i \int_{-\infty}^{\infty} \phi^* \frac{\partial V}{\partial x'} \phi dx' \end{aligned}$$

We here utilize the following relations and the boundary conditions:

$$\begin{aligned} \int_{-\infty}^{\infty} (\phi^* \phi_{x'x'x'} - \phi_{x'x'}^* \phi'_x) dx' &= 0 \\ \int_{-\infty}^{\infty} b(\phi^{*2} \phi \phi'_x + \phi^* \phi^2 \phi'_x) dx' &= 0 \\ \lim_{|x'| \rightarrow \infty} \phi(x', t') &= \lim_{|x'| \rightarrow \infty} \phi_{x'}(x', t') = 0 \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \phi^* \phi dx' &= \text{const.} \\ \lim_{|x'| \rightarrow \infty} \phi^* x' \phi'_x &= \lim_{|x'| \rightarrow \infty} \phi_x^* x' \phi = 0 \end{aligned}$$

where  $\phi_{x'} = \frac{\partial \phi}{\partial x'}$ ,  $\phi_{x'x'x'} = \frac{\partial^3 \phi}{\partial x'^3}$ . Thus, we can get

$$\begin{aligned} \frac{d}{dt'} \int_{-\infty}^{\infty} \phi^* x' \phi dx' &= \int_{-\infty}^{\infty} \left( \frac{\partial \phi^*}{\partial t'} \right) x' \phi + \phi^* x' \left( \frac{\partial \phi}{\partial t'} \right) dx' \\ &= -2i \int_{-\infty}^{\infty} \phi^* \phi'_x dx' \end{aligned}$$

In the systems, the average position (or position of mass centre) of MIP can be represented by  $\langle x' \rangle = \int_{-\infty}^{\infty} \phi^* x' \phi dx' / \int_{-\infty}^{\infty} \phi^* \phi dx'$ , the average velocity (or ve-

locity of mass centre) of MIP is defined by

$$\begin{aligned} \frac{d}{dt'} \langle x' \rangle &= \frac{\partial}{\partial t'} \left( \int_{-\infty}^{\infty} \phi^* x' \phi dx' / \int_{-\infty}^{\infty} \phi^* \phi dx' \right) \\ &= -2 \int_{-\infty}^{\infty} \phi^* \phi'_x dx' / \int_{-\infty}^{\infty} \phi^* \phi dx' \end{aligned}$$

Thus, the average acceleration (or acceleration of mass centre) of MIP can also be denoted by

$$\begin{aligned} \frac{d^2}{dt'^2} \langle x' \rangle &= -2i \frac{d}{dt'} \left( \int_{-\infty}^{\infty} \phi^* \phi'_x dx' / \int_{-\infty}^{\infty} \phi^* \phi dx' \right) \\ &= -2 \int_{-\infty}^{\infty} \phi^* V'_x \phi dx' = -2 \left\langle \frac{\partial V}{\partial x'} \right\rangle \end{aligned} \quad (52)$$

If  $\phi$  is normalized, i.e.,  $\int_{-\infty}^{\infty} \phi^* \phi dx' = 1$ , then the above conclusions also are not changed. However,  $\frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} \neq \left\langle \frac{\partial V(x')}{\partial x'} \right\rangle$ , thus we have to expand  $\frac{\partial V}{\partial x'}$  at  $x' = \langle x' \rangle$  by

$$\begin{aligned} \frac{\partial V(x')}{\partial x'} &= \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} + (x' - \langle x' \rangle) \frac{\partial^2 V(\langle x' \rangle)}{\partial \langle x' \rangle^2} \\ &\quad + \frac{1}{2!} (x' - \langle x' \rangle)^2 \frac{\partial^3 V(\langle x' \rangle)}{\partial \langle x' \rangle^3} + \dots \end{aligned}$$

Finding the expectation value to the above formula, thus we get

$$\left\langle \frac{\partial V(x')}{\partial x'} \right\rangle = \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} + \frac{1}{2!} (\langle x' - \langle x' \rangle \rangle^2) \frac{\partial^3 V(\langle x' \rangle)}{\partial \langle x' \rangle^3}$$

For the microscopic particle, described by Eq. (9), the position of the particle, that is, the position of the mass centre of the particle is known, is just  $\langle x' \rangle = x_0 = \text{constant}$ , or 0, thus  $\langle (x' - \langle x' \rangle)^2 \rangle = 0$ . Thus,  $\left\langle \frac{\partial V(x')}{\partial x'} \right\rangle = \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle}$ . Finally, we can get

$$\frac{d^2}{dt'^2} \langle x' \rangle = -2 \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} \quad \text{or} \quad m \frac{d^2 x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \quad (53)$$

where  $x_0 = \langle x \rangle$  is the position of the mass centre of MIP. Eq. (53) is a Newton-type classical equation of motion. This shows clearly that the motion of the mass centre of MIP satisfies the Newton law in nonlinear quantum mechanics [26, 27]. Therefore, we can say that the microscopic particle has the property of the classical particle. If  $V = \text{constants}$  in Eq. (5) with  $A(\phi) = 0$  we can get from Eqs. (52) and (53) that  $m \frac{d^2}{dt'^2} \langle x' \rangle = -2 \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} = 0$ . This shows that the MIP moves in uniform velocity in space-time. For the solution Eq. (9) we can get that the acceleration of the mass centre of MIP is just zero because  $V = 0$ . Therefore, the velocity

of the particle is a const. In fact, if we insert Eq. (9) into Eq. (19) we can obtain  $v_g = d\langle x' \rangle / dt' = v_e = \text{const.}$  This shows clearly that the velocity of the uniform motion of MIP is just the group velocity of the soliton. This property of the motion of microscopic particle shows that its energy and momentum can be retained in the motion process.

The above equation of motion of microscopic particles can also be derived from the nonlinear Schrödinger equation (5) with  $A(\phi) = 0$  by means of another method. As is known, the energy  $E$  and momentum of MIP described by the nonlinear Schrödinger equation (5) with  $A(\phi) = 0$  are denoted by Eq. (18) and  $P = -i \int_{-\infty}^{\infty} (\phi^* \phi'_x - \phi'^*_x \phi) dx'$ , respectively. For this system, the energy  $E$  and quantum number  $N_s = \int_{-\infty}^{\infty} |\phi|^2 dx'$  are integral invariant. However, the momentum  $P$  is not conserved and has the following property:

$$\begin{aligned} \frac{dP}{dt'} &= \int_{-\infty}^{\infty} 2V(x') \frac{\partial}{\partial x'} |\phi|^2 dx' \\ &= -2 \int_{-\infty}^{\infty} \frac{\partial V}{\partial x'} |\phi|^2 dx' \end{aligned} \quad (54)$$

where the boundary condition is  $\phi(x') \rightarrow 0$  as  $|x'| \rightarrow \infty$ . For slowly varying inhomogeneities (in comparison with particle scale (soliton)), i.e.  $W_s \gg L$ , where  $L$  is the inhomogeneity scale,  $W_s$  is its width, expanding Eq. (54) into a power series in  $W_s/L$  and keeping only the leading term, we can get

$$\frac{dP}{dt'} = -2 \frac{\partial V(x'_0)}{\partial x'_0} N_s \quad (55)$$

where  $x'_0$  is the position of the center of the mass of the macroscopic particle. Eq. (54) or (55) is essentially consistent with Eqs. (52) and (53) which are in the form of the equation of motion for a classical particle. Indeed, if we write the particle (soliton) solution as

$$\phi(x', t') = \varphi(x' - x'_0, t') e^{ip(x' - x'_0) + i\theta} \quad (56)$$

we assume that it is a solution of Eq. (5) at  $A(\phi) = 0$ . Inserting Eq. (56) into the representation of  $P(x, t)$ , we get  $P = pN_s$ . Let  $p = \frac{dx'_0}{dt'}$  be the velocity of the center of the particle, then Eq. (55) and  $P = pN_s$  indicate that the center of mass of the microscopic particle moves like a classical particle in a weakly inhomogeneous potential field  $V(x'_0)$  according to

$$\frac{d^2 x'_0}{dt'^2} = -2 \frac{\partial V}{\partial x'_0} \quad \text{or} \quad m \frac{d^2 x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \quad (57)$$

This is the same as Eq. (53) and it is Newton's equation

for a classical particle.

### 3.7 Features of motion of MIPs in several special potentials

We now consider some particular cases. Let  $V(x') = \alpha x'$  in Eq. (5) with  $A(\phi) = 0$ , where  $\alpha$  is constant, and make the following transformation [26, 27, 36, 37]:

$$\begin{aligned} \phi(x', t') &= \phi'(\tilde{x}', \tilde{t}') e^{-i\alpha \tilde{x}' \tilde{t}' - i\alpha^2 (\tilde{t}')^3 / 3} \\ x' &= \tilde{x}' - \alpha \tilde{t}'^2, \quad t' = \tilde{t}' \end{aligned}$$

Then Eq. (5) with  $A(\phi) = 0$  becomes

$$i\phi'_{\tilde{t}'} + \phi'_{\tilde{x}' \tilde{x}'} + 2|\phi'|^2 \phi' = 0 \quad (58)$$

where  $b = 2$ . The exact and complete solution of the above equation is well known. We can thus obtain the complete solution of the nonlinear Schrödinger equation with  $V(x') = \alpha x'$ . Its single soliton solution is given by

$$\begin{aligned} \phi &= 2\eta \operatorname{sech} [2\eta (x' - 4\xi t' + 2\alpha t'^2 - x'_0)] \\ &\cdot \exp \left\{ i \left[ 2\xi x' \cos 2\alpha (t' - t'_0) - \frac{\xi^2}{\alpha} \sin [4\alpha (t' - t'_0)] \right. \right. \\ &\quad \left. \left. + 4\eta^2 (t' - t'_0) + \theta'_0 \right] \right\} \end{aligned} \quad (59)$$

When  $V_0(x') = \alpha^2 x'^2$ , we can get

$$\begin{aligned} \phi &= 2\eta \operatorname{sech} \left\{ 2\eta x' - \frac{4\xi\eta}{\alpha} \sin [2\alpha (t' - t'_0)] \right\} \\ &\cdot \exp \left\{ i \left[ 2\xi x' \cos 2\alpha (t' - t'_0) - \frac{\xi^2}{\alpha} \sin [4\alpha (t' - t'_0)] \right. \right. \\ &\quad \left. \left. + 4\eta^2 (t' - t'_0) + \theta'_0 \right] \right\} \end{aligned} \quad (60)$$

In each of the above two cases, with two different external potential fields, the characteristics of motion of the microscopic particle can be determined according to Eq. (57). The accelerations of the center of mass of the microscopic particle is given by

$$\frac{d^2 x'_0}{dt'^2} = -2 \frac{\partial V(\langle x' \rangle)}{\partial \langle x' \rangle} = -2\alpha = \text{const.} \quad (61)$$

for  $V(x') = \alpha x'$ , and

$$\frac{d^2 x'_0}{dt'^2} = -4\alpha^2 x'_0 \quad (62)$$

for  $V(x') = \alpha^2 x'^2$  [26, 27, 36, 37], respectively.

These results can also be obtained using the following method. From de Broglie relation  $E = h\nu = \hbar\omega$  and  $\mathbf{P} = \hbar\mathbf{k}$  for microscopic particles which represent the

wave-corpucle duality in quantum theory, the frequency  $\omega$  retains its role as the Hamiltonian of the system even in this complicated and nonlinear systems and

$$\frac{d\omega}{dt'} = \frac{\partial\omega}{\partial k} \Big|_{x'} \frac{dk}{dt} + \frac{\partial\omega}{\partial x'} \Big|_k \frac{\partial x'}{\partial t} = 0$$

as in the usual stationary media [26, 27]. From the above result we also know that the usual Hamilton equation in Eq. (34) for the nonlinear quantum mechanical systems remain valid for the microscopic particles. Thus, the Hamilton equation in Eq. (34) can be now represented by another form:

$$\frac{dk}{dt'} = -\frac{\partial\omega}{\partial x'} \Big|_k, \quad \frac{dx'}{dt'} = \frac{\partial\omega}{\partial k} \Big|_{x'} \quad (63)$$

in the energy picture, where  $k = \partial\theta/\partial x'$  is the time-dependent wave number of the microscopic particle,  $\omega = -\partial\theta/\partial t'$  is its frequency,  $\theta$  is the phase of the wave function of the microscopic particles. From Eqs. (59) and (60), we know that

$$\theta = 2(\xi - \alpha t')x' + \frac{4\alpha^2 t'^3}{3} - 4\alpha\xi t'^2 + 4(\xi^2 - \eta^2)t' + \theta_0$$

for  $V(x') = \alpha x'$  and

$$\theta = 2\zeta x' \cos \left[ 2a(t' - t'_0) + \left( \frac{\xi^2}{a} \right) \sin 4a(t' - t'_0) \right] \\ \cdot 4\eta^2(t' - t'_0) + \theta'_0$$

for  $V(x') = a^2 x'^2$ , respectively. From Eq. (59) we can find that for  $V(x') = \alpha x'$ ,

$$k = 2(\xi - \alpha t') \\ \omega = 2\alpha x' - 4(2\xi - \alpha t')^2 + (2\eta)^2 = 2\alpha x' - k^2 + (2\eta)^2$$

Thus, the group velocity of the microscopic particle is

$$v_g = \frac{d\tilde{x}'}{dt'} = \frac{\partial\omega}{\partial k} \Big|_{x'} = 4(2\xi - \alpha t')$$

and its acceleration is given by

$$\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'} = -2\alpha = \text{const.}, \text{ here } x'_0 = \tilde{x}' \quad (64)$$

For Eq. (60), we have

$$V(x') = a^2 x'^2, \quad k = 2\xi \cos 2\alpha(t' - t'_0) \\ \omega = 4\alpha\xi x' \sin 2\alpha(t' - t'_0) - 4\xi^2 \cos 4\alpha(t' - t'_0) - 4\eta^2 \\ = 2\alpha x' (4\xi^2 - k^2)^{1/2} - 2k^2 + 4(\xi^2 - \eta^2)$$

Thus, the group velocity of the microscopic particle is

$$v_g = \frac{\partial\omega}{\partial k} \Big|_{x'} = \frac{\alpha x'}{\xi} \frac{k}{\sqrt{1 - k^2/4\xi^2}} - 2k$$

$$= 2\alpha x' \cot [2\alpha(t' - t'_0)] - 4\xi \cos [2\alpha(t' - t'_0)]$$

while its acceleration is

$$\frac{dk}{dt'} = -\frac{\partial\omega}{\partial x'} \Big|_k = -2\alpha\sqrt{4\xi^2 - k^2} \\ = -4\xi\alpha \sin [2\alpha(t' - t'_0)]$$

since  $\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'}$ , here  $(\tilde{x}' = x'_0)$ . We have

$$\frac{dk}{dt'} = \frac{d^2\tilde{x}'}{dt'^2} = -4\xi\alpha \sin [2\alpha(t' - t'_0)]$$

and

$$\tilde{x}' = \frac{2\xi}{\alpha} \sin [2\alpha(t' - t'_0)]$$

Finally, the acceleration of the microscopic particle is

$$\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'} = -4\alpha^2\tilde{x}' \quad (65)$$

Eqs. (64) and (65) are exactly the same as Eqs. (61) and (62), respectively, which shows that Eqs. (53) or (55) and (57) have the same effects and function as Eqs. (51) and (63) in nonlinear quantum mechanics. On the other hand, it is well known that a macroscopic object moves with a uniform acceleration, when  $V(x') = ax'$  which corresponds to the motion of a charge particle in a uniform electric field, and when  $V(x') = \alpha^2 x'^2$  which is a harmonic potential, the macroscopic object performs localized vibration with a frequency of  $2\alpha$  and an amplitude of  $2\xi/\alpha$ , and the corresponding classical vibrational equation is  $x' = x'_0 \sin \omega t'$ , with  $\omega = 2\alpha$  and  $x'_0 = \xi/\alpha$ . The equations of motion of the macroscopic object are consistent with Eq. (57) and Eqs. (61) and (62) or Eqs. (64) and (65) for the center of mass of microscopic particles in the nonlinear quantum mechanics. These correspondence between a microscopic particle and a macroscopic object shows [26, 27] that microscopic particles in nonlinear quantum mechanics have exactly the same properties as classical particles, and their motion satisfy the classical laws of motion. We have thus demonstrated clearly from the dynamic equations (nonlinear Schrödinger equation), the Hamiltonian or Lagrangian of the systems, and the solutions of equations of motion, in both uniform and inhomogeneous systems, that microscopic particles in nonlinear quantum mechanics really have the corpuscle property.

### 3.8 Mechanism and rules of collision of the microscopic particle

As is known, the most obvious feature of macroscopic particles is meeting the collision law or conservation law of momentum. Therefore, we often also use the law to

determine the particulate feature of macroscopic particles. In Fig. 1(d), we show also the collision feature by numerical simulation method for the solution of the nonlinear Schrödinger equation (5). From this figure, we see that microscopic particle satisfies the collision law of macroscopic particles. As a matter of fact, Zakharov *et al.* [31] demonstrated that the solutions of Eq. (5) with  $V(x, t) = A(\phi) = 0$  obey also the collision law of macroscopic particles by theoretical analysis at both  $b > 0$  and  $b < 0$ . Their results show that when microscopic particles collide with other particles, the faster particle moves forward by an amount of phase shift, and the slower one shifts backwards by an amount of phase. The total shift of the particles is equal to the algebraic sum of those of the pair during the paired collisions. At the same time, experiments and numerical simulations also show during the collision that the MIPs interact and exchange positions in the space-time trajectory as if they had passed through each other. After the collision, the two MIPs may appear to be instantly translated in space and/or time but otherwise unaffected by their interaction. The translation is called a phase shift as mentioned above. In one dimension, this process results from two MIPs colliding head-on from opposite directions, or in one direction between two particles with different amplitudes. This is possible because the velocity of a particle depends on the amplitude. The two MIPs surviving a collision completely unscathed demonstrates clearly the corpuscle feature of the microscopic particles. This property separates nonlinear quantum mechanical microscopic particles (solitons) from particles in the linear quantum mechanical regime. Therefore, the rule of collision of MIPs is the same as that of classical particles.

In the following, we describe a series of laboratory and numerical experiments dedicated to investigate the detailed structure, mechanism and rules of collision between the microscopic particles described by the nonlinear Schrödinger equation in nonlinear quantum mechanics. The properties and rules of such collision between two microscopic particles have been first studied by Aossey *et al.* [38]. Both the phase shift of the microscopic particles after their interaction and the range of the interaction are functions of the relative amplitude of the two colliding microscopic particles. The microscopic particles preserve the shape after the collision.

For the microscopic particles described by the nonlinear Schrödinger equation (5) with  $V(x, t) = A(\phi) = 0$ , we will limit our discussion to the hole (dark) spatial particles (solitons) with  $b < 0$  [26, 27]. The hole-particle is now given by

$$\phi(x', t') = \phi_0 \sqrt{1 - B^2 \text{sech}^2 \xi'} e^{\pm i \theta \xi'} \quad (66)$$

where

$$\Theta(\xi') = \arcsin \frac{B \tanh \xi'}{\sqrt{1 - B^2 \text{sech}^2 \xi'}}, \quad \xi' = \mu(\xi' - v_t t')$$

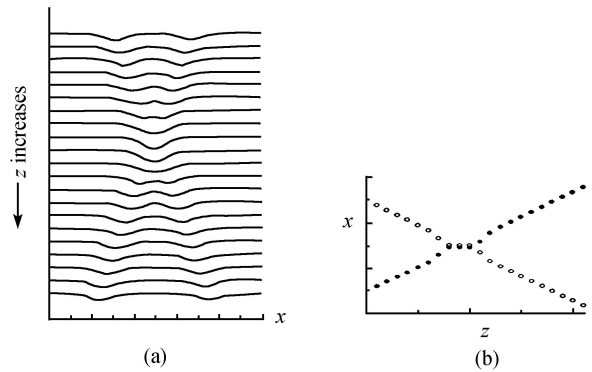
Here,  $B$  is a measure of the amplitude (“blackness”) of the solitary wave (hole or dark soliton) and can take a value between  $-1$  and  $1$ ,  $v_t$  is the dimensionless transverse velocity of the particle center, and  $\mu$  is the shape factor of the particle. The intensity ( $I_d$ ) of the solitary wave (or the depth of the irradiance minimum of the dark soliton) is given by  $B^2 \phi_0^2$ . Aossey *et al.* showed that the shape factor  $\mu$  and the transverse velocity  $v_t$  are related to the amplitude of the particles, which can be obtained from the nonlinear Schrödinger equation in the optical fiber to be

$$\mu^2 = n_0 |n_2| \mu_0^2 B^2 \phi_0^2, \quad v_t \approx \pm \sqrt{(1 - B^2) \frac{|n_2| \phi_0^2}{n_0}}$$

where  $n_0$  and  $n_2$  are the linear and nonlinear indices of refraction for the optical fiber material. We have assumed  $|n_2| \phi_0^2 \ll n_0$ . When two microscopic particles (solitons) described by NLSE collide, their individual phase shifts are given by

$$\delta x_j = \sqrt{\frac{n_0}{|n_2| \phi_0^2}} \frac{1}{2\mu_0 n_0 B_j} \cdot \ln \frac{(\sqrt{1 - B_1^2} + \sqrt{1 - B_2^2})^2 + (B_1 + B_2)^2}{(\sqrt{1 - B_1^2} + \sqrt{1 - B_2^2})^2 + (B_1 - B_2)^2} \quad (67)$$

The MIP (soliton) interaction can be easily investigated numerically by using a split-step propagation algorithm which was found, by Thusrston *et al.* [39], to closely predict experimental results. The results of a simulated collision between two equi-amplitude microscopic particles (solitons) are shown in Fig. 2 (a), which are similar to that of general MIPs (bright solitons) as shown in



**Fig. 2** Numerical simulation of an overtaking collision of equi-amplitude dark particles. (a) Sequence of the waves at equal intervals in the longitudinal position  $z$ . (b) Time-of-flight diagrams of the signal.

Fig. 1(d). We note that the two particles interpenetrate each other, retain their shape, energy and momentum, but experience a phase shift at the point of collision. In addition, there is also a well-defined interaction length in  $z$  along the axis of time  $t$  that depends on the relative amplitude of two colliding MIPs. This case occurs also in the collision of two KdV solitons. Cooney *et al.* [40] studied the overtaking collision, to verify the KdV soliton nature of an observed signal in the plasma experiment. In the following, we discuss a fairly simple model which was used to simulate and to interpret the experimental results on the MIPs (solitons) described by NLSE and KdV solitons.

The model is based on the fundamental property of solitons that two MIPs (solitons) can interact and collide, but survive the collision and remain unchanged. Rather than using the exact functional form of  $\text{sech } \xi$  for MIPs (solitons) described by NLSE, the MIPs are represented by rectangular pulses with an amplitude  $A_j$  and a width  $W_j$  where the subscript  $j$  denotes the  $j$ th microscopic particles. An evolution of the collision of two MIPs is shown in Fig. 3(a). In this case, Aosey *et al.* [38] considered two MIPs with different amplitudes. The details of what occurs during the collision need not concern us here other than to note that the MIPs with the larger-amplitude has completely passed through the one with the smaller amplitude. In regions which can be considered external to the collision, the MIPs do not overlap as there is no longer an interaction between them. The microscopic particles are separated by a distance,  $D = D_1 + D_2$ , after the interaction. This manifests itself in a phase shift in the trajectories de-

icted in Fig. 3(b). This was noted in the experimental and numerical results. The minimum distance is given by the half-widths of the two microscopic particles,  $D \geq W_1/2 + W_2/2$ . Therefore,

$$D_1 \geq \frac{W_1}{2} \text{ and } D_2 \geq \frac{W_2}{2} \tag{68}$$

Another property of the microscopic particles (solitons) is that their amplitude and width are related. For the microscopic particles described by the nonlinear Schrödinger equation with  $b < 0$  in Eq. (5) of  $V(x, t) = A(\phi) = 0 (W \approx 1/\mu)$ , we have

$$B_j W_j = \text{const.} = K_1 \tag{69}$$

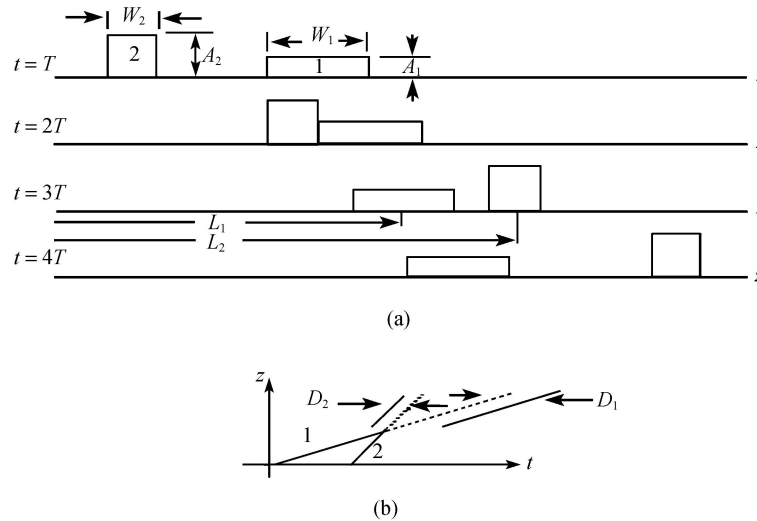
Using the minimum values in Eq. (68), we find that the ratio of the repulsive shifts for the microscopic particles described by the nonlinear Schrödinger equation is given by

$$\frac{D_1}{D_2} = \frac{B_2}{B_1} \tag{70}$$

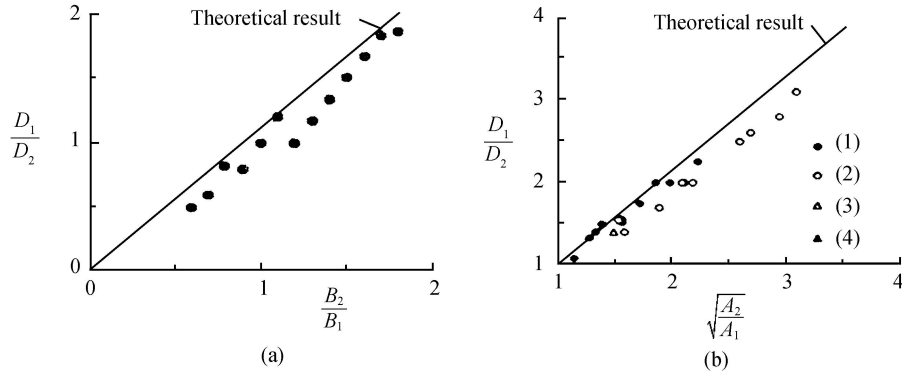
Results obtained from simulation of the kind of microscopic particle are presented in Fig. 4(a). The solid line in the figure corresponds to Eq. (70).

In addition to predicting the phase shift that results from the collision of two microscopic particles, the model also allows us to estimate the size of the collision region or duration of the collision. Each microscopic particles depicted in Fig. 3 travels with its own amplitude-dependent velocity  $v_j$ . For the two microscopic particles to interchange their positions during a time  $\Delta T$ , they must travel a distance  $L_1$  and  $L_2$ ,

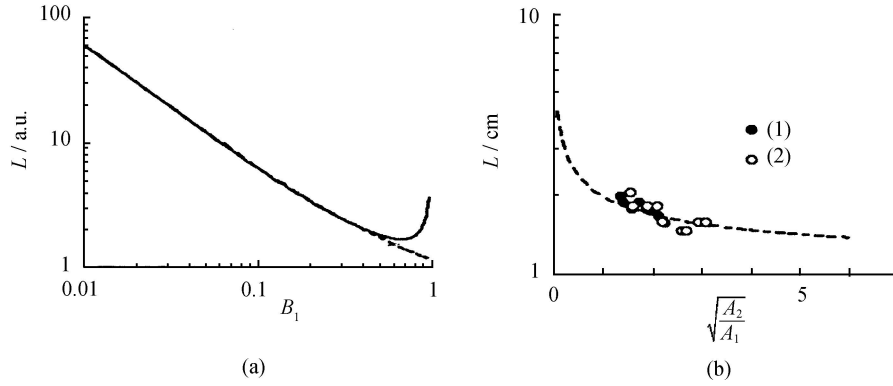
$$L_1 = v_1 \Delta T \text{ and } L_2 = v_2 \Delta T \tag{71}$$



**Fig. 3** Overtaking collision of two MIPs. (a) Model of the interaction just prior to the collision and just after the collision. After the collision, the two MIPs are shifted in phase. (b) Time-of-light diagram of the signals. The phase shifts are indicated.



**Fig. 4** Summary of the ratio of the measured phase shifts as a function of the ratio of amplitudes. **(a)** The particle described by NLSE, the solid line corresponds to Eq. (67). **(b)** KdV solitons, the data are from (1) this experiment, (2) Zabusky *et al.* [31], (3) Lamb's [41] and (4) Ikezi *et al.*'s [42] results. The solid line corresponds to Eq. (9).



**Fig. 5** Summary of the measured interaction length as a function of the amplitudes. **(a)** The particles described by NLSE, the dashed line corresponds to Eq. (70) with  $B_2=1$  and  $K_1=6$ . **(b)** KdV solitons, the data are from (1) this experiment and (2) Zabusky *et al.*'s result [31]. The dashed line corresponds to Eq. (72) with  $K_2 = A_1 = 1$ .

The interaction length must then satisfy the relation

$$L = L_2 - L_1 = (v_2 - v_1) \Delta T \geq W_1 + W_2 \quad (72)$$

Eq. (71) can be written in terms of the amplitudes of the two MIPs. For the MIPs described by NLSE, combining Eqs. (68) and (72), Aossey *et al.* obtained

$$L \geq K_1 \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \quad (73)$$

In Fig. 5(a), the results for the MIPs described by NLSE are presented. The dashed line corresponds to Eq. (73) with  $B_2 = 1$  and  $K_1 = 6$ . The interaction time (solid line) is the sum of the widths of the two microscopic particles, minus their repulsive phase shifts, and multiplied by the transverse velocity of MIP 1. Since the longitudinal velocity is a constant, this scales as the interaction length. From the figure, we see that the theoretical result obtained using the simple collision model is in good agreement with that of the numerical simulation.

The discussion presented above and the corresponding formulae reveal the mechanism and rule of the collision between MIPs depicted by NLSE in the nonlinear quan-

tum mechanics.

To verify the validity of this simple collision model, Aossey *et al.* studied the collision of the solitons using the exact form of  $\text{sech}^2\xi$  for the KdV equation  $u_t + uu_x + d'u_{xxx} = 0$ , and the collision model shown in Fig. 3. For the KdV soliton they found that

$$A_j (W_j)^2 = \text{const.} = K_2 \quad \text{and} \quad \frac{D_1}{D_2} = \frac{W_1/2}{W_2/2} = \sqrt{\frac{A_2}{A_1}} \quad (74)$$

where  $A_j$  and  $W_j$  are the amplitude and width of the  $j$ th KdV soliton, respectively. Corresponding to the above, Aossey *et al.* obtained

$$L \geq K_2 \left( \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} \right) = \frac{K_2}{\sqrt{A_1}} \left( 1 + \sqrt{\frac{A_1}{A_2}} \right) \quad (75)$$

for the interaction length.

Aossey *et al.* [38] compared their results for the ratio of the phase shifts as a function of the ratio of the amplitudes for the KdV solitons, with those obtained in the experiments of Ikezi, Taylor, and Baker [42], and those

obtained from numerical work of Zabusky and Kruskal [31] and Lamb [41], as shown in Fig. 4(b). The solid line in Fig. 4(b) corresponds to Eq. (74). Results obtained by Aosey *et al.* for the interaction length are shown in Fig. 5(b) as a function of amplitudes of the colliding KdV solitons. Numerical results (which were scaled) from Zabusky and Kruskal are also shown for comparison. The dashed line in Fig. 5(b) corresponds to Eq. (75), with  $A_1 = l$  and  $K_2 = l$ .

Since the theoretical results obtained by the collision model based on macroscopic bodies in Fig. 3 are consistent with experimental data for the KdV soliton, shown in Figs. 4(b) and 5(b), it is reasonable to believe the validity of the above theoretic results of model of collision presented above, and results shown in Figs. 4(a) and 5(a) for the microscopic particles described in the nonlinear Schrödinger equation which are obtained using the same model as that shown in Fig. 3. Thus, the above colliding mechanism for the microscopic particles shows clearly the classical corpuscle feature of the microscopic particles in nonlinear quantum mechanics.

### 3.9 Features of reflection and transmission of microscopic particles at interfaces

As mentioned above, microscopic particles in nonlinear quantum mechanics represented by Eq. (5) also have wave property, in addition to the corpuscle property. This wave feature can be conjectured from the following reasons.

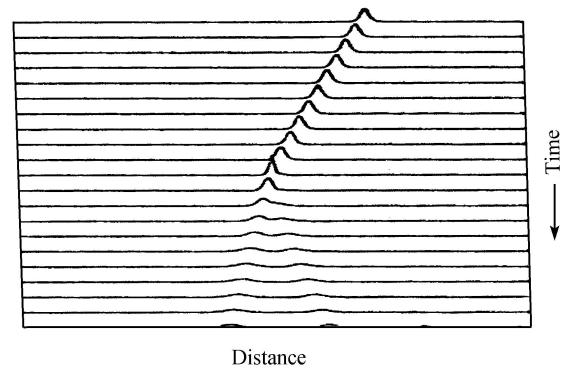
(1) Eqs. (5)–(8) are wave equations and their solutions, Eqs. (9), (10) and (15) are solitary waves having the features of traveling waves. A solitary wave consists of a carrier wave and an envelope wave, has certain amplitude, width, velocity, frequency, wavevector, and so on, and satisfies the principles of superposition of waves, although the latter are different when compared with classical waves or the de Broglie waves in linear quantum mechanics.

(2) The solitary waves have reflection, transmission, scattering, diffraction and tunneling effects, just as that of classical waves or the de Broglie waves in linear quantum mechanics. At present, we study the reflection and transmission of the microscopic particles at an interface.

The propagation of microscopic particles (solitons) in a nonlinear nonuniform media is different from that in uniform media. The nonuniformity can be due to a physical confining structure or two nonlinear materials being juxtaposed. One could expect that a portion of microscopic particles that was incident upon such an interface from one side would be reflected and a portion would be transmitted to the other side due to its wave feature.

Lonngren *et al.* [43] observed the reflection and transmission of microscopic particles (solitons) in a plasma consisting of a positive ion and a negative ion interface, and numerically simulated the phenomena at the interface of two nonlinear materials. To illustrate the rules of reflection and transmission of microscopic particles, we discuss here the work of Lonngren *et al.* [43]

Lonngren *et al.* [41] simulated numerically the behaviors of microscopic particles (MIP) (soliton) described by NLSE. They found that the signal had the property of a soliton. These results are in agreement with numerical investigations of similar problems by Aceves *et al.* A sequence of pictures obtained by Lonngren *et al.* [41] at uniform temporal increments of the spatial evolution of the signal are shown in Fig. 6. From this figure, we note that the incident microscopic particles propagating toward the interface between the two nonlinear media splits into a reflected and transmitted soliton at the interface. From the numerical values used in producing the figure, the relative amplitudes of the incident, the reflected and the transmitted solitons can be deduced.



**Fig. 6** Simulation results showing the collision and scattering of an incident MIPs described by NLSE (top) onto an interface. The peak nonlinear refractive index change is 0.67 % of the linear refractive index for the incident MIPs and the linear offset between the two regions is also 0.67 %.

They assumed that the energy that is carried by the incident MIP (soliton) is all transferred to either the transmitted or the reflected MIPs and none is lost through radiation. Thus

$$E_{\text{inc}} = E_{\text{ref}} + E_{\text{trans}} \quad (76)$$

Lonngren *et al.* gave approximately the energy of MIP by

$$E_j = \frac{A_j^2}{Z_c} W_j$$

where the subscript  $j$  refers to the incident, reflected or transmitted MIPs. The amplitude of the MIP is  $A_j$  and its width is  $W_j$ . The characteristic impedance of a

material is given by  $Z_c$ . Hence, Eq. (76) can be written as

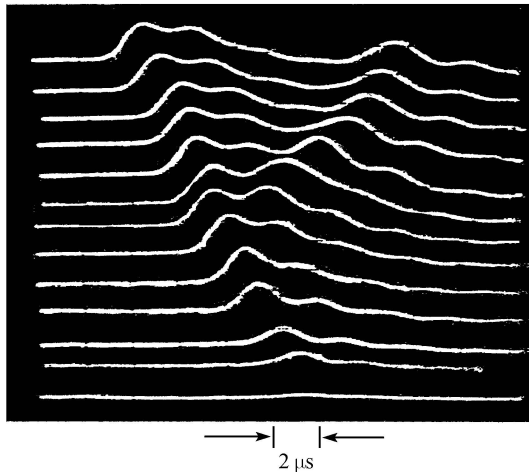
$$\frac{A_{\text{inc}}^2}{Z_{\text{cI}}} W_{\text{inc}} = \frac{A_{\text{ref}}^2}{Z_{\text{cI}}} W_{\text{ref}} + \frac{A_{\text{trans}}^2}{Z_{\text{cII}}} W_{\text{trans}} \quad (77)$$

Since  $A_j W_j = \text{const.}$  for the MIP described NLSE (see Eq. (69) in which  $B_j$  is represented by  $A_j$ ), we obtain the following relation between the reflection coefficient  $R = A_{\text{ref}}/A_{\text{inc}}$  and the transmission coefficient  $T = A_{\text{trans}}/A_{\text{inc}}$ ,

$$1 = R + \frac{Z_{\text{cI}}}{Z_{\text{cII}}} T \quad (78)$$

for the MIPs described by NLSE.

To verify further this idea, Lonngren *et al.* [44] conducted experiments with KdV soliton. They found that the detected signal had the characteristics of a KdV soliton. Lonngren *et al.* [44] showed a sequence of pictures taken using a small probe at equal spatial increments starting initially in a homogeneous plasma sheath adjacent to a perturbing biased object, as shown in Fig. 7. From this figure, we see that the probe first detects the incident soliton and some time later the reflected soliton. The signals are observed, as expected, to coalesce together as the probe passed through the point where the soliton was actually reflected. Beyond this point which was at the location where the density started to decrease in the steady-state sheath, a transmitted soliton was observed. From Fig. 7, the relative amplitudes of incident, the reflected and the transmitted solitons can be deduced, which was done by the author.

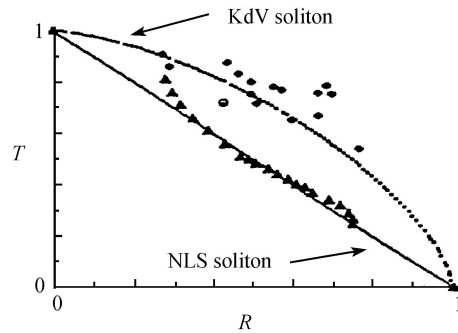


**Fig. 7** Sequence of the signals detected as the probe is moved in 2 mm increments from 30 to 6 mm in front of the reflector. The incident and reflected KdV solitons coalesce at the point of reflection, which is approximately 16 mm in front of the reflector. A transmitted soliton is observed closer to the disc. The amplitude scale at 8 and 6 mm is increased by 2 from the previous traces.

For the KdV solitons, there is also  $A_j W_j^2 = \text{const.}$  [see Eq. (74)]. Thus, for the KdV solitons, we have

$$1 = R^{3/2} + \frac{Z_{\text{eI}}}{Z_{\text{eII}}} T^{3/2}$$

The relations between the reflection and the transmission coefficients for the MIP described by NLSE and KdV soliton are shown in Fig. 8, with the ratio of characteristic impedances set to one. The experimental results on KdV solitons and results of the numerical simulation of MIPs described by NLSE are also given in this figure. The computed data are shown using triangles. Good agreement between the analytic results and simulation results can be seen. The oscillatory deviation from the analytic result is due to the presence of radiation modes in addition to the soliton modes. The interference between these two types of modes results in the oscillation in the soliton amplitude. In the asymptotic limit, the radiation will spread and damp the oscillation, and result in the reflection — transmission coefficient curve falling on the analytic curve.



**Fig. 8** The relationship between the reflection and transmission coefficients of a microscopic particle (soliton) given in Eq. (78). The solid circles are results from the laboratory experiment on KdV solitons and the hollow circle is Nishida's result. The solid triangles are Lonngren *et al.*'s numerical results for the particle (soliton) described by NLSE.

The above rule of propagation of the microscopic particles in nonlinear quantum mechanics is different from that of linear waves in classical physics. Lonngren *et al.* [45] found that a linear wave obeyed the following relation:

$$1 = R^2 + \frac{Z_{\text{eI}}}{Z_{\text{eII}}} T^2 \quad (79)$$

This can be also derived from Eq. (74), by assuming the linear waves. The width of the incident, reflected and transmitted pulses  $W_j$  will be the same. For the linear waves

$$R = \frac{Z_{\text{cII}} - Z_{\text{cI}}}{Z_{\text{cII}} + Z_{\text{cI}}} \text{ and } T = \frac{2Z_{\text{cII}}}{Z_{\text{cII}} + Z_{\text{cI}}}$$

Eq. (79) is satisfied. Obviously, Eq. (79) is different from Eq. (78). This shows clearly that the microscopic particles in nonlinear quantum mechanics have a wave feature, but it is different from that of linear classical waves and the de Broglie waves in linear quantum mechanics.

### 3.10 The uncertainty relation in linear and nonlinear quantum mechanics

#### 3.10.1 The uncertainty relation in linear quantum mechanics

The uncertainty relation in quantum mechanics is an important representation and also a problem that has troubled many scientists. Whether this is an intrinsic feature of microscopic particles or a result brought by the linear quantum mechanics or by the measuring instruments, it results in a long-term controversy in physics. How do we understand this correctly? Obviously, it is closely related to the wave-corpucle duality of microscopic particles. Since we have established nonlinear quantum mechanics which differs from linear quantum mechanics, we could expect that the uncertainty relation in the nonlinear quantum mechanics is different from that in linear quantum mechanics [26, 27]. The significance of the uncertainty relation can be revealed by comparing the linear and nonlinear quantum theories.

It is well known that the uncertainty relation in the linear quantum mechanics can be obtained from [1, 6]

$$I(\xi') = \int |(\xi' \Delta \hat{A} + i \Delta \hat{B})\psi(\mathbf{r}, t)|^2 d\mathbf{r} \geq 0 \quad (80)$$

or

$$\bar{F}(\xi) = \int \psi^*(\mathbf{r}, t) \mathbf{F}[\hat{A}(\mathbf{r}, t), \hat{B}(\mathbf{r}, t)]\psi(\mathbf{r}, t) d\mathbf{r}$$

In the coordinate representation,  $\mathbf{A}$  and  $\mathbf{B}$  are operators of two physical quantities, for example, position and momentum, or energy and time, and satisfy the commutation relation  $[\hat{A}, \hat{B}] = i\hat{C}$ ,  $\psi(x, t)$  and  $\psi^*(x, t)$  are wave functions of the microscopic particle satisfying the linear Schrödinger equation and its conjugate equation, respectively,  $\hat{F} = (\Delta A \xi' + \Delta B)^2$ ,  $[\Delta \hat{A} = \hat{A} - \bar{A}, \Delta \hat{B} = \hat{B} - \bar{B}, \bar{A}$  and  $\bar{B}$  are the average values of the physical quantities in the state denoted by  $\psi(x, t)$ ], is an operator of physical quantity related to  $\bar{A}$  and  $\bar{B}$ ,  $\xi'$  is a real parameter.

After some simplifications, we can get

$$I = \bar{F} = \overline{\Delta \hat{A}^2 \xi'^2} + 2\overline{\Delta \hat{A} \Delta \hat{B} \xi'} + \overline{\Delta \hat{B}^2} \geq 0$$

or

$$\overline{\Delta \hat{A}^2 \xi'^2} + \overline{\hat{C} \xi'} + \overline{\Delta \hat{B}^2} \geq 0 \quad (81)$$

Using mathematical identities, this can be written as

$$\overline{\Delta \hat{A}^2 \Delta \hat{B}^2} \geq \frac{\overline{\hat{C}^2}}{4} \quad (82)$$

This is the uncertainty relation in linear quantum mechanics. From the above derivation we see that the uncertainty relation was obtained based on the fundamental hypotheses of linear quantum mechanics, including properties of operators of the mechanical quantities, the state of particle represented by the wave function, which satisfies the linear Schrödinger equation (1), the concept of average values of mechanical quantities and the commutation relations and eigenequation of operators. Therefore, we can conclude that the uncertainty relation Eq. (82) is a necessary result of the linear quantum mechanics. Since the linear quantum mechanics only describes the wave nature of microscopic particles, the uncertainty relation is a result of the wave feature of microscopic particles, and it inherits the wave nature of microscopic particles. This is why its coordinate and momentum cannot be determined simultaneously. This is an essential interpretation for the uncertainty relation Eq. (82) in linear quantum mechanics. It is not related to measurement, but closely related to the linear quantum mechanics. In other words, if linear quantum mechanics could correctly describe the states of microscopic particles, then the uncertainty relation should also reflect the peculiarities of microscopic particles.

Eq. (81) can be written in the following form:

$$\hat{F} = \overline{\Delta \hat{A}^2} \left( \xi' + \frac{\overline{\Delta \hat{A} \Delta \hat{B}}}{\overline{\Delta \hat{A}^2}} \right)^2 + \overline{\Delta \hat{B}^2} - \frac{(\overline{\Delta \hat{A} \Delta \hat{B}})^2}{\overline{\Delta \hat{A}^2}} \geq 0$$

or

$$\overline{\Delta \hat{A}^2} \left( \xi' + \frac{\bar{C}}{4\overline{\Delta \hat{A}^2}} \right)^2 + \overline{\Delta \hat{B}^2} - \frac{\bar{C}^2}{4\overline{\Delta \hat{A}^2}} \geq 0 \quad (83)$$

This shows that  $\overline{\Delta \hat{A}^2} \neq 0$ , if  $\overline{(\Delta \hat{A} \Delta \hat{B})^2}$  or  $\bar{C}^2/4$  is not zero, else, we cannot obtain Eq. (82) and  $\overline{\Delta \hat{A}^2 \Delta \hat{B}^2} > \overline{(\Delta \hat{A} \Delta \hat{B})^2}$  because when  $\overline{\Delta \hat{A}^2} = 0$ , Eq. (83) does not hold. Therefore,  $\overline{(\Delta \hat{A}^2)} \neq 0$  is a necessary condition for the uncertainty relation Eq. (82),  $\overline{\Delta \hat{A}^2}$  can approach zero, but cannot be equal to zero. Therefore, in linear quantum mechanics, the right uncertainty relation should take the form [26, 27]:

$$\overline{\Delta \hat{A}^2 \Delta \hat{B}^2} > \frac{(\bar{C}^2)^2}{4} \quad (84)$$

#### 3.10.2 The uncertainty relation in nonlinear quantum mechanics

We now return to the uncertainty relation in nonlin-

ear quantum mechanics. Since microscopic particles in NLQM is a soliton and they have wave-corpucle duality, and there is no fundamental hypothesis in the nonlinear quantum mechanics, derivation of the uncertainty relation should be different from that in the linear quantum theory given above. We can also expect that the uncertainty relation in nonlinear quantum mechanics [26, 27] is different from Eq. (84).

We now derive this relation for position and momentum of a microscopic particle depicted by the nonlinear Schrödinger equation (5) with  $V = 0$ , and  $A=0$ , with a solution,  $\phi_s$ , as given in Eq. (9), which is now represented by

$$\phi_s(x', t') = 2\eta\sqrt{2/b}\text{sech}[2\eta(x' - x'_0) + 8\eta\xi t'] \cdot e^{-4i(\xi^2 - \eta^2)t' - 2i\xi x' + i\theta} \tag{85}$$

where  $\xi$  is related to velocity of particle (soliton),  $\eta$  denotes its amplitude,  $x'_0$  is the position of mass center,  $\theta$  is an any constant.

The function  $\phi_s(x', t')$  is a square integrable function localized at  $x'_0 = 0$  in the position space. If the microscopic particle is localized at  $x'_0 \neq 0$ , it satisfies the nonlinear Schrödinger equation,

$$i\phi_{t'} + \frac{1}{2}\phi_{x'x'} + |\phi|^2\phi = 0 \tag{86}$$

for  $b = 1$ . The Fourier transform of this function [26, 27] is

$$\phi_s(p, t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x', t') e^{-ipx'} \tag{87}$$

It shows that  $\phi_s(p, t')$  is localized at  $p$  in momentum space. For Eq. (87), the Fourier transform is explicitly given by

$$\phi_s(p, t') = \sqrt{\frac{\pi}{2}} \text{sech} \left[ \frac{\pi}{2\eta} (p - 2\sqrt{2}\xi) \right] \cdot e^{4i[\eta^2 + \xi^2 - p\xi/2\sqrt{2}t' - i(p - 2\sqrt{2}\xi)x'_0]} \tag{88}$$

The results in Eqs. (87) and (88) show that the microscopic particle is localized not only in position space in the shape of soliton, but also in the momentum space in a soliton. For convenience, we introduce the normalization coefficient  $A_0$  in Eqs. (85) and (88), then obviously  $A_0^2 = \frac{1}{4\sqrt{2}}\eta$ , the position of the certain of mass of the microscopic particle,  $\langle x' \rangle$ , and its square,  $\langle x'^2 \rangle$ , at  $t' = 0$  are given by

$$\langle x' \rangle = \int_{-\infty}^{\infty} dx' |\phi_s(x')|^2$$

$$\langle x'^2 \rangle = \int_{-\infty}^{\infty} dx' x'^2 |\phi_s(x')|^2$$

We can thus find that

$$\langle x' \rangle = 4\sqrt{2}\eta A_0^2 x'_0, \quad \langle x'^2 \rangle = \frac{A_0^2 \pi^2}{12\sqrt{2}\eta} + 4\sqrt{2}A_0^2 \eta x_0'^2$$

respectively. Similarly, the momentum of the center of mass of the microscopic particle,  $\langle p \rangle$ , and its square,  $\langle p^2 \rangle$ , are given by

$$\langle p \rangle = \int_{-\infty}^{\infty} p |\hat{\phi}_s(p)|^2 dp, \quad \langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\hat{\phi}_s(p)|^2 dp$$

which yield

$$\langle p \rangle = 16A_0^2 \eta \xi, \quad \langle p^2 \rangle = \frac{32\sqrt{2}}{3} A_0^2 \eta^3 + 32\sqrt{2} A_0^2 \eta \xi^3 \tag{89}$$

The standard deviations of position  $\Delta x' = \sqrt{\langle x'^2 \rangle - \langle x' \rangle^2}$  and momentum  $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$  are given by

$$\begin{aligned} (\Delta x')^2 &= A_0^2 \left[ \frac{\pi^2}{12\eta} + 4\eta x_0'^2 (1 - 4\sqrt{2}\eta A_0^2) \right] = \frac{\pi^2}{96\eta^2} \\ (\Delta p)^2 &= 32\sqrt{2} A_0^2 \left[ \frac{1}{3}\eta^3 + \eta \xi^3 (1 - 4\sqrt{2}\eta A_0^2) \right] = \frac{8}{3}\eta^2 \end{aligned} \tag{90}$$

respectively. Thus we obtain the uncertainty relation between position and momentum for the microscopic particle depicted by nonlinear quantum mechanics, Eq. (86),

$$\Delta x' \Delta p = \frac{\pi}{6} \tag{91}$$

This result is not related to the features of the microscopic particle (soliton) depicted by the nonlinear Schrödinger equation in nonlinear quantum mechanics because Eq. (91) has nothing to do with characteristic parameters of the nonlinear Schrödinger equation.  $\pi$  in Eq. (91) comes from of the integral coefficient  $1/\sqrt{2\pi}$ . For a quantized microscopic particle,  $\pi$  in Eq. (91) should be replaced by  $\pi\hbar$ , because Eq. (87) is replaced by

$$\phi_s(p, t') = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx' \phi_s(x', t') e^{-ipx'/\hbar}$$

The corresponding uncertainty relation of the quantum microscopic particle in nonlinear quantum mechanics is given by

$$\Delta x \Delta p = \frac{\pi\hbar}{6} = \frac{h}{12} \tag{92}$$

The uncertainty relation in Eq. (92) or Eq. (91) [26, 27] is different from that in linear quantum mechanics Eq. (84), i.e.,  $\Delta x \Delta p > h/2$ . However, the minimum value

$\Delta x \Delta p = \hbar/2$  has not been observed in practical system in linear quantum mechanics up to now except for the coherent and squeezed states of microscopic particles. The relation Eq. (91) cannot be obtained from the solutions of linear Schrödinger equation. Practically, we can only get  $\Delta x \Delta p > \hbar/2$  from Eq. (84), but not  $\Delta x \Delta p = \hbar/2$ , in linear quantum mechanics.

### 3.10.3 The uncertainty relations of coherent states

As a matter of fact, we can represent one-quantum coherent state of harmonic oscillator by [26, 27]

$$|\alpha\rangle = \exp(\alpha \hat{b}^\dagger - \alpha^* \hat{b})|0\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{b}^{\dagger n} |0\rangle$$

in the number picture, which is a coherent superposition of a large number of microscopic particles (quanta). Thus

$$\langle \alpha | \hat{x} | \alpha \rangle = \sqrt{\frac{\hbar}{2\omega m}} (\alpha + \alpha^*)$$

$$\langle \alpha | \hat{p} | \alpha \rangle = i\sqrt{\hbar m \omega} (\alpha - \alpha^*)$$

and

$$\langle \alpha | \hat{x}^2 | \alpha \rangle = \frac{\hbar}{2\omega m} (\alpha^{*2} + \alpha^2 + 2\alpha\alpha^* + 1)$$

$$\langle \alpha | \hat{p}^2 | \alpha \rangle = \frac{\hbar m \omega}{2} (\alpha^{*2} + \alpha^2 - 2\alpha\alpha^* - 1)$$

where

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega m}} (\hat{b} + \hat{b}^\dagger), \quad \hat{p} = i\sqrt{\frac{\hbar m \omega}{2}} (\hat{b}^\dagger - \hat{b})$$

and  $\hat{b}^\dagger$  ( $\hat{b}$ ) is the creation (annihilation) operator of the microscopic particle (quantum),  $\alpha$  and  $\alpha^*$  are some unknown functions,  $\omega$  is the frequency of the particle,  $m$  is its mass. Thus we can get

$$(\Delta x)^2 = \frac{\hbar}{2\omega m}, \quad (\Delta p)^2 = \frac{\hbar m \omega}{2}, \quad \langle \Delta x \rangle^2 \langle \Delta p \rangle^2 = \frac{\hbar^2}{4}$$

$$\frac{\Delta x}{\Delta p} = \frac{1}{\omega m}, \quad \text{or} \quad \Delta p = (\omega m) \Delta x \quad (93)$$

For the squeezed state of the microscopic particle:  $|\beta\rangle = \exp[\beta(\hat{b}^{\dagger 2} - \hat{b}^2)]|0\rangle$ , which is a two-microscopic particle (quanta) coherent state, we can find that

$$\langle \beta | \Delta x^2 | \beta \rangle = \frac{\hbar}{2m\omega} e^{4\beta}, \quad \langle \beta | \Delta p^2 | \beta \rangle = \frac{\hbar m \omega}{2} e^{-4\beta}$$

using a similar approach as the above. Here  $\beta$  is the squeezed coefficient and  $|\beta| < 1$ . Thus,

$$\Delta x \Delta p = \frac{\hbar}{2}, \quad \frac{\Delta x}{\Delta p} = \frac{1}{m\omega} e^{8\beta}$$

$$\text{or } \Delta p = \Delta x (\omega m) e^{-8\beta} \quad (94)$$

This shows that the momentum of the microscopic particle (quantum) is squeezed in the two-quanta coherent state compared to that in the one-quantum coherent state.

From the above results, we see that both one-quantum and two-quanta coherent states satisfy the minimal uncertainty principle. This is the same as that of the nonlinear quantum states in nonlinear quantum mechanics. We can conclude that a coherent state is a kind of nonlinear quantum state, and the coherence of quanta is a nonlinear phenomenon, instead of a linear effect.

As is known, the coherent state satisfies the classical equation of motion, in which the fluctuation in the number of particles approaches zero, which is a classically steady wave. In fact, according to quantum theory, the coherent state of a harmonic oscillator at time  $t$  can be represented by

$$|\alpha, t\rangle = e^{-i\hat{H}t/\hbar} |\alpha\rangle = e^{-i\omega(\hat{b} + \hat{b}^\dagger)t} |\alpha\rangle$$

$$= e^{-i\omega t/2 - |\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle, \quad |n\rangle = (b^\dagger)^n |0\rangle$$

This shows that the shape of a coherent state can be retained during its motion. This is the same as that of a microscopic particle (soliton) in nonlinear quantum mechanics. The mean position of the particle in the time-dependent coherent state is

$$\langle \alpha, t | x | \alpha, t \rangle = \langle \alpha | e^{iHt/\hbar} x e^{-iHt/\hbar} | \alpha \rangle$$

$$= \langle \alpha | x - \frac{it}{\hbar} [x, H] + \frac{(-it)^2}{2!\hbar^2} [[x, H], H] + \dots | \alpha \rangle$$

$$= \langle \alpha | x + \frac{pt}{m} - \frac{1}{2!} t^2 \omega^2 x + \dots | \alpha \rangle$$

$$= \langle \alpha | x \cos \omega t + \frac{p}{m\omega} \sin \omega t | \alpha \rangle$$

$$= \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos(\omega t + \theta) \quad (95)$$

where  $\theta = \arctan \frac{y}{x}$ ,  $x + iy = \alpha$ ,  $[x, H] = \frac{i\hbar p}{m}$ ,  $[p, H] = -i\hbar m \omega^2 x$ .

Comparing (95) with the solution of a classical harmonic oscillator

$$x = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t + \theta), \quad E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

we find that they are similar, with

$$E = \hbar \omega \alpha^2 = \langle \alpha | H | \alpha \rangle - \langle 0 | H | 0 \rangle$$

$$H = \hbar\omega \left( b^+b + \frac{1}{2} \right)$$

Thus, we can say that the center of the coherent state-packet indeed obeys the classical law of motion, which is the same as the law of motion of microscopic particles in nonlinear quantum mechanics discussed in Eqs. (70), (71).

We can similarly obtain

$$\begin{aligned} \langle \alpha, t | p | \alpha, t \rangle &= -\sqrt{2m\hbar\omega} |\alpha| \sin(\omega t + \theta) \\ \langle \alpha, t | x^2 | \alpha, t \rangle &= \frac{2\hbar}{\omega m} \left[ |\alpha|^2 \cos^2(\omega t + \theta) + \frac{1}{4} \right] \\ \langle \alpha, t | p^2 | \alpha, t \rangle &= 2m\hbar\omega \left[ |\alpha|^2 \sin^2(\omega t + \theta) + \frac{1}{4} \right] \end{aligned}$$

and

$$[\Delta x(t)]^2 = \frac{\hbar}{2\omega m}, \quad [\Delta p(t)]^2 = \frac{1}{2} m\omega\hbar, \quad \Delta x(t)\Delta p(t) = \frac{\hbar}{2} \quad (96)$$

This is the same as Eq. (92). It shows that the minimal uncertainty principle for the coherent state is retained at all times, i.e., the uncertainty relation does not change with time  $t$ .

The mean number of quanta in the coherent state is given by

$$\begin{aligned} \bar{n} &= \langle \alpha | \hat{N} | \alpha \rangle = \langle \alpha | \hat{b}^+ \hat{b} | \alpha \rangle = \alpha^2 \\ \langle \alpha | \hat{N}^2 | \alpha \rangle &= |\alpha|^4 + |\alpha|^2 \end{aligned}$$

Therefore, the fluctuation of the quantum in the coherent state is

$$\Delta n = \sqrt{\langle \alpha | \hat{N}^2 | \alpha \rangle - (\langle \alpha | \hat{N} | \alpha \rangle)^2} = |\alpha|$$

which leads to

$$\frac{\Delta n}{n} = \frac{1}{|\alpha|} \ll 1$$

It is thus obvious that the fluctuation of the quantum in the coherent state is very small. The coherent state is quite close to the feature of soliton and solitary wave.

These properties of coherent states are also similar to those of microscopic particles described by the nonlinear Schrödinger equation, the  $\phi^4$ -equation, or the Sine-Gordon equation in nonlinear quantum mechanics. In practice, the state of a microscopic particle in nonlinear quantum mechanics can always be represented by a coherent state, for example, the Davydov's wave functions, both  $|D_1\rangle$  and  $|D_2\rangle$  [46], and Pang's wavefunction [47, 48] of exciton-solitons in protein molecules and the wavefunction in acetanilide [49, 50]; the wavefunction of

proton transfer in hydrogen-bonded systems [51, 53] and the BCS's wave function in superconductors [54], etc. Hence, the coherence of particles is a kind of nonlinear phenomenon that occurs only in nonlinear quantum mechanics. It does not belong to systems described by linear quantum mechanics, because the coherent state cannot be obtained by superposition of linear waves, such as plane wave, de Broglie wave, or Bloch wave, which are solutions of the linear Schrödinger equation in linear quantum mechanics. Therefore, the minimal uncertainty relation Eq. (92), as well as Eqs. (94) and (96), are only applicable to microscopic particles in nonlinear quantum mechanics. In other words, only microscopic particles in nonlinear quantum mechanics satisfy the minimal uncertainty principle. It reflects the wave-corpucle duality of microscopic particles because it holds only if the duality exists.

This uncertainty principle also suggests that the position and momentum of the microscopic particle can be simultaneously determined in a certain degree and range. A rough estimate for the size of the uncertainty can be given. If it is required that  $\phi_s(x, t)$  in Eq. (85) or  $\phi_s(p, t)$  in Eq. (87) satisfies the admissibility condition, i.e.,  $\phi_s(0) \approx 0$ , we choose  $\xi = 140$ ,  $\eta = \sqrt{300/0.253}/2\sqrt{2}$  and  $\mathbf{x}_0 = 0$  in Eq. (85) (In fact, in such a case we can get  $\phi_s(0) \approx 10^{-6}$ , thus the admissibility condition can be satisfied). We then get  $\Delta x \approx 0.02624$  and  $\Delta p \approx 19.893$ , according to Eqs. (91) and (92). These results show that the position and momentum of microscopic particles in nonlinear quantum mechanics can be simultaneously determined within a certain approximation.

Pang *et al.* [55, 56] also calculated the uncertainty relation and quantum fluctuations and studied their properties in nonlinearly coupled electron-phonon systems based on the Holstein model by a new ansatz including the correlations among one-phonon coherent and two-phonon squeezing states and polaron state proposed by himself. Many interesting results were obtained. The minimum uncertainty relation takes different forms in different systems which are related to the properties of the microscopic particles. Nevertheless, the minimum uncertainty relation in Eq. (92) holds for both the one-quantum coherent state and two-quanta squeezed state. These works enhanced our understanding of the significance and nature of the minimum uncertainty relation.

#### 3.10.4 Quantum fluctuation of particles described by quantum nonlinear Schrödinger equation

Finally, we determine the uncertainty relation of the microscopic particle described by quantum nonlinear Schrödinger equation, arising from the quantum fluctu-

ation effect in the nonlinear quantum field theory. The quantum theory was discussed by Lai and Haus *et al.* [26, 27, 57] based on the nonlinear Schrödinger equation. A superposition of a subclass of bound state  $|n, P\rangle$ , characterized by number of the boson, for example, photon or phonon, and the momentum of the center of the mass  $P$ , can reproduce the expectation values of the microscopic particle (soliton) in the limit where the average number of the bosons (phonons) are larger; we refer to these states formed by the superposition of  $|n, P\rangle$  as a fundamental soliton states. In quantum theory, the quantized dynamic equation in the second quantized picture is given by

$$i\hbar \frac{\partial}{\partial t} \hat{\phi}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \hat{\phi}(x, t) + 2b\hat{\phi}^+(x, t)\hat{\phi}(x, t)\hat{\phi}(x, t) \quad (97)$$

The operators  $\hat{\phi}(x, t)$  and  $\hat{\phi}^+(x, t)$  are the annihilation and creation operators of field of a quantum at a “point”  $x$  and “time”  $t$ , they satisfy the commutation relation:

$$[\hat{\phi}(x'', t), \hat{\phi}^+(x, t)] = \delta(x - x'') \quad (98)$$

$$[\hat{\phi}(x'', t), \hat{\phi}(x, t)] = [\hat{\phi}^+(x'', t), \hat{\phi}^+(x, t)] = 0$$

The corresponding quantum Hamiltonian is given by

$$\hat{H} = \frac{\hbar^2}{2m} \int \hat{\phi}_x^+(x, t)\hat{\phi}_x(x, t)dx + b \int \hat{\phi}^+(x, t)\hat{\phi}^+(x, t)\hat{\phi}(x, t)\hat{\phi}(x, t)dx \quad (99)$$

In the Schrödinger picture, the time evolution of the system is described by

$$i\hbar \frac{d}{dt} |\Phi\rangle = \hat{H}_s |\Phi\rangle \quad (100)$$

with the commutation relation:

$$[\hat{\phi}(x''), \hat{\phi}^+(x)] = \delta(x - x'') \quad (101)$$

$$[\hat{\phi}(x''), \hat{\phi}(x)] = [\hat{\phi}^+(x''), \hat{\phi}^+(x)] = 0$$

where  $\hat{\phi}(x)$  and  $\hat{\phi}^+(x)$  are the field operators in the Schrödinger representation. The corresponding quantum Hamiltonian is given by

$$\hat{H}_s = \frac{\hbar^2}{2m} \int \hat{\phi}_x^+(x)\hat{\phi}_x(x)dx + b \int \hat{\phi}^+(x)\hat{\phi}^+(x)\hat{\phi}(x)\hat{\phi}(x)dx \quad (102)$$

The many-particle state  $|\Phi\rangle$  can be built up from the  $n$ -particle states given by

$$|\Phi\rangle = \sum_n a_n \int \frac{1}{\sqrt{n!}} f_n(x_1, \dots, x_n, t)$$

$$\cdot \hat{\phi}^+(x_1) \cdots \hat{\phi}^+(x_n) dx_1 \cdots dx_n |0\rangle \quad (103)$$

The quantum theory based on Eq. (103) describes an ensemble of bosons interacting via a  $\delta$ -potential. Note that  $\hat{H}$  preserves both the particle number.

$$\hat{N} = \int \hat{\phi}^+(x)\hat{\phi}(x)dx \quad (104)$$

and the total momentum

$$\hat{P} = i\frac{\hbar}{2} \int \left[ \frac{\partial}{\partial x} \hat{\phi}^+(x)\hat{\phi}(x) - \hat{\phi}^+(x)\frac{\partial}{\partial x} \hat{\phi}(x) \right] dx \quad (105)$$

Lai *et al.* [55] proved that the boson number and momentum operator commute, so that common eigenstates of  $\hat{H}$ ,  $\hat{P}$  and  $\hat{N}$  exist in such a case. In the case of a negative ratio  $b$ , the interaction between the bosons is attractive and Hamiltonian Eq. (97) has bound states. A subset of these bound states is characterized solely by the eigenvalues of  $\hat{N}$  and  $\hat{P}$ :

$$f_{n,p} = N_n \exp \left( ip \sum_{j=1}^{\infty} x_j + \frac{b}{2} \sum_{1 \leq i, j < n} |x_i - x_j| \right) \quad (106)$$

where

$$N_n = \sqrt{\frac{(n-1)! |b|^{n-1}}{2\pi}}$$

Thus

$$f_n(x_1, \dots, x_n, t) = \int dp g_n(p) f_{np}(x_1, \dots, x_n, t) e^{-iE(n,p)t} \quad (107)$$

where

$$g_n = \sqrt{g(p)} e^{-inp x_0}$$

and

$$g(p) = \frac{e\phi\{- (p-p_0)^2 / [2(\Delta p)^2]\}}{\sqrt{2\pi(\Delta p)^2}}$$

Using  $f_{n,p}$  given in Eq. (106), we find that  $|n, P\rangle$  decays exponentially with separation between an arbitrary pair of bosons. It describes an  $n$ -particle soliton moving with momentum  $P = \hbar np$  and energy  $E(n, p) = np^2 - |b|^2(n^2 - 1)n/12$ . By construction, the quantum number  $p$  in this wave function is related to the momentum of the center of mass of the  $n$  interacting bosons, which is now defined as

$$\hat{X} = \lim_{\epsilon \rightarrow 0} \int x \hat{\phi}^+(x)\hat{\phi}(x) dx (\epsilon + \hat{N})^{-1} \quad (108)$$

with

$$[\hat{X}, \hat{P}] = i\hbar$$

The limit of  $\epsilon \rightarrow 0$  is introduced to regularize the position operator for the vacuum state.

We are interested in the fluctuations of Eqs. (104), (105) and (106) for a state  $|\epsilon(t)\rangle$  with a large average Boson number and a well-defined mean field. Kartner and Boiven [56] decomposed the field operator in its mean value and a remainder which is responsible for the quantum fluctuations,

$$\begin{aligned} \hat{\phi}(x) &= \langle \psi'(0) | \hat{\phi}^+(x) | \psi(0) \rangle + \hat{\phi}_1(x), [\hat{\phi}_1(x), \hat{\phi}_1^+(x')] \\ &= \delta(x - x'), \quad [\hat{\phi}_1(x), \hat{\phi}_1^+(x')] = 0 \end{aligned} \quad (109)$$

Since the field operator  $\hat{\phi}$  is time independent in the Schrödinger representation, we can then choose  $t = 0$  for definiteness. Inserting Eq. (109) into Eqs. (104), (105) and (108) and neglecting terms of second and higher order in the noise operator, Kartner *et al.* obtained that

$$\begin{aligned} \hat{N} &= n_0 + \Delta\hat{n}, \quad n_0 = \int dx (\langle \hat{\phi}^+(x) \rangle \langle \hat{\phi}(x) \rangle) \\ \Delta\hat{n} &= \int dx (\langle \hat{\phi}^+(x) \rangle \hat{\phi}_1(x)) + \text{c.c.} \\ \hat{P} &= \hbar n_0 p_0 + \hbar n_0 \Delta\hat{p}, \quad p_0 = \frac{i}{n_0} \int dx \langle \hat{\phi}_x^+(x) \rangle \langle \hat{\phi}(x) \rangle \\ \Delta\hat{p} &= \frac{i}{n_0} \int dx \langle \hat{\phi}_x^+(x) \rangle \hat{\phi}_1(x) + \text{c.c.} \\ \hat{X} &= x_0 \left( 1 - \frac{\Delta\hat{n}}{n_0} \right) + \Delta\hat{x} \\ x_0 &= \frac{1}{n_0} \int dx x \langle \hat{\phi}^+(x) \rangle \langle \hat{\phi}(x) \rangle \\ \Delta\hat{x} &= \frac{1}{n_0} \int dx x \langle \hat{\phi}^+(x) \rangle \hat{\phi}_1(x) + \text{c.c.} \end{aligned}$$

where  $\Delta\hat{x}$  is the deviation from the mean value of the position operator,  $\Delta\hat{n}$ ,  $\Delta\hat{p}$ , and  $\Delta\hat{x}$  are linear in the noise operator. Because the third- and fourth-order correlators of  $\hat{\phi}_1$  and  $\hat{\phi}_1^+$  are very small, they can be neglected in the limit of large  $n_0$ . Note that  $\Delta\hat{n}$ ,  $\Delta\hat{p}$ , and  $\Delta\hat{x}$  are all quadratures of the noise operator with  $\Delta\hat{p}$  and  $\Delta\hat{x}$  being conjugate variables. To complete this set, we introduce a quadrature variable conjugate to  $\Delta\hat{n}$ ,

$$\begin{aligned} \Delta\hat{\theta} &= \frac{1}{n_0} \int dx \{ i[\hat{\phi}^+(x) + x \langle \hat{\phi}_x^+(x) \rangle] \\ &\quad - p_0 x \langle \hat{\phi}_x^+(x) \rangle \} \hat{\phi}_1(x) + \text{c.c.} \end{aligned}$$

As is known, if the propagation distance is not too large, the mean value of the field is given to the first order by the classical soliton solution

$$\langle \hat{\phi}(x) \rangle = \phi_{0,n_0}(x, t) \left[ 1 + O\left(\frac{1}{n_0}\right) \right]$$

with

$$\begin{aligned} \phi_{0,n_0}(x, t) &= \frac{n_0 \sqrt{|b|}}{2} \\ &\quad \cdot \exp [i\Omega_{nl} - ip_0^2 t + ip_0(x - x_0) + i\theta_0] \\ &\quad \cdot \text{sech} \left[ \frac{n_0 |b|}{2} (x - x_0 - 2p_0 t) \right] \end{aligned} \quad (110)$$

and the nonlinear phase shift  $\Omega_{nl} = n_0^2 |b|^2 t/4$ . If  $p_0 = x_0 = \theta_0 = 0$ , we obtain the following for the fluctuation operators in the Heisenberg picture,

$$\begin{aligned} \Delta\hat{n}(t) &= \int dx [f_{-n}(x)^* F'_{nl} + \text{c.c.}] \\ \Delta\hat{\theta}(t) &= \int dx [f_{-\theta}(x)^* F'_{nl} + \text{c.c.}] \\ \Delta\hat{p}(t) &= \int dx [f_{-p}(x)^* F'_{nl} + \text{c.c.}] \\ \Delta\hat{x}(t) &= \int dx [f_{-x}(x)^* F'_{nl} + \text{c.c.}] \end{aligned}$$

with

$$F'_{nl} = e^{i\Omega_{nl}} \hat{\phi}_1(x, t)$$

and the set of adjoint functions

$$\begin{aligned} f_{-n}(x) &= \frac{n_0 \sqrt{|b|}}{2} \text{sech } x_{n_0} \\ f_{-\theta}(x) &= \frac{i\sqrt{|b|}}{2} \left[ \text{sech } x_{n_0} + x_{n_0} \frac{d}{dx_{n_0}} \text{sech } x_{n_0} \right] \\ f_{-p}(x) &= -\frac{in_0 \sqrt{|b|^3}}{4} \frac{d}{dx_{n_0}} \text{sech } x_{n_0} \\ f_{-x}(x) &= \frac{1}{n_0 \sqrt{|b|}} x_{n_0} \text{sech } x_{n_0} \end{aligned}$$

where

$$x_{n_0} = \frac{1}{2} n_0 |b| x$$

For a coherent state defined by

$$\hat{\phi}(x) | \Phi_{0,n_0} \rangle = \phi_{0,n_0}(x) | \Phi_{0,n_0} \rangle, \quad \hat{\phi}_1(x) | \Phi_{0,n_0} \rangle = 0$$

where

$$\begin{aligned} | \Phi_{0,n_0} \rangle &= \\ \exp \left\{ \int dx [\phi_{0,n_0}(x) \hat{\phi}^+(x) - \phi_{0,n_0}^*(x) \hat{\phi}(x)] \right\} | 0 \rangle \end{aligned}$$

$\phi_{0,n_0}$  has been given Eq. (110). Kartner *et al.* further obtained that

$$\langle \Delta\hat{n}_0^2 \rangle = n_0, \quad \langle \Delta\hat{\theta}_0^2 \rangle = \frac{0.6075}{n_0}$$

$$\langle \Delta \hat{p}_0^2 \rangle = \frac{1}{3n_0\tau_0^2}, \quad \langle \Delta \hat{x}_0^2 \rangle = \frac{1.645\tau_0^2}{2n_0}$$

where  $\tau_0^2 = 2/n_0 |b|$  is the width of the microscopic particle (soliton). The uncertainty products of Boson number and phase, momentum and position are, respectively,

$$\langle \Delta \hat{n}_0^2 \rangle \langle \Delta \hat{\theta}_0^2 \rangle = 0.6075 \geq 0.25$$

$$n_0^2 \langle \Delta \hat{p}_0^2 \rangle \langle \Delta \hat{x}_0^2 \rangle = 0.27 \geq 0.25$$

Here the quantum fluctuation of the coherent state is white, i.e.,

$$\langle \hat{\phi}_1(x) \hat{\phi}_1(y) \rangle = \langle \hat{\phi}_1^+(x) \hat{\phi}_1(y) \rangle = 0$$

However, the quantum fluctuation of the soliton cannot be written because the particle interaction introduces correlations between them. Thus, Kätner and Boivin [58] assumed a fundamental soliton state with a Poissonian distribution for the boson number  $p_n = \frac{n_0^n}{n!} e^{-n_0}$  and a Gaussian distribution for the momentum Eq. (73) with a width  $\langle \Delta p_0^2 \rangle = n_0 |b|^2 / 4\mu$ , where  $\mu$  is a parameter of the order of unity compared to  $n_0$ . They finally obtained the minimum uncertainty values:

$$\langle \Delta \hat{\theta}_0^2 \rangle = \frac{0.25}{\langle \Delta \hat{n}_0^2 \rangle} = \frac{0.25}{n_0} \left[ 1 + O\left(\frac{1}{n_0}\right) \right]$$

and

$$\langle \Delta \hat{x}_0^2 \rangle = \frac{0.25}{\langle n_0^2 \rangle} = \frac{0.25\mu\tau_0^2}{n_0} \left[ 1 + O\left(\frac{1}{n_0}\right) \right]$$

up to order  $1/n_0$  for the corresponding initial fluctuations in MIP (soliton) phase and timing. Thus, at  $t = 0$  the fundamental soliton with the given Boson number and momentum distributions is a minimum uncertainty state in the four collective variables, the Boson number, phase momentum and position, up to the terms of  $O(1/n_0)$ , which are of the form [26, 27, 57]

$$\langle \Delta \hat{n}_0^2 \rangle \langle \Delta \hat{\theta}_0^2 \rangle = 0.25 \left[ 1 + O\left(\frac{1}{n_0}\right) \right]$$

and

$$n_0^2 \langle \Delta p_0^2 \rangle \langle \Delta \hat{x}_0^2 \rangle = 0.25 \left[ 1 + O\left(\frac{1}{n_0}\right) \right] \quad (111)$$

These are the uncertainty relations arising from the quantum fluctuations in nonlinear quantum mechanics of MIP described by NLSE. They are the same as Eqs. (92)–(94). Therefore, we can conclude that the uncertainty relation in NLQM takes the minimum values regardless whether a state is coherent or squeezed, a system is classical or quantum.

In light of the above discussion, we can distinguish the motions of particles in the linear quantum mechanics,

nonlinear quantum mechanics, and classical mechanics by means of the uncertainty relation. When the motion of the particles satisfies  $\Delta x \Delta p > h/2$  or  $\pi/6$ , the particles obey laws of motion in linear quantum mechanics, and the particles are some waves. When the motion of the particles satisfies  $\Delta x \Delta p = h/12$  or  $\pi/6$ , the particles obey laws of motion in nonlinear quantum mechanics, and the particles are solitons, exhibiting wave-corpucle duality. If the motion of the particles satisfies  $\Delta x \Delta p = 0$ , then the particles can be treated as classical particles, with only corpucle feature. The nonlinear quantum mechanics introduced here makes physics more complete. Therefore, we can say that the nonlinear quantum mechanics is a new theory which is a new physical branch, it bridges the gap between the classical and linear quantum mechanics.

## 4 Eigenvalue problem of microscopic particles in NLQM

4.1 The eigenenergy spectrum of the Hamiltonian of the systems

In linear quantum mechanics (LQM), because the Hamiltonian of the systems is independent of the state wavefunction of the particle, the eigenenergy spectrum of the Hamiltonian operator of the systems can be easily obtained from its eigenequation,  $H |\psi(x, t)\rangle = E |\psi(x, t)\rangle$ , where  $|\psi(x, t)\rangle$  is its eigenwave-function in coordinate or particle number representation. It also is just a time-independent linear Schrödinger equation in the coordinate representation. However, in the NLQM we find that this method fails in the coordinate representation because the wavefunction of state of MIP is contained in the Hamiltonian operator of the systems; that is, the Hamiltonian operator depends on the state wave vector of the MIP, thus exact eigenvalues cannot be obtained. Therefore, we must use other methods to find the eigenenergies of the Hamiltonian operator of the MIPs. We have two ways to get the eigenenergies of the Hamiltonian operator. The first is that the energy of the MIP satisfying the NLSE can be obtained from  $E = \int_{-\infty}^{\infty} \langle \phi(x, t) | \hat{H}' | \phi(x, t) \rangle dx$  as used in Eq. (18), where  $H' = \mathcal{H}$  is the Hamiltonian density of the systems which depends on the wave function  $\phi(x, t)$ . The second is that the Hamiltonian operator and state wavefunction of particles are all given in particle number representation, then we can find the eigenenergy spectrum of the Hamiltonian operator from its eigenequation,  $\hat{H} |\phi(x, t)\rangle = E |\phi(x, t)\rangle$ , whether one mode motion or many mode motion. We often use the latter to find the

eigenenergy of the Hamiltonian operator of the system.

We know that the wave function of a microscopic particle can be quantized by the creation and annihilation operators of the particle in the second quantum representation in NLQM. Then the Hamiltonian of a system described by the wave function  $\phi(x, t)$  can be quantized by introducing creation and annihilation operators in the particle number representation or second quantization representation. Thus, we can calculate the eigenenergy spectrum by using the eigenequation of the quantum Hamiltonian and corresponding wavevector in number representation. This is basically how the eigenenergy spectra in the NLQM can be obtained. For convenience, we express the nonlinear Schrödinger Eq. (5) with  $A(\phi) = 0$  in the following discrete form:

$$i\hbar \frac{\partial \phi_j}{\partial t} = -\frac{\hbar}{2mr_0^2}(\phi_{j+1} - 2\phi_j + \phi_{j-1}) - b|\phi_j|^2\phi_j + V(j, t)\phi, \quad j = 1, 2, 3, \dots, J \quad (112)$$

in lattice field, where  $r_0$  is a spacing between two neighboring lattice points,  $j$  labels the discrete lattice points,  $J$  is total number of lattice points in the lattice field of the system. The vector form of the above equation in the lattice field is

$$\left[ i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{mr_0^2} - V(j, t) \right] \bar{\phi} = -\varepsilon M \bar{\phi} - b \text{diag.} (|\phi_1|^2, |\phi_2|^2, \dots, |\phi_\alpha|^2) \bar{\phi} \quad (113)$$

where  $\bar{\phi}(x, t)$  is the column vector,  $\bar{\phi}(x, t) = \text{Col.} (\phi_1, \phi_2, \dots, \phi_\alpha)$ , whose complex components, Eq. (113) is a vector NLSE with  $\alpha$  modes of motion. In Eq. (113),  $b$  is a nonlinear parameter and  $\alpha$  is a number of motion modes that exist in the systems.  $M = [M_{nl}]$  is an  $\alpha \times \alpha$  real symmetric dispersion matrix,  $\varepsilon = \hbar^2/2mr_0^2$ . Here,  $n$  and  $l$  are integers denoting the modes of motion. The Hamiltonian and the particle number corresponding to Eq. (113), respectively, are

$$H = \sum_{N=1}^{\alpha} \left( \hbar\omega_0 |\phi_n|^2 - \frac{1}{2}b|\phi_n|^4 \right) - \varepsilon \sum_{n \neq l} M_{nl} \phi_n \phi_l$$

and

$$N = \sum_{N=L}^{\alpha} |\phi_n|^2 \quad (114)$$

where  $\hbar\omega_0 = \hbar^2/2mr_0^2 + V(j, t)$ .

We have assumed that  $V(j, t)$  are independent of  $j$  and  $t$ . In the canonical second quantization theory, the complex amplitude ( $\phi_n^*$  and  $\phi_n$ ) become boson creation and annihilation operators ( $\hat{B}_n^+$  and  $\hat{B}_n$ ) in the number representation. If  $|m_n\rangle$  is an eigenfunction of a particular mode, then

$$\hat{B}_n^+ |m_n\rangle = \sqrt{m_n + 1} |m_n + 1\rangle$$

$$\hat{B}_n |m_n\rangle = \sqrt{m_n} |m_n - 1\rangle$$

and

$$\hat{B}_n |0\rangle = 0$$

Since no particular ordering is specified in Eq. (114) we use the averages:

$$|\phi|^2 \rightarrow \frac{1}{2}(\hat{B}_n^+ \hat{B}_n^+ \hat{B}_n \hat{B}_n^+)$$

and

$$|\phi_n|^4 \rightarrow \frac{1}{6}(\hat{B}_n^+ \hat{B}_n^+ \hat{B}_n \hat{B}_n + \hat{B}_n^+ \hat{B}_n \hat{B}_n^+ \hat{B}_n^+ \hat{B}_n \hat{B}_n \hat{B}_n^+ + \hat{B}_n \hat{B}_n^+ \hat{B}_n \hat{B}_n^+ + \hat{B}_n \hat{B}_n \hat{B}_n^+ \hat{B}_n^+ + \hat{B}_n \hat{B}_n^+ \hat{B}_n^+ \hat{B}_n)$$

with the Boson commutation rule  $\hat{B}_n \hat{B}_n^+ - \hat{B}_n^+ \hat{B}_n = 1$ , Eq. (114) then becomes

$$\hat{H} = \sum_{n=1}^{\alpha} \left[ \left( \hbar\omega_0 - \frac{1}{2}b \right) \left( \hat{B}_n^+ \hat{B}_n + \frac{1}{2} \right) - \frac{1}{2}b \hat{B}_n^+ \hat{B}_n \hat{B}_n^+ \hat{B}_n \right] - \varepsilon \sum_{n \neq l} M_{nl} \hat{B}_n^+ \hat{B}_l \quad (115)$$

$$\hat{N} = \sum_{n=1}^{\alpha} \left( \hat{B}_n^+ \hat{B}_n + \frac{1}{2} \right) \quad (116)$$

From now on, we will use the notation  $[m_1, m_2, \dots, m_\alpha]$  to denote the products of number states  $|m_1\rangle|m_2\rangle \dots |m_\alpha\rangle$ . Thus, stationary states of the vector NLSE (114) must be eigenfunctions of both  $\hat{N}$  and  $\hat{H}$ . Consider an  $m$ -quantum state (i.e., the  $n$ th excited level,  $m = m_1 + m_2 + \dots + m_j$ ), with  $m < \alpha$ . An eigenfunction of  $\hat{N}$  can be established as

$$|\phi_m\rangle = C_1[m, 0, 0, \dots, 0] + \dots + C_2[0, m, 0, 0, \dots, 0] + \dots + C_i[0, 0, 0, \dots, m, \dots] + \dots + C_{i+1}[m-1, m, 0, 0, \dots, 0] + \dots + C_p[0, 0, 0, \dots, 0, 1, 1, \dots, 1] \quad (117)$$

The number of terms in Eq. (117) is equal to the number of ways that  $m$  quanta can be placed on  $\alpha$  sites, which is given by  $P = \frac{(m + \alpha - 1)!}{m!(\alpha - 1)!}$ . The wave function  $|\phi_m\rangle$

in Eq. (117) is an eigenfunction of  $\hat{N}$  for any values of the  $C'_\alpha$ . Thus, we are free to choose these coefficients so that

$$\hat{H}|\phi_m\rangle = E|\phi_m\rangle \quad (118)$$

Eq. (118) requires that the column vector  $C = \text{Col.} (C_1, C_2, \dots, C_p)$  satisfies the matrix equation:

$$[H - IE]C = 0 \quad (119)$$

where  $H$  is a  $p \times p$  symmetric matrix with real elements.  $I$  is a  $p \times p$  identity matrix,  $E$  is just the eigenenergy. Eq. (118) is an eigenvalue equation of quantum Hamiltonian operator (115) of the systems. We can find the eigenenergy spectra  $E_m$  of the systems from Eq. (119) for given parameters,  $\varepsilon, \omega_0$ , and  $b$ . Scott *et al.* [59 – 61] and Pang *et al.* [62 – 69] used this method to calculate the energy-spectra of vibrational excitations (quanta) in many nonlinear systems, for example, small molecules or organic molecular crystals and biomolecules. These results can be compared with the experimental data.

#### 4.2 Eigenvalue problem of the nonlinear Schrödinger equation

In LQM we know that the time-independent linear Schrödinger equation is an eigenequation of the Hamiltonian operator in the coordinate representation. However, we do not know the meaning of the eigenvalue problem of the nonlinear Schrödinger equation, which is therefore a new problem. This problem comes from the Lax method. According to this method, for any nonlinear equation,  $\frac{\partial}{\partial t}\phi(\mathbf{r}, t) = K(\phi(\mathbf{r}, t))$ , where  $K(\phi(\mathbf{r}, t))$  is a nonlinear operator. If  $K(\phi(\mathbf{r}, t))$  is related to two linear operators  $\hat{L}$  and  $\hat{B}$ , which depend on  $\phi$  and satisfy the Lax operator equation:

$$i\hat{L}_t = \hat{B}\hat{L} - \hat{L}\hat{B} = [\hat{B}, \hat{L}] \tag{120}$$

where  $t' = t/\hbar$ , then the eigenvalue  $E$ , which does not vary with time, and eigenfunction  $\psi$  of the nonlinear equation is determined by the eigenequation of operator  $\hat{L}$  as follows:

$$\hat{L}\psi = \lambda\psi; \quad i\psi'_t = \hat{B}\psi \tag{121}$$

Thus, the eigenvector and eigenvalue of nonlinear systems are determined by the eigenvector and eigenvalue of the above linear operators. In general, concerning any types of nonlinear equation, its corresponding linear eigenequation and time-independent eigenvalue can always be found. For the NLSE in Eq. (5) with  $V(\mathbf{r}, t) = A(\phi) = 0$ , the two linear operators  $\hat{L}$  and  $\hat{B}$  are

$$\begin{aligned} \hat{L} &= \begin{pmatrix} 1+s & 0 \\ 0 & 1-s \end{pmatrix} \frac{\partial}{\partial x'} + \begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix} \\ \hat{B} &= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x'^2} + \begin{pmatrix} |\phi|^2/(1+s) & i\phi_{x'} \\ -i\phi_{x'} & -|\phi|^2/(1-s) \end{pmatrix} \end{aligned} \tag{122}$$

where  $s^2 = 1 - 2/b$ ,  $x' = x\sqrt{2m/\hbar^2}$ . Thus the eigenvalue of NLSE is determined by

$$\hat{L}\psi = \lambda\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{123}$$

Its corresponding solution can be found by use of inverse-scattering or another method.

The eigenequation corresponding to the NLSE (5) with  $V(x, t) = A(\phi) = 0$  and the Galilei invariance is found by the linear Zakharov-Shabat equation [31]:

$$i\psi_{x'} + \Phi\psi = \lambda\sigma_3\psi \tag{124}$$

This is an eigenequation of eigenfunction  $\psi$  with an eigenvalue  $\lambda$  and potential  $\Phi$ , where,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix} \tag{125}$$

where  $\phi$  satisfies Eq. (5) with  $V(x, t) = A(\phi) = 0$ . It evolves with time according to Eq. (121). However, what are the properties of the eigenvalue problems determined by these relations? This deserves further consideration.

As is known, the eigenequation is invariant under the Galilei transformation. As a matter of fact, if we substitute the following Galilei transformation:

$$\phi'(\tilde{x}, \tilde{t}) = e^{ivx' - iv^2/2} \phi(x', t'), \quad \tilde{x} = x' - vt', \quad \tilde{t} = t' \tag{126}$$

into Eq. (125), then  $\Phi$  is transformed into

$$\Phi'(\tilde{x}) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \Phi(x') \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{127}$$

where  $\theta = vx' - \frac{1}{2}v^2t' + \theta_0$ , here  $\theta_0$  is an arbitrary constant. If the eigenfunction  $\psi(x')$  is transformed as

$$\psi'(\tilde{x}) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \psi(x') \tag{128}$$

then Eq. (124) becomes

$$i\psi'_{x'} + \Phi'\psi' = \left(\lambda - \frac{v}{2}\right) \sigma_3\psi' \tag{129}$$

It is clear that in the reference frame that is moving with the velocity  $v$ , the eigenvalue is reduced to  $v/2$  compared with that in the rest frame. It shows that the velocity of the MIP (soliton) is given by  $2 \Re(\lambda_k)$ . When  $\theta$  is constant, i.e.,  $\theta = \theta_0$ , the eigenvalue is unchanged because  $v = 0$ . This implies that the NLSE is invariant under the gauge transformation,  $\phi' = e^{i\theta_0} \phi(x')$ .

Satsuma and Yajima [70] studied the eigenfunction of Eq. (124) and its properties, where the eigenfunction satisfied the boundary condition,  $\psi = 0$  at  $|x| \rightarrow \infty$ . The eigenvalues and the corresponding eigenfunctions were denoted by  $\lambda_1, \lambda_2, \dots, \lambda_N$  and  $\psi_1, \psi_2, \dots, \psi_N$ . For a given eigenfunction,  $\psi_n(x')$ , Eq. (124) reads

$$i \frac{d\psi_n(x')}{dx'} + \Phi(x')\psi_n(x') = \lambda_n \sigma_3 \psi_n(x'), \quad n = 1, 2, \dots, N \quad (130)$$

$\psi(x')$  was expressed in terms of Pauli's spin matrices  $\sigma_1$  and  $\sigma_2$ ,

$$\psi(x') = \Re[\psi(x')] \sigma_1 - \Im[\psi(x')] \sigma_2 \quad (131)$$

Multiplying Eq. (130) by  $\sigma_2$  from the left and taking the transpose of the resulting equation, we get

$$-i \frac{d\psi_m^T}{dx'} \sigma_2 - \psi_m^T \Phi^* \sigma_2 = i \lambda_m \psi_m^T \sigma_1 \quad (132)$$

where the superscript "T" denotes transpose. Multiplying the above equation by  $\psi_n$  from right and Eq. (129) by  $\psi_m^T \sigma_2$  from the left and subtracting one from the other, Satsuma and Yajima [70] obtained the following equation:

$$(\lambda_n - \lambda_m) \int_{-\infty}^{\infty} \psi_m^T \sigma_1 \psi_n dx' = 0$$

The boundary conditions,  $\psi_n, \psi_m \rightarrow 0$  as  $|x'| \rightarrow \infty$ , were used in obtaining the above equation. The following orthonormal condition was then derived:

$$\int_{-\infty}^{\infty} \psi_m^T \sigma_1 \psi_n dx' = \delta_{nm} \quad (133)$$

Satsuma and Yajima further demonstrated that Eq. (130) has the following symmetry properties.

(I) If  $\phi(x')$  satisfies  $\phi(-x') = \phi^*(x')$ , then replacing  $x'$  by  $-x'$  in Eq. (130) and multiplying it by  $\sigma_2$  from left, we can get

$$i \frac{d}{dx'} [\sigma_2 \psi_n(-x')] + \Phi(x') [\sigma_2 \psi_n(-x')] = \lambda_n \sigma_3 [\sigma_2 \psi_n(-x')]$$

Since  $\sigma_2 \psi_n(-x')$  is also an eigenfunction associated with  $\lambda_n$ , its behavior resembles that of  $\psi_n(x')$  in the asymptotic region, i.e.,  $\sigma_2 \psi_n(-x') \rightarrow 0$  as  $|x'| \rightarrow \infty$ , thus  $\psi_n$  has the following symmetry

$$\sigma_2 \psi_n(-x') = \delta \psi_n(x')$$

or

$$\psi_n(-x') = \delta \sigma_2 \psi_n(x'), \quad \delta = \pm 1$$

Therefore, if  $\phi(-x') = \phi^*(x')$ , then  $\psi(x')$  satisfies the symmetry property  $\psi_n(-x') = \delta \sigma_1 \psi_n(x')$  with  $\delta = \pm 1$ . This can easily be verified by replacing  $\sigma_1$  with  $\sigma_2$  in the above derivations.

(II) If  $\phi(x')$  is a symmetric (or antisymmetric) function of  $x'$ , i.e.,  $\phi(-x') = \pm \phi(x')$ , then  $\psi_n^{(s)}(x') = \sigma_1 \psi_n^*(-x')$  is

the eigenfunction belonging to the eigenvalue  $-\lambda_n^*$ , and  $\psi_n^{(a)}(x') = \sigma_2 \psi_n^*(-x')$  is the eigenfunction belonging to the eigenvalue  $\lambda_n^*$ . The suffix  $s$  (or  $a$ ) to the eigenfunction  $\psi_n$ , indicates that  $\phi$  is symmetric (or antisymmetric). Since  $\phi(-x') = \phi(x')$ , replacing  $x'$  with  $-x'$  in Eq. (130) and taking complex conjugate, we get

$$i \frac{d}{dx'} [\sigma_1 \psi_n^*(-x')] + \psi_n(x') [\sigma_1 \psi_n^*(-x')] = -\lambda_n \sigma_3 [\sigma_1 \psi_n^*(-x')]$$

Compared with Eq. (130), the above equation implies that  $-\lambda_n^*$  is also an eigenvalue and the associated eigenfunction  $\psi_n^{(s)}(x')$  is just  $\sigma_1 \psi_n^*(-x')$ , with the arbitrary constant. For  $\phi(-x') = -\phi(x')$ , the same conclusion is obtained by replacing  $\sigma_1$  with  $\sigma_2$  in the above derivations.

These symmetry properties are useful in providing a general view of the solution of Eq. (5) with  $V(x, t) = A(\phi) = 0$ . As is known, the real part of the eigenvalue,  $\xi_n$ , corresponds to the velocity of a soliton and the imaginary part,  $\eta_n$ , the amplitude. Then, if  $\phi(x', t' = 0)$ , whose initial value has the symmetry  $\phi(x', t' = 0) = \pm \phi(-x', t' = 0)$ , breaks into the series of solutions, the decay is bisymmetric, corresponding to the eigenvalues  $-\lambda_n^*$ . If  $\phi(x')$  is real, the above symmetry property yields

$$\begin{aligned} \psi_n^{(s)}(-x') &= \sigma_1 [-\delta \sigma_2 \psi_n^*(-x')] = \delta \sigma_2 \psi_n^{(s)}(x') \\ \psi_n^{(a)}(-x') &= \sigma_2 [-\delta \sigma_1 \psi_n^*(-x')] = \delta \sigma_1 \psi_n^{(a)}(x') \end{aligned}$$

i.e.,  $\psi_n^{(s)}(x')$  has the same parity as  $\psi_n(x')$ , while  $\psi_n^{(a)}(x')$  has the opposite one. When  $\phi(-x') = -\phi(x')$  and  $\lambda_n$  is pure imaginary ( $\lambda_n = -\lambda_n^*$ ), the eigenvalues corresponding to the positive and negative parity eigenfunctions degenerate.

(III) If  $\phi(x')$  is real, but not antisymmetric, then the eigenvalue  $\lambda_n$  is pure imaginary, i.e.,  $\Re(\lambda_n) = 0$ . From Eq. (130) and its Hermitian conjugate, Satsuma *et al.* [70] found that

$$\Re(\lambda_n) \langle n | \sigma_2 | n \rangle = \langle n | \Im[\phi(x')] \sigma_3 | n \rangle \quad (134)$$

with

$$\langle m | \sigma_2 | n \rangle = \int_{-\infty}^{\infty} \psi_m^+ \sigma_2 \psi_n dx' \quad (135)$$

where  $[\Phi, \sigma_1] = 2i \Im(\phi) \sigma_3$  was used. We see from Eq. (134) that  $\Re(\lambda_n)$  vanishes if  $\phi$  is real and  $\langle m | \sigma_2 | n \rangle \neq 0$ . When  $\phi$  is a real and an antisymmetric function of  $x'$ , the symmetry property (I) gives

$$\begin{aligned} \langle m | \sigma_2 | n \rangle &= \delta^2 \int_{-\infty}^{\infty} \psi_m^+(-x') \sigma_1 \sigma_2 \sigma_1 \psi_n(-x') dx' \\ &= -\langle | \sigma_2 | \rangle \end{aligned}$$

Thus  $\langle n|\sigma_2|n\rangle = 0$ .

(IV) If the initial value takes the form of  $\phi = e^{ivx'}R(x')$ , where  $R(x')$  is a real, but not antisymmetric function of  $x'$ , then all the eigenvalues have the common real part,  $-v/2$ . This can be easily shown by the Galilei transformation. In fact, when  $\phi(x', t' = 0) = e^{ivx'}R(x')$ , the solution does not decay to the series of solitons moving with the different velocities, but form a bound state. In this case, the real parts are common to all the eigenvalues, i.e., the relative velocities of the solitons vanish.

(V) If  $\phi$  is a real non-antisymmetric function of  $x'$ , it can be shown that

$$\psi_n^*(x') = i\delta\sigma_3\psi_n(x') \quad (136)$$

where  $\delta = \pm 1$ . Because  $\Re(\lambda_n) = 0$ , from the complex conjugate of Eq. (130), one can get  $\psi_n^*(x') \propto \sigma_3\psi$ . Substituting Eq. (124) into the normalization condition Eq. (132), one then has  $\delta = \pm 1$ . If the eigenvalue of Eq. (124) is real, i.e.,  $\lambda = \xi$  is real, then

$$i\frac{d\psi}{dx'} + \Phi\psi = \xi\sigma_3\psi \quad (137)$$

and the adjoint function of  $\psi$ ,  $\bar{\psi} = i\sigma_2\psi^*$ , is also a solution of Eq. (137), i.e.,

$$i\frac{d\bar{\psi}}{dx'} + \Phi\bar{\psi} = \xi\sigma_3\bar{\psi}$$

From this and Eq. (137), Satsuma and Yajima obtained the following relations:

$$\begin{aligned} \frac{d}{dx'}(\psi^+\psi) &= \frac{d}{dx'}(\bar{\psi}^+\psi) = \frac{d}{dx'}(\psi^+\bar{\psi}) \\ &= \frac{d}{dx'}(\bar{\psi}^+\bar{\psi}) = 0 \end{aligned} \quad (138)$$

Using the above boundary conditions, they found that the solutions of Eq. (124)  $\psi_1(x', \xi)$ ,  $\psi_2(x', \xi)$ , and  $\bar{\psi}_2(x', \xi)$  satisfy the following relations:

$$\psi_1^+\psi_1 = \psi_2^+\psi_2 = \bar{\psi}_2^+\bar{\psi}_2 = 1, \quad \bar{\psi}_2^+\psi_2 = \psi_2^+\bar{\psi}_2 = 0$$

From  $\psi_1 = a(\xi)\bar{\psi}_2 + b(\xi)\psi_2$ , we get  $a = \psi$  and  $b = \bar{\psi}_2^+\psi_1$ ,

where  $\psi_1 = (x', \xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x'}$ , as  $x' = -\infty$  and

$\psi_2 = (x', \xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+i\xi x'}$ ,  $\bar{\psi}_2(x', \xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x'}$  as

$x' = \infty$ . As pointed out earlier, if a real (not antisymmetric) initial value is considered, the microscopic particle does not decay into moving solitons, but forms a bound states of solitons pulsating with the proper frequency. Satauma and Yajima developed a perturbation approach to investigate the conditions for the solutions to evolve and decay into moving solitons.

If the wave function  $\phi$  in Eq. (124) undergoes a small change, i.e.,  $\phi \rightarrow \phi' = \phi + \Delta\phi$ , the corresponding change in  $\Phi$  is given by

$$\Delta\Phi = \begin{pmatrix} 0 & \Delta\phi \\ \Delta\phi^* & 0 \end{pmatrix}$$

Then,  $\lambda_n$  and  $\psi_n$  changes as  $\lambda_n + \Delta\lambda_n$  and  $\psi_n + \Delta\psi_n$ , respectively. To the first order in the variation, Eq. (124) becomes

$$\left[ i\frac{d}{dx'} + (\Phi - \lambda_n\sigma_3) \right] \Delta\psi_n + (\Delta\Phi - \Delta\lambda_n\sigma_3)\psi_n = 0$$

Multiplying the above equation by  $\psi_n^T\sigma_2$  from the left and integrating with respect to  $x'$  over  $(-\infty, \infty)$ , we get

$$\begin{aligned} \Delta\lambda_n &= -i \int_{-\infty}^{\infty} \psi_n^T\sigma_2\Delta\Phi\psi_n dx' \\ &= - \int_{-\infty}^{\infty} \psi_n^T\Re(\Delta\phi)\sigma_3\psi_n dx' \\ &\quad + i \int_{-\infty}^{\infty} \psi_n^T\Im(\Delta\phi)\psi_n dx' \end{aligned}$$

If  $\phi$  is a real and non-antisymmetric function of  $x'$ , Eq. (137) holds and

$$\Delta\lambda_n = \delta\langle n|\Im(\Delta\phi)\sigma_3|n\rangle + i\delta\langle n|\Re(\Delta\phi)|n\rangle \quad (139)$$

Eq. (139) indicates that if  $\langle n|\Im(\Delta\phi)\sigma_3|n\rangle \neq 0$ , the perturbation  $\Delta\phi$  makes the real part of the eigenvalue finite. That is, for the initial value,  $\phi(x') + \Delta\phi(x')$ , the solution of the NLSE in Eq. (5) with  $V(x, t) = A(\phi) = 0$  breaks up into moving solitons with velocity  $2\Re(\Delta\lambda_n)$ . If  $\phi$  is real and is either a symmetric or an antisymmetric function of  $x'$ , the above symmetry properties of eigenvalues of the NLSE lead to

$$\langle n|\Im(\Delta\phi(x'))\sigma_3|n\rangle = -\langle n|\Im(\Delta\phi(-x))\sigma_3|n\rangle$$

Therefore, if  $\Im(\Delta\phi)$  is a symmetric function,  $\langle n|\Im(\Delta\phi)\sigma_3|n\rangle$  vanishes, i.e.,  $\Re(\Delta\lambda_n) = 0$ , and the soliton bound state does not resolve into moving solitons even in the presence of the perturbation  $\Delta\phi$ .

Satsuma and Yajima [70] also obtained the shifts of the eigenvalues of Eq. (124) under the double-humped initial values,  $\phi(x', t' = 0) = \phi_0(x' - x'_0) + e^{i\theta_0}\phi(x' + x'_0)$ , where  $\phi_0$  is a real and symmetric function of  $x'$ ,  $x'_0$  and  $\theta_0$  are real. The shifts of the eigenvalues were finally written as

$$\begin{aligned} \Delta\lambda_n^\pm &= \delta \left[ \sin\theta \langle n|\sigma_3\phi_0(x' + 2x'_0)|n\rangle \mp \sin\frac{\theta_0}{2} \right. \\ &\quad \left. \cdot \langle n|\sigma_3\phi_0(x')e^{2x'_0(d/dx')}|n\rangle \right] \end{aligned}$$

$$+i\delta \left[ \cos \theta_0 \langle n | \phi_0(x' + 2x'_0) | n \rangle \pm \cos \frac{\theta_0}{2} \cdot \langle n | \phi_0(x') e^{2x'_0(d/dx')} | n \rangle \right]$$

where

$$\begin{aligned} & -\delta \cos \frac{\theta_0}{2} \langle n | \phi_0(x') e^{2x'_0(d/dx')} | n \rangle \\ & -i\delta \sin \frac{\theta_0}{2} \langle n | \sigma_3 \phi_0(x') e^{2x'_0(d/dx')} | n \rangle \\ & = \int_{-\infty}^{\infty} \psi''_2{}^{(n)T} \sigma_2 \Phi \psi''_1{}^{(n)} dx' \\ & = \int_{-\infty}^{\infty} \psi''_1{}^{(n)T} \sigma_2 \Phi \psi''_2{}^{(n)} dx' \\ & \quad -\delta \cos \theta_0 \langle n | \phi_0(x' + 2x'_0) | n \rangle \\ & \quad -i\delta \sin \theta_0 \langle n | \sigma_3 \phi_0(x' + 2x'_0) | n \rangle \\ & = \int_{-\infty}^{\infty} \psi''_1{}^{(n)T} \sigma_2 \Phi_2 \psi''_1{}^{(n)} dx' \\ & = \int_{-\infty}^{\infty} \psi''_2{}^{(n)T} \sigma_2 \Phi_1 \psi''_2{}^{(n)} dx', \end{aligned}$$

here

$$\Phi(x') = \Phi_1(x') + \Phi_2(x'), \quad \Phi_1(x') = \sigma_1 \phi_0(x' - x'_0)$$

$$\Phi_2(x') = [(\cos \theta_0 \sigma_1 - \sin \theta_0 \sigma_2) \phi_0(x' + x'_0)]$$

The corresponding eigenvalue equation is given by

$$i \frac{d}{dx'} \psi''_n + \Phi(x') \psi''_n = \lambda_n \sigma_3 \psi''_n(x')$$

The eigenfunction  $\psi''_n(x')$  satisfies the following symmetry and orthogonality requirements:

$$\psi''_{n\pm}(-x') = \pm \delta \left( \cos \frac{\theta_0}{2} \sigma_2 + \sin \frac{\theta_0}{2} \sigma_1 \right)$$

$$\psi''_{n\pm}(x'), \quad \delta = \pm 1$$

$$\int_{-\infty}^{\infty} \psi''_{n+}{}^T(x') \sigma_1 \psi''_{n-}(x') dx' = 0$$

When  $\theta_0 = 0$ ,  $\phi(x')$  is real and symmetric,  $\Delta \lambda_n^{(\pm)}$  is pure imaginary, when  $\theta_0 = \pi$ ,  $\phi(x')$  is real and antisymmetric,  $\Delta \lambda_n^{(\pm)}$  is real,

$$\Re[\Delta \lambda_n^{(\pm)}(\theta_0 = \pi)] = \mp \delta \langle n | \sigma_3 \phi_0(x') e^{2x'_0(d/dx')} | n \rangle$$

$$\Im[\Delta \lambda_n^{(\pm)}(\theta_0 = \pi)] = -\delta \langle n | \sigma_3 \phi_0(x' + 2x'_0) | n \rangle$$

(140)

Thus, the solution of the NLSE (5) with  $V(x, t) = A(\phi) = 0$  decays into paired solitons and each pair consists of solitons with equal amplitude and moving in the

opposite directions with the same speed. For arbitrary  $\theta'_0$ , we can see from Eq. (140) that the solution of Eq. (5) with  $V(x, t) = A(\phi) = 0$  breaks up into an even number of moving solitons with different speeds and amplitudes. Therefore, the eigenvalues of NLSE in nonlinear quantum mechanics are a very complicated problem.

## 5 Conclusions

In this paper, we give the elementary principles of nonlinear quantum mechanics (NLQM) according to the difficulties in linear quantum mechanics. Using these elementary principles, we study in detail some main properties of microscopic particles in nonlinear quantum systems. We give the invariance and conservation laws of mass, energy and momentum and angular momentum for the microscopic particles, which are also some elementary and universal laws of matter in the NLQM. We also find the classical laws of motion of microscopic particles and that motions of microscopic particles satisfy the Lagrangian and Hamilton equations. From dynamical equation — nonlinear Schrödinger equation and their solutions, the collision processes of many microscopic particles and their reflection and transmission features on the interface as well as the uncertainty relation of the motion of particles and quantum fluctuation of particle numbers, we discuss their wave-corpucle duality. Finally, we discuss further the eigenvalue problem of particles, etc. From these investigations and discussions, we see clearly that microscopic particles in nonlinear quantum mechanical systems have a lot of new properties, their laws, rules and features of motion are completely different from that in the linear quantum mechanics (LQM), for example, the particles possess the real wave-corpucle duality, obey the classical rule of motion and conservation laws of energy, momentum and mass, satisfy minimum uncertainty relation, can be localized due to the nonlinear interaction, and its position and momentum can be also determined in certain degree, etc. Therefore, it is very necessary to establish the NLQM, which is a new physical theory and a necessary result of the development of quantum mechanics. Its correct representation describes microscopic particles in nonlinear systems. The NLQM could solve problems disputed by scientists in the LQM field for about a century. The NLQM established can promote the development of physics and can enhance and raise the knowledge and recognition levels to the essences of microscopic matter. We can predict that nonlinear quantum mechanics has extensive applications in physics, chemistry, biology, polymers, and so on.

**Acknowledgements** The author would like to acknowledge the Major State Basic Research Development Program (973 program) of China for the financial support (Grant No. 2007CB936103).

## References

1. D. Bohr and J. Bub, *Phys. Rev.*, 1935, 48: 169
2. E. Schrödinger, *Collected Paper on Wave Mechanics*, London: Blackie and Son, 1928
3. E. Schrödinger, *Proc. Cambridge Phil. Soc.*, 1935, 31: 555
4. W. Heisenberg, *Z. Phys.*, 1925, 33: 879; W. Heisenberg and Euler H., *Z. Phys.* 1936, 98:714
5. M. Born and L. Infeld, *Proc. Roy. Soc. A*, 1934, 144: 425
6. P. A. Dirac, *The Principles of Quantum Mechanics*, Oxford: Clarendon Press, 1967
7. S. Diner, D. Farque, G. Lochak, and F. Selleri, *The Wave-Particle Dualism*, Dordrecht: Riedel, 1984
8. M. Ferrero and A. Van der Merwe, *New Developments on Fundamental Problems in Quantum Physics*, Dordrecht: Kluwer, 1997
9. M. Ferrero and A. Van der Merwe, *Fundamental Problems in Quantum Physics*, Dordrecht: Kluwer, 1995
10. L. de Broglie, *Nonlinear Wave Mechanics: A Causal Interpretation*, Amsterdam: Elsevier, 1960
11. P. B. Burt, *Quantum Mechanics and Nonlinear Waves*, New York: Harwood Academic Publisher, 1981
12. X. F. Pang, *J. Low. Temp. Phys.*, 1985, 58: 334
13. X. F. Pang, *Problems of Nonlinear Quantum Mechanics*, Chengdu: Sichuan Normal University Press, 1985
14. X. F. Pang, *Chin. J. Low. Temp. Supercon.*, 1982, 4: 33; *ibid*, 1981, 3: 62
15. X. F. Pang, *Chin. J. Nature*, 1982, 5: 254
16. X. F. Pang, *Phys. Bulletin Sin.*, 1983, 3: 17
17. X. F. Pang, *Chin. J. Potential Science*, 1985, 5: 16
18. X. F. Pang, *J. Res. Met. Mat. Sin.*, 1986, 12: 31
19. X. F. Pang, *J. Science Exploration Sin.*, 1986, 4: 70
20. X. F. Pang, *The Principle and Theory for Nonlinear Quantum Mechanics*, Proc. ICNLP, Shanghai, 1989: 1098
21. X. F. Pang, *Nonlinear Quantum Mechanics*, Proc. ICIPES, Beijing, 1991: 34
22. X. F. Pang, *The Principle and Theory of Nonlinear Quantum Mechanics*, Proc. 4th APPC, Seoul, Korea, 1990: 996
23. X. F. Pang, *The Elementary Principle and Theory for Nonlinear Quantum Mechanics*, Proc. ICMP, Wuhan, 1994: 991, PF-1
24. X. F. Pang, *The Theory of Nonlinear Quantum Mechanics*. In: *Research of New Sciences*, edited by Lui Hong, Beijing: Chin. Science and Tech. Press, 1991: 10
25. X. F. Pang, et al., *The Properties of Soliton Motion for Superconductivity Electrons*, Proc. ICNP., Shanghai, 1989: 139
26. X. F. Pang, *Quantum Mechanics in Nonlinear Systems*, Singapore: World Scientific Publishing Co., 2005
27. X. F. Pang, *The Theory of Nonlinear Quantum Mechanics*, Chongqing: Chinese Chongqing Press, 1994
28. X. F. Pang, *Establishment of Nonlinear Quantum Mechanics*, Research and Development and of World Science and Technology, 2006, 28:11
29. X. F. Pang, *Rules of Motion of Microscopic Particles in Nonlinear Systems*, Research and Development and of World Science and Technology, 2003, 24:54
30. X. F. Pang, *Features of Motion of Microscopic Particles in Nonlinear Systems and Nonlinear Quantum Mechanics in Scientific Proceeding — Physics and Others*, Beijing: Atomic Energy Press, 2006
31. V. E. Zakharov and A. B. Shabat, *Sov. Phys. JETP*, 1972, 34: 62; *ibid*, 1973, 37: 823
32. X. F. Pang, *Soliton Physics*, Chengdu: Sichuan Science and Technology Press, 2003
33. B. L. Go and X. F. Pang, *Solitons*, Beijing: Chinese Science Press, 1987
34. C. Sulem and P.L.Sulem, *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse*, Berlin, Heidelberg: Springer-Verlag, 1999
35. X. F. Pang, *Phys. Stat. Sol.(b)*, 2003, 236: 34
36. H. H. Chen and C. S. Liu, *Phys. Rev. Lett.*, 1976, 37: 693
37. H. H. Chen, *Phys. Fluids*, 1978, 21: 377
38. D. W. AOssey, S. R. Skinner, J. T. Cooney, J. E. Williams, J.M. T. Gavin, D. R. Andersen, and K. E. Lonngren, *Phys. Rev. A*, 1992, 45: 2606
39. R.N.Thurston and A. M. Weiner, *J. Opt. Soc. Am. B*, 1991, 8: 471
40. J. L. Cooney, M. T. Gavin, J. E. Williams, D. W. AOssey, and K. E. Lonngren, *Phys. Fluids B*, 1991, 3: 3277
41. G. L. Lamb, *Elements of Soliton Theory*, New York: Wiley, 1980: 118
42. H. Ikezi, R. J. Taylor, and D. R. Baker, *Phys. Rev. Lett.*, 1970, 25: 11
43. K. E. Lonngren, *Plasma Phys.*, 1983, 25: 943
44. K. E. Lonngren, D. R. Andersen and J. L. Cooney, *Phys. Lett. A*, 1991, 156: 441
45. K. E. Lonngren and A. C. Scott, *Solitons in Action*, New York: Academic, 1978: 153
46. A. S. Davydov, *Solitons in Molecular Systems*, Dordrecht: Reidel Publishing Comp., 1985; 2nd Ed., 1991
47. X. F. Pang, *Phys. Rev. E*, 2000, 62: 6989
48. X. F. Pang, *European Phys. J. B*, 2001, 19: 297
49. X. F. Pang, *European Phys. J. B*, 1999, 10: 415
50. X. F. Pang, *Physica D*, 2001, 154: 138
51. X. F. Pang, *J. Phys. Condens. Matter*, 1990, 2: 9541
51. X. F. Pang and H. J. W. Müller-Kirsten, *J. Phys. Condens. Matter*, 2000, 12: 885
53. X. F. Pang and Y.-P. Yuan, *Chem. Phys. Lett.*, 2003, 373: 392
54. L. N. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.*, 1957, 108: 1175
55. X. F. Pang, *Phys. Lett. A*, 1999, 259: 466
56. X. F. Pang, *J. Phys. Chem. Sol.*, 2001, 62:491
57. Y. Lai and H. A. Haus, *Phys. Rev. A*, 1989, 40: 844, 854
58. F. X. Kartner and L. Boivin, *Phys. Rev. A*, 1996, 53: 454

59. A. C. Scott and J. C. Eilbeck, *Phys. Lett. A*, 1986, 119: 60
60. A. C. Scott and J. C. Eilbeck, *Chem. Phys. Lett*, 1986, 132: 23
61. A. C. Scott, P. S. Lomdahl, and J. C. Eilbeck, *Chem. Phys. Lett.*, 1985, 113: 29
62. X. F. Pang, *Acta Phys. Sin.*, 1994, 43: 1987
63. X. F. Pang, *Chin. J. Phys. Chem.*, 1995, 12: 1062
64. X. F. Pang and X. R. Chen, *Chinese Physics*, 2000, 9: 108
65. X. F. Pang and X. R. Chen, *Commun. Theor. Phys.*, 2001, 35: 313
66. X. F. Pang and X. R. Chen, *Commun. Theor. Phys.*, 2001, 35: 763
67. X. F. Pang and X. R. Chen, *J. Phys. Chem. Solid.*, 2001, 62: 793
68. X. F. Pang and H. W. Zhang, *Modern Phys. Lett. B*, 2006, 20: 1923
69. X. F. Pang and X. R. Chen, *Inter. J. Mod. Phys. B*, 2006, 20: 2505
70. J. Satsuma and N. Yajima, *Prog. Theor. Phys., Suppl.*, 1974, 55: 284