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RESEARCH ARTICLE

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# Snyder's quantized space-time and de Sitter special relativity



**Abstract** There is a one-to-one correspondence between Snyder's model in de Sitter space of momenta and the d$\mathcal{S}$-invariant special relativity as well as a minimum uncertainty-like relation. This indicates that physics at the Planck length $\ell_P$ and the scale $R = (3/\Lambda)^{1/2}$ should be dual to each other and there is in-between gravity of local d$\mathcal{S}$-invariance characterized by a dimensionless coupling constant $g = \ell_P /R \sim 10^{-61}$.

**Keywords** Snyder's model, d$\mathcal{S}$ special relativity, correspondence, d$\mathcal{S}$ gravity

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## 1 Introduction

A long time ago, Snyder [1] proposed a quantized space-time model in a projective geometry approach to the de Sitter (d$\mathcal{S}$)-space of momenta with a scale $a$ near or at the Planck length. The energy and momentum of a particle were identified with the inhomogeneous projective coordinates. Then, the space-time coordinates became operators $\hat{x}\mu$ given by 4-"translation" generators of d$\mathcal{S}$-algebra, being noncommutative.

Recently, the "doubly special relativity" or the "deformed special relativity" (DSR) has been proposed [2, 3]. There is also a large scale $\kappa$ near the Planck energy scale, related to $a$ in [1]. Since some DSR models can be realized by the identification of 4-momentum with certain coordinates on a d$\mathcal{S}$- or $\mathcal{A}$d$\mathcal{S}$-space of momenta [4, 5], Snyder's model may be viewed as the first of them.

The projective geometry approach is basically equivalent to the Beltrami model [6] of d$\mathcal{S}$-space ($\mathcal{B}$d$\mathcal{S}$). Importantly, the Beltrami coordinates of a d$\mathcal{S}$-hyperboloid, or inhomogeneous projective ones, without the antipodal identification, play a similar role as the Minkowski coordinates in a Minkowski-space. In these coordinates, particles and light signals move along the time-like or null geodesics being straight world-lines with *constant* coordinate velocities in each patch, respectively. Among these systems, the properties are invariant under the fractionally linear transformations with common denominators ($\mathcal{FLT}$ s) of d$\mathcal{S}$-group. These motions and the systems could be regarded as inertia without gravity. Then, there should be the principle of relativity in d$\mathcal{S}$/$\mathcal{A}$d$\mathcal{S}$-spacetime, respectively. Lu [7] emphasized the issue and began to study the special relativity in d$\mathcal{S}$/$\mathcal{A}$d$\mathcal{S}$-space, with his collaborators [8–11]. Promoted by recent observations on dark universe [12–15], further studies have been made [16–24].

In fact, in Einstein's special relativity, the assumptions are made [25] that rest rigid ruler is Euclidean and that time flow itself is homogeneous. However, these are not supported by

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the asymptotic behavior of our universe [12−15]. Just as weakening the fifth axiom leads to non-Euclidean geometry, giving up the assumptions leads to two kinds of the d$\mathcal{S}$/$\mathcal{A}$d$\mathcal{S}$-invariant special relativity in d$\mathcal{S}$/$\mathcal{A}$d$\mathcal{S}$-spacetime, which are on an almost equal footing with Einstein's [7, 16−23].

It is important that from two fundamental constants, the Planck length $\ell_P := (G\hbar c^{-3})^{1/2}$ and the d$\mathcal{S}$-radius $R = (3/\Lambda)^{1/2}$, it follows a dimensionless constant

$$g := \sqrt{3}\ell_P R^{-1} \quad \text{or} \quad g^2 = \frac{G\hbar\Lambda}{3c^3} \sim 10^{-122} \tag{1}$$

Since Newton's constant $G$ is present in Eq.(1), $g$ should describe gravity. A simple gauge-like model for the d$\mathcal{S}$-gravity showed this feature [26−33].

In this letter, we show that there is an interesting and important one-to-one correspondence between d$\mathcal{S}$-invariant special relativity and Snyder's model. In addition, there is also a minimum uncertainty-like relation between them. These indicate that the physics at the Planck scale and the scale $R$ should be dual to each other and there is in-between the local d$\mathcal{S}$-invariant gravity characterized by the dimensionless coupling constant $g$.

## 2 The Beltrami model

The 4-d Riemann sphere $\mathcal{S}^4$ can be embedded in a 5-d Euclidian space $\mathcal{E}^5$:

$$\mathcal{S}^4: \quad \delta_{AB}\xi^A\xi^B = \ell^2 > 0, \quad A, B = 0, \cdots, 4 \tag{2}$$
$$\mathrm{d}s_E^2 = \delta_{AB}\mathrm{d}\xi^A\mathrm{d}\xi^B = \mathrm{d}\xi^t\mathcal{I}\mathrm{d}\xi \tag{3}$$

where superscript $t$ represents transpose. They are invariant under rotations of $SO(5)$:

$$\xi \to \xi' = S\xi, \quad S^t\mathcal{I}S = \mathcal{I}, \ \forall\, S \in SO(5) \tag{4}$$

A Beltrami model $\mathcal{B}$ of $\mathcal{S}^4$ is the intrinsic geometry of $\mathcal{S}^4$ with Beltrami coordinate atlas. In a patch, say,

$$x^\mu := \ell\xi^\mu/\xi^4, \quad \xi^4 \neq 0, \quad \mu = 0, \cdots, 3 \tag{5}$$

with

$$\sigma_E(x) := \sigma_E(x, x) = 1 + \ell^{-2}\delta_{\mu\nu}x^\mu x^\nu > 0 \tag{6}$$
$$\mathrm{d}s_E^2 = [\delta_{\mu\nu}\sigma_E^{-1}(x) - \ell^{-2}\sigma_E^{-2}(x)\delta_{\mu\sigma}x^\sigma\delta_{\nu\rho}x^\rho]\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{7}$$

it is invariant under $\mathcal{FLT}$s of $SO(5)$ with a transitive form sending the point $A(a^\mu)$ to the origin $O(o^\mu = 0)$,

$$\begin{aligned} &x^\mu \to \tilde{x}^\mu = \pm\sigma_E^{1/2}(a)\sigma_E^{-1}(a, x)(x^\nu - a^\nu)N_\nu^\mu \\ &N_\nu^\mu = O_\nu^\mu - \ell^{-2}\delta_{\nu\sigma}a^\sigma a^\rho[\sigma_E(a) + \sigma_E^{1/2}(a)]^{-1}O_\rho^\mu \\ &O := (O_\nu^\mu) \in SO(4) \end{aligned} \tag{8}$$

There is an invariant for two points $A(a^\mu)$ and $B(b^\nu)$

$$\Delta_{E,\ell}^2(a, b) = \ell^2[1 - \sigma_E^{-1}(a)\sigma_E^{-1}(b)\sigma_E^2(a, b)] \tag{9}$$

The proper length between $A$ and $B$ is the integral of $\mathrm{d}s_E$ over the geodesic segment $\overline{AB}$:

$$L(a, b) = \ell \arcsin(|\Delta_E(a, b)|/l) \tag{10}$$

The geodesics in $\mathcal{B}$ are straight-lines and equivalent to

$$\frac{\mathrm{d}q^\mu}{\mathrm{d}s} = 0, \quad q^\mu := \sigma_E^{-1}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{11}$$

from which it follows the constant ratios

$$\frac{q^i}{q^0} = \frac{\mathrm{d}x^i}{\mathrm{d}x^0} = consts, \quad i = 1, 2, 3 \tag{12}$$

And they can be integrated out:

$$x^i(s) = \alpha^i x^0 + \beta^i, \quad \alpha^i, \beta^i = consts \tag{13}$$

In view of the gnomonic projection, the great circles on Eq. (2) are mapped to the straight-lines, the geodesics (13) in $\mathcal{B}$, and vice versa. It is also the case for Lobachevski space $\mathcal{L}^4$ as the original model [6] is just for the Lobachevski plane.

## 3 d$\mathcal{S}$-invariant special relativity via an inverse Wick rotation

From an inverse Wick rotation of Riemann sphere $\mathcal{S}^4$ with $\ell = R$ [18], it follows

$$H_R: \eta_{AB}\xi^A\xi^B = \xi^t\mathcal{J}\xi = -R^2 < 0 \tag{14}$$

$$\mathrm{d}s^2 = \eta_{AB}\mathrm{d}\xi^A\mathrm{d}\xi^B = \mathrm{d}\xi^t\mathcal{J}\mathrm{d}\xi \tag{15}$$

with $\mathcal{J} = (\eta_{AB}) = \mathrm{diag}(1, -1, -1, -1, -1)$ and the projective boundary $\partial_P H_R: \xi^t\mathcal{J}\xi = 0$. Under

$$\xi \to \xi' = S\xi, \ S^t\mathcal{J}S = \mathcal{J}, \ \forall\, S \in SO(1, 4) \tag{16}$$

they are invariant. Great circles in $\mathcal{S}^4$ now become a kind of uniform "great-circular" motions with a conserved 5-d angular momentum on $H_R$

$$\frac{\mathrm{d}\mathcal{L}^{AB}}{\mathrm{d}s} = 0, \quad \mathcal{L}^{AB} := m_R\left(\xi^A\frac{\mathrm{d}\xi^B}{\mathrm{d}s} - \xi^B\frac{\mathrm{d}\xi^A}{\mathrm{d}s}\right) \tag{17}$$

with an Einstein-like formula for mass $m_R$

$$-\frac{1}{2R^2}\mathcal{L}^{AB}\mathcal{L}_{AB} = m_R^2, \quad \mathcal{L}_{AB} = \eta_{AC}\eta_{BD}\mathcal{L}^{CD} \tag{18}$$

Further, a "simultaneous" 3-hypersurface

$$\delta_{ab}\xi^a\xi^b = R^2 + (\xi^0)^2, \quad a, b = 1, \cdots, 4 \tag{19}$$

is an expanding $S^3$.

The 5-d angular momentum operators, proportional to the generators of the d$\mathcal{S}$-algebra $so(1,4)$, or the Killing vector fields acting on the d$\mathcal{S}$-hyperboloid, read ($\hbar = 1$)

$$\hat{\mathcal{L}}_{AB} = \frac{1}{\mathrm{i}}\left(\xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A}\right), \;\; \xi_A = \eta_{AB}\xi^B \tag{20}$$

And they are globally defined on the d$\mathcal{S}$-hyperboloid.

Via the inverse Wick rotation, the Beltrami model of Riemann-sphere becomes the $\mathcal{B}$d$\mathcal{S}$-space covered with Beltrami coordinate atlas patch by patch. The condition and Beltrami metric with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ in each patch

$$\sigma(x) = \sigma(x,x) := 1 - R^{-2}\eta_{\mu\nu}x^\mu x^\nu > 0 \tag{21}$$

$$\mathrm{d}s^2 = [\eta_{\mu\nu}\sigma^{-1}(x) + R^{-2}\eta_{\mu\sigma}\eta_{\nu\rho}x^\sigma x^\rho \sigma^{-2}(x)]\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{22}$$

are invariant under $\mathcal{FLT}$s of $SO(1,4)$

$$\begin{aligned} & x^\mu \to \tilde{x}^\mu = \pm\sigma^{1/2}(a)\sigma^{-1}(a,x)(x^\nu - a^\nu)D^\mu_\nu \\ & D^\mu_\nu = L^\mu_\nu - R^{-2}\eta_{\nu\sigma}a^\sigma a^\rho[\sigma(a) + \sigma^{1/2}(a)]^{-1}L^\mu_\rho \\ & L := \{L^\mu_\nu\} \in SO(1,3) \end{aligned} \tag{23}$$

In such a $\mathcal{B}$d$\mathcal{S}$-space, the generators of $\mathcal{FLT}$s, or the Killing vectors, read

$$\begin{aligned} & \hat{q}_\mu = (\delta^\nu_\mu + R^{-2}x_\mu x^\nu)\partial_\nu, \;\; x_\mu := \eta_{\mu\nu}x^\nu \\ & \hat{L}_{\mu\nu} = x_\mu \hat{q}_\nu - x_\nu \hat{q}_\mu = x_\mu\partial_\nu - x_\nu\partial_\mu \in SO(1,3) \end{aligned} \tag{24}$$

and form an $SO(1,4)$ algebra

$$[\hat{q}_\mu, \hat{q}_\nu] = -R^{-2}\hat{L}_{\mu\nu}, \;\; [\hat{L}_{\mu\nu}, \hat{q}_\sigma] = \eta_{\nu\sigma}\hat{q}_\mu - \eta_{\mu\sigma}\hat{q}_\nu$$

$$[\hat{L}_{\mu\nu}, \hat{L}_{\sigma\rho}] = \eta_{\nu\sigma}\hat{L}_{\mu\rho} - \eta_{\nu\rho}\hat{L}_{\mu\sigma} + \eta_{\mu\rho}\hat{L}_{\nu\sigma} - \eta_{\mu\sigma}\hat{L}_{\nu\rho} \tag{25}$$

There are inertial motions with a set of conserved observable along geodesics

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}s} = 0 \quad \text{with} \quad p^\mu = \sigma^{-1}(x)m_R\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{26}$$

$$\frac{\mathrm{d}L^{\mu\nu}}{\mathrm{d}s} = 0 \quad \text{with} \quad L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{27}$$

or equivalent to

$$m_R\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = 0, \qquad t = x^0/c \tag{28}$$

The pseudo 4-momentum $p^\mu$ and pseudo 4-angular-momentum $L^{\mu\nu}$ constitute a conserved 5-d angular momentum (17). Obviously, Eq.(26) is the counterpart of Eq.(11) and Eq.(18) becomes

$$E^2 = m_R^2c^4 + p^2c^2 + \frac{c^2}{R^2}l^2 - \frac{c^4}{R^2}k^2 \tag{29}$$

which is a generalized Einstein's formula with energy $E = cp^0$, momentum $p^i$, $p_i = \delta_{ij}p^j$, boosts $k^i = L^{0i}$, $k_i = \delta_{ij}k^j$ and 3-angular momentum $l^i = \frac{1}{2}\epsilon^i{}_{jk}L^{jk}$, $l_i = \delta_{ij}l^j$. It can be proved that they are Noether's charges with respect to the Killing vectors (24).

It should be emphasized that since the generators in Eq. (20) are globally defined on the d$\mathcal{S}$-hyperboloid, they should also be globally defined in the Beltrami atlas patch by patch. Thus, there is a set of globally defined ten Killing vectors in the Beltrami atlas and correspondingly, there is a set of ten Noether's charges forming a 5-d angular momentum $\mathcal{L}^{AB}$ in Eq. (17) *globally* in the Beltrami atlas, though the physical meaning of each Noether's charge depends on the Beltrami coordinate patch used.

The interval and thus light-cone can be well defined by the counterparts of Eqs. (9) and (10).

Thus, d$\mathcal{S}$-invariant special relativity can be set up on the relativity principle [7–11] and the universal constant postulate for the speed of light $c$ and radius $R$ [16, 17].

## 4 Snyder's model and DSR

Snyder considered the homogenous quadratic form

$$-\eta^2 := \eta^{AB}\eta_A\eta_B < 0 \tag{30}$$

It may be regarded as a hyperboloid in 5-d space of momenta with the line-element,

$$\mathrm{d}s_p^2 = \eta^{AB}\mathrm{d}\eta_A\mathrm{d}\eta_B \tag{31}$$

and is identical to the inverse Wick rotation of the Eq.(2) after identifying $\eta_A$ with $\rho\xi_A$ with a common factor $\rho \neq 0$ in relativistic units $[\rho] = L^{-2}$. Thus, a Beltrami model of d$\mathcal{S}$-space of momenta may also be set up on a space of momenta. In fact, Snyder defines the energy-momentum with help of a constant $a$, which may be taken as the Planck length,

$$p_0 = a^{-1}\eta_0/\eta_4 = a^{-1}\xi^0/\xi^4, \quad p_i = a^{-1}\eta_i/\eta_4 = a^{-1}\xi^i/\xi^4$$

Quantum mechanically, in this "momentum picture", the operators for the space-time-coordinates $\hat{x}^i, \hat{t}$ should be given by:

$$\hat{x}^i := \mathrm{i}\left[\frac{\partial}{\partial p_i} + a^2p^ip_\nu\frac{\partial}{\partial p_\nu}\right] \tag{32}$$

$$\hat{t} = \hat{x}^0/c := \frac{\mathrm{i}}{\mathrm{c}}\left[\frac{\partial}{\partial p_0} + a^2p^0p_\nu\frac{\partial}{\partial p_\nu}\right], \quad p^\mu = \eta^{\mu\nu}p_\nu$$

Together with "boost" $\hat{M}^i = \hat{x}^0p^i + \hat{x}^ip^0$ and "3-angular momentum" $\hat{L}^i = \frac{1}{2}\epsilon^i{}_{jk}\hat{x}^j p^k$, they form an $so(1,4)$ algebra

$$[\hat{x}^i, \hat{x}^j] = -\mathrm{i}a^2 \epsilon_k^{\ ij} L^k, \quad [\hat{x}^0, \hat{x}^i] = -\mathrm{i}a^2 \hat{M}^i \tag{33}$$

$$[\hat{L}^i, \hat{L}^j] = \epsilon_k^{\ ij} \hat{L}^k, [\hat{M}^i, \hat{M}^j] = \epsilon_k^{\ ij} \hat{M}^k; \ \ etc$$

Since $p^\mu$ are inhomogeneous (projective) coordinates or Beltrami coordinates, one patch in the model is not enough. And since the projective space $RP^4$ is non-orientable, the antipodal identification should not be taken to preserve the orientation. The operators $\hat{x}^\mu$ are just 4-generators of the $\mathrm{d}\mathcal{S}$-algebra (24), and $\hat{L}^i, \hat{M}^i$ are the remaining 6-generators $\hat{L}_{\mu\nu}$ in (25) of $so(1,3)$ algebra. Actually, the algebra (33) is the same as (25).

Similar to Snyder's model, a quantized space-time model on $\mathcal{A}\mathrm{d}\mathcal{S}$-space of momenta can be constructed. Actually, some other DSR models can also be described in other coordinates in a $\mathrm{d}\mathcal{S}$- or $\mathcal{A}\mathrm{d}\mathcal{S}$-space of momenta [4, 5].

It is important that the correspondence of the ratio (12) in $\mathrm{d}\mathcal{S}$-space of momenta may be viewed as the inverse of "group velocity" components of some "wave-packets". If so, one may define in Snyder's model a new kind of uniform motions with constant component "group velocity". In particular, when the correspondences of $\beta^i$ in (13) vanish, the "group velocity" of a "wave-packet" coincides its "phase velocity". This is similar to the case for a light pulse propagating in vacuum Minkowski spacetime.

Furthermore, $\mathrm{d}\mathcal{S}$-space of momenta also has a horizon. Thus, one may imitate the study of $\mathrm{d}\mathcal{S}$-space in general relativity to introduce "temperature" $\tilde{T}_p$ and "entropy" $\tilde{S}_p$ for the horizon. But the question is, do they make sense?

## 5 On thermodynamics

In the viewpoint of $\mathrm{d}\mathcal{S}$-invariant special relativity, there is no gravity in $\mathrm{d}\mathcal{S}$-space. Therefore, the thermodynamics is not originated from gravity.

Since there exist inertial motions and inertial observers in $\mathrm{d}\mathcal{S}$-invariant special relativity, one may set up inertial reference frame. In the viewpoint of inertial observers in an inertial reference frame, the horizon in $\mathrm{d}\mathcal{S}$-space is at $T = 0$ without entropy. The temperature $T = \hbar c/(2\pi R k_{\mathrm{B}})$ and entropy $S = 4\pi R^2 c^3 k_{\mathrm{B}}/(G\hbar)$ in the static $\mathrm{d}\mathcal{S}$-coordinates or other coordinates arise from non-inertial motions and/or non-inertial parameterization rather than gravity [24].

Similarly, $\tilde{T}_p$ and $\tilde{S}_p$ in Snyder's model vanish even if the horizon in the $\mathrm{d}\mathcal{S}$-space of momenta exists. Thus, we may circumvent the difficulty in the explanation of the physical meaning of $\tilde{T}_p$ and $\tilde{S}_p$ in Snyder's model. However, these quantities do exist in some DSR models in $\mathrm{d}\mathcal{S}$-space of momenta and DSR advocators may have to face the problem of how to explain their physical meaning.

## 6 The Planck scale-$\Lambda$ duality

It is straightforward to see that there is an interesting and important one-to-one correspondence between Snyder's model and the $\mathrm{d}\mathcal{S}$-invariant special relativity as shown in the following Table:

| $\mathrm{d}\mathcal{S}$ special relativity | Snyder's model |
|---|---|
| Coordinate "picture" | Momentum "picture" |
| $\mathcal{B}\mathrm{d}\mathcal{S}$-spacetime | $\mathcal{B}\mathrm{d}\mathcal{S}$-space of momenta |
| $R \sim$ cosmic radius | $1/a \sim$ Planck momentum |
| Constant 3-velocity | Constant "group velocity" |
| "Quantized" momenta | Quantized space-time |
| $\hat{p}_\alpha, \quad \hat{E}$ | $\hat{x}_\alpha, \quad \hat{t}$ |
| $T = 0$ without $S$ | $\tilde{T}_p = 0$ without $\tilde{S}_p$ |
| No gravity | No gravity |

The one-to-one correspondence should not be considered to happen accidentally.

In fact, there is also a minimum uncertainty-like relation between them and indicate why there should be a one-to-one correspondence. We now present an argument for the relation. Quantum mechanically, the coordinates and momenta cannot be determined exactly at the same time if the uncertainty principle, which reads

$$\Delta\xi^I \Delta\eta_I \geqslant \hbar \tag{34}$$

where $I = 1, \cdots, 4$ and the sum over $I$ is not taken, is valid in the embedded space†. Limited on the hyperboloid in embedded space, $\Delta\xi^I \leqslant R$. Suppose that the momentum $\eta_I$ conjugate to $\xi^I$ also takes values on a hyperboloid. Then, $\Delta\eta_I \leqslant \eta$ and the uncertainty relation implies $R\eta \sim \hbar$. Here $R$ and $\eta$ are two free parameters. We may write it in a covariant form

$$\eta_{AB}\xi^A\xi^B\eta^{CD}\eta_C\eta_D = \hbar^2 \tag{35}$$

and refer it as an uncertainty-like relation. When the size of the hyperboloid in the space of coordinates is Planck length, namely,

$$\eta_{AB}\xi^A\xi^B = -\ell_P^2 = -G\hbar c^{-3} \tag{36}$$

the hyperboloid in the space of momenta then has Planck scale,

† Here we simply employ the same notation of some observable for the expectation value of its operator over wave function in quantum mechanics.

$$\eta^{AB}\eta_A\eta_B = -E_P^2/c^2 = -\hbar c^3/G < 0 \tag{37}$$

which is equivalent to the Snyder's relation (30). On the contrary, when the scale of the hyperboloid in the space of momenta is

$$\eta^{AB}\eta_A\eta_B = -\frac{\Lambda\hbar^2}{3} \tag{38}$$

then we have relation (14). Therefore, the relation (35) may indicate a kind of the UV-IR connection and the correspondence listed in the Table should reflect some dual relation between the physics at these two scales. Of course, the argument here should be further demonstrated. We will provide it in detail elsewhere.

Furthermore, both Snyder's model and the $\mathrm{d}\mathcal{S}$-invariant special relativity deal with the motion of relativistic particles. In $\mathrm{d}\mathcal{S}$-invariant special relativity, the momenta of a particle are quantized and noncommutative, while in Snyder's model, the coordinates of a particle are quantized and noncommutative. There is no gravity in both. As was mentioned in the beginning, however, the dimensionless constant $g = \ell_P/R$ in (1) contains the gravitational constant and thus should describe some gravity. Therefore, we may make a conjecture that the physics at such two scales should be dual to each other in some "phase" space and there is in-between the gravity characterized by $g$.

## 7 How to describe the gravity characterized by $g$?

It is the core of the equivalence principle that gravity should be based on localized special relativity. In general relativity, however, there are only local $SO(1,3)$ Lorentz frames without local translations. One may expect that the gravity should be based on the equivalence principle with full localized symmetry of special relativity, similar to the gauge principle, and be governed by gauge-like dynamics.

Now, there are three kinds of special relativity on Minkowski, $\mathrm{d}\mathcal{S}$ and $\mathcal{A}\mathrm{d}\mathcal{S}$ space with Poincaré, $\mathrm{d}\mathcal{S}$ and $\mathcal{A}\mathrm{d}\mathcal{S}$ group, respectively. Thus, there should be three kinds of gravity with relevant localized special relativity with full local symmetry.

These requirements have been indicated by a kind of simple models of $\mathrm{d}\mathcal{S}/\mathcal{A}\mathrm{d}\mathcal{S}$-gravity [26−33]. For the $\mathrm{d}\mathcal{S}$-model, the gauge-like action with the constant $g$ of $\mathrm{d}\mathcal{S}$-gravity in Lorentz gauge reads [26, 27]

$$\begin{aligned}S_G &= -\frac{\hbar}{4g^2}\int \mathrm{d}^4x e(\mathcal{F}^{AB}_{\ \ \mu\nu}\mathcal{F}_{AB}^{\ \ \mu\nu})\\ &= \int \mathrm{d}^4x e\left[\frac{c^3}{16\pi G}(F-2\Lambda)\right.\\ &\left.\quad -\frac{\hbar}{4g^2}F^{ab}_{\ \ \mu\nu}F_{ab}^{\ \ \mu\nu} + \frac{c^3}{32\pi G}T^a_{\ \mu\nu}T_a^{\ \mu\nu}\right]\end{aligned} \tag{39}$$

where $e = \det(e^a_\mu)$, $\mathcal{F}^{AB}_{\ \ \mu\nu}$ is the curvature of a $\mathrm{d}\mathcal{S}$-connection $\mathcal{B}^{AB}_{\ \ \mu} \in so(1,4)$, with $\mathcal{B}^{ab}_{\ \ \mu} = B^{ab}_{\ \ \mu}, \mathcal{B}^{a4}_{\ \ \mu} = R^{-1}e^a_{\ \mu}$, $F$, $F^{ab}_{\ \ \mu\nu}$ and $T^a_{\ \mu\nu}$ Cartan's scalar curvature, curvature, and torsion, respectively, on Riemann-Cartan manifolds with metric $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, Lorentz frame and connection $e^a_{\ \mu}, B^{ab}_{\ \ \mu} \in so(1,3)$.

It can be shown that the $\mathrm{d}\mathcal{S}$-space and thus the $\mathrm{d}\mathcal{S}$-invariant special relativity do fit this model. In addition, the terms in the action other than the Einstein-Hilbert term $R$, which can be picked up from the Einstein-Cartan term $F$, should play an important role as some "dark matter" in the viewpoint of general relativity. Thus, this model may provide a new platform for the data analysis of dark universe.

To show whether the Snyder's model also fits the model of gravity, one needs to study the quantization of the model of gravity in a nonperturbative procedure. Undoubtedly, there is a long way to go. Fortunately, it has been shown that the model of gravity is renormalizable perturbatively [28−31]. Also, the Euclidean version of action (35) is $SO(5)$ gauge-like and the Riemann sphere is its solution as an instanton. So, the quantum tunneling scenario should support $\Lambda > 0$. Furthermore, asymptotic freedom may imply that the coupling constant $g$ should be very tiny and it should link $\Lambda$ as an infrared cut-off with $\ell_P$ as an ultraviolet cut-off providing a fixed point.

Finally, note that $g^2$ is in the same order of difference between $\Lambda$ and the theoretical quantum "vacuum energy"; the big difference is no longer a puzzle in the viewpoint of the $\mathrm{d}\mathcal{S}$-invariant special relativity and local $\mathrm{d}\mathcal{S}$-invariant gravity. Since $\Lambda$ is a fundamental constant as $c$, $G$ and $\hbar$, a further question should be: what are the origins of these fundamental constants or the origin of the dimensionless constant $g$ and is $g$ calculable?

## 8 Concluding remarks

We have shown the one-to-one correspondence between Snyder's model and the $\mathrm{d}\mathcal{S}$-invariant special relativity as well as the minimum uncertainty-like relation. Based on this correspondence and the relation, we have made a conjecture that there should be a duality in physics at the Planck scale and at the cosmological scale $R$ and that there is in-between gravity characterized by a dimensionless constant $g$.

The gravity between the two scales should be based on the localization of the $\mathrm{d}\mathcal{S}$-invariant special relativity with gauge-like dynamics. A simple model of $\mathrm{d}\mathcal{S}$-gravity in the Lorentz gauge may support this point of view.

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