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RESEARCH ARTICLE

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An alternative quantum theory for single particles and a proposed experimental test

Abstract An alternative quantum theory for single particles bounded in the external field proposed in 1986 (Huang X. Y., Phys. Lett. A., 1986, 115: 310) is further developed from which the energy of the state for the single particle takes one of the eigenvalues of the quantum Hamiltonian, and the usual quantum mechanics for the particle in a stationary state holds only in the statistical sense. In light of the theory, the particle of definite energy, ground-state-energy for instance, can exhibit a novel periodic behavior. This result for the ground-state-energy state neutron in the Earth's gravitational field is experimentally testable using ultracold neutron beam passing through the same apparatus that was devised in 2002 to identify the energy quantization of neutron in the field (Nesvizhevsky V. V., et al., Nature, 2002, 415: 297).

Keywords quantum mechanics, proposed experimental test

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1 Introduction

In orthodox quantum mechanics, when a single particle, such as hydrogen, is bounded in an external field, it most probably remains in the ground state at relatively lower temperature, and goes over to a superposition of stationary states (even the mixed state) at relatively higher temperature. In appropriate limits, quantum mechanics can reproduce the results of classical mechanics. Though quantum mechanics is supported by experiments up to the present, many physicists feel uneasy about its foundation. For instance, Einstein thought that the classically-mechanical state for the single particle should be the classical limit of the state for a reasonably defined “sharply defined energy” in quantum mechanics [1], while orthodox quantum mechanics considers that it should be the classical limit of a superposition of many stationary states, or the mixed state [1, 2]. Moreover, the statistical interpretation of quantum mechanics insists that the orthodox quantum mechanics applies only for the ensemble of similarly prepared systems, and there must be a theory that can be applicable for each single system [3, 4]. Such a theory for a single free particle was proposed in 1986 [5]. Despite this proposal winning some citations, it has not developed far enough to yield an experimentally testable prediction. As far as my knowledge goes, the apparatus [6, 7] devised in 2002 to identify the quantization of neutron’s energy in the Earth's gravitational field offers the only experimentally feasible scheme to test the proposal.

In order to get a quick look, let us discuss how this theory describes the free particles. With μ denoting the particle’s mass and p_0 denoting the particle’s momentum, there are four identities for a single free particle [5]:

$$\begin{aligned} &H(x) = x_0 + p_0 t/\mu, H(x^2) = (x_0 + p_0 t/\mu)^2 \\ &H(p) = p_0, \quad H(p^2) = p_0^2 \end{aligned} \tag{1}$$

where $H(f)$ is defined by

$$H(f) \equiv \mathrm{Re}\int_{-\infty}^{\infty} \varphi^*(x,t) f \psi p_0(x,t)\mathrm{d}x \tag{2}$$

where $\psi_{p_0}(x,t)$ is the usual plane wave, and $\varphi(x, t)$ is the superposition of all plane waves with unit weight,

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$$\psi_{p_0}(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{\mathrm{i}p_0(x-x_0)}{\hbar} - \frac{\mathrm{i}p_0^2 t}{2\mu\hbar}\right] \quad (3)$$

$$\varphi(x,t) = \int_{-\infty}^{\infty} \psi_p(x,t)\mathrm{d}p \quad (4)$$

These identities (1) establish an entirely new kind of relation between classical quantities and quantum mechanical wave functions and operators. It can also be expected that for the free particle, $H(f)$ (2) for quantities other than x, x^2, p, p^2 will certainly give results that are very similar to those given by classical mechanics. The physical interpretation of these relations is straightforward as it is in classical mechanics: Every physical quantity f evolves over time in accordance with $H(f)$ (2). In other words, if a single measurement of physical quantity f is performed at instant t, without in any sense disturbing the free particle, a record $H(f)$ will be obtained provided $H(f^2) = H(f)^2$. Moreover, if a single measurement does not suffice to produce the useful information for some reasons, for example the uncontrollable effects from the measuring apparatus and/or from the measurement processes, etc., many repetitions of measurement must be performed on the same particle, or on an ensemble of the similarly prepared particles. Then the usual quantum mechanical expectation value of f is formally $f_{p_0p_0}$,

$$f_{p_0p_0} = \lim_{X\to\infty} \frac{1}{2X} \int_{-X}^{X} H(f)\mathrm{d}x_0 \quad (5)$$

Clearly, $f_{p_0p_0}$ is the arithmetic mean of $H(f)$, averaged over initial position x_0, which is distributed uniformly in the infinite interval $(-\infty,\infty)$. Then, a consistent interpretation of the plane wave follows. The plane wave $\psi_{p_0}(x,t)$ describes an ensemble of the free particles (which are not classical ones!) each of which has a definite initial position, which is distributed uniformly in the infinite interval $(-\infty,\infty)$. It is exactly what the interpretation of the plane wave asserts in the statistical interpretation of quantum mechanics [4].

The identities (1) above were found by Prof. Huang of Peking University, China, 21 years ago [5], which in mathematics are nothing but a new kind of mapping from an operator representing a physical quantity f to the C-number $H(f)$ (2). The new mapping was sometimes called Huang's expectation value [8] for the quantity f. It must be kept in mind that no matter how beautiful the new mapping in mathematics is, it must be excluded from physics unless it can give something new in experiments. Why this approach did not attract much attention from the international community for such a long time is two-fold. First, the theory kept its primitive form, wherein Huang's expectation value $H(f)$ in the general case was directly interpreted as a directly measurable quantity [9]. This interpretation of $H(f)$ renders the theory not only deterministic but also locally realistic, which is problematic from either theoretical or experimental considerations. However, the self-consistency of the theory requires that the quantity $H(f)$ be in general not directly measurable unless $H(f^2) = [H(f)]^2$, which will be seen in Section 2. Secondly, the theory had not offered any result for experimental confirmation or rejection.

As we will see shortly, it is impossible to formulate the results (1) into orthodox quantum mechanics. In Section 3 of this paper, the self-consistent generalization of Huang's approach for single particles bounded in the external field will be made, and a new experimentally testable result is then carried out. This paper will be closed with a brief conclusion in Section 4.

2 An alternative quantum theory for a single particle

In general, for a single particle bounded in an external field, it could be of a definite energy E_n whose value is one of the eigenvalues of the system's Hamiltonian $H = p^2/2 + V$, and the time evolution of the physical quantity f is postulated to be given by following Huang's expectation value $H(f)$:

$$\begin{aligned} H(f) &= \mathrm{Re}\int_{-\infty}^{\infty} \varphi^*(x,t) f \psi_n(x,t)\mathrm{d}x \\ &= \mathrm{Re}\sum_m f_{mn} \exp\left[\mathrm{i}\frac{(E_m - E_n)(t-t_0)}{\hbar}\right] \end{aligned} \quad (6)$$

where

$$\psi_n(x,t) = \psi_n(x)\exp\left[-\mathrm{i}\frac{E_n(t-t_0)}{\hbar}\right] \quad (7)$$

is the usual stationary state, and

$$\begin{aligned} \varphi(x,t) &= \sum_m \psi_m(x,t) \\ &= \sum_m \psi_m(x)\cdot\exp\left[-\mathrm{i}\frac{E_m(t-t_0)}{\hbar}\right] \end{aligned} \quad (8)$$

Based on the postulate (6), the following consequences ensue. In the rest of paper $t - t_0$ is simply written as t unless specified.

(1) The particle is free of energy dispersion because we have

$$\begin{aligned} H(H^j) &= \mathrm{Re}\int_{-\infty}^{\infty} \varphi^*(x,t) H^j \psi_n(x,t)\mathrm{d}x = (E_n)^j \\ &= [H(H)]^j, \quad j = 1,2,\cdots \end{aligned} \quad (9)$$

For any physical quantity f that is independent of time $\partial f/\partial t = 0$ and commutes with the quantum Hamiltonian $[H, f] = 0$, we also have $H(f^j) = [H(f)]^j$, $j = 1, 2, \cdots$. A set of eigenvalues given by the complete set of mutual commutable quantities can be used to specify the state. At this point, our theory is compatible with the conventional one. However, our theory considers that the state is of "sharply defined energy" E_n and the usual stationary state $\psi_n(x,t)$ is insufficient to specify the state. The particle always remains in the state of definite energy, whether its energy spectrum is discrete or continuous.

Therefore, the *ground-state-energy state* in our theory and *ground state* in the usual stationary ground state differ. The latter can be represented by one stationary state $\psi_n(x)$, while the former cannot.

(2) Quantum mechanical expectation value f_{nn} is nothing but the statistical average over initial time t_0 in interval $(-\infty, \infty)$, for we can easily see

$$f_{nn} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H(f)\mathrm{d}y, \quad y = t, \text{or } t_0 \tag{10}$$

The statistical average can be equivalently taken in the statistical ensemble (over t_0) or time ensemble (over t).

The superposition of stationary states in usual quantum mechanics is not recoverable from our theory that deals with only the energy dispersion-free state. In this sense, our theory bears some resemblance to classical mechanics.

(3) The particle exhibits a new kind of periodic, collapse and revival behavior by viewing the time evolution of $H(f)$, similar to that in orthodox quantum mechanics by viewing the expectation value for the quantity f on a quasiclassical wave packet [10]. Note that $H(f)$ (6) can be rewritten into

$$\begin{aligned} H(f) &= \mathrm{Re} \sum_{m=0}^{\infty} f_{mn} \exp\left[\mathrm{i}\frac{(E_m - E_n)t}{\hbar}\right] \\ &= \mathrm{Re} \sum_{m=-n}^{\infty} f_{m+n,n} \exp\left[\mathrm{i}\frac{(E_{m+n} - E_n)t}{\hbar}\right] \end{aligned} \tag{11}$$

in which $(E_{m+n} - E_n)/\hbar$ in the time-dependent factor can be expanded in terms of the so-called quasiclassical period T_{cl}, fractional revival period T_{fr}, and super-revival period $T_{sr} \cdots$ which are defined as follows [11]:

$$\frac{E_{m+n} - E_n}{\hbar} = m\frac{2\pi}{T_{cl}} + m^2\frac{2\pi}{T_{fr}} + m^3\frac{2\pi}{T_{sr}} + \cdots \tag{12}$$

where [11],

$$\begin{aligned} &T_{cl} = 2\pi\left(\frac{\mathrm{d}E_n}{\mathrm{d}n}\right)^{-1}, \quad T_{fr} = 2!2\pi\left(\frac{\mathrm{d}^2E_n}{\mathrm{d}n^2}\right)^{-1} \\ &T_{sr} = 3!2\pi\left(\frac{\mathrm{d}^2E_n}{\mathrm{d}n^3}\right)^{-1} \end{aligned} \tag{13}$$

Note that these definitions (13) hold better for larger n and the system in general behaves quasi-periodically. In the extreme quantum case, the ground-state-energy state for instance, the periodic structure exists but it can only be approximately represented by the periods defined via Eq. (13).

(4) In the mathematically classical limit:

$$n \to \infty, \hbar \to 0, n\hbar = \text{an appropriate classical action} \tag{14}$$

the $H(f)$ is exactly the classical quantity f in general. This fact is evident for we have

$$\lim_{(14)} H(f) = \sum_{m=-\infty}^{\infty} f_{m+n,n} \exp\left(\mathrm{i}m\omega t\right) = f(t) \tag{15}$$

where we used a well-known semiclassical result where matrix element $f_{m+n,n}$ in the classical limit goes to the mth Fourier series component for a classical quantity f, and the classical frequency is $\omega = \lim_{(14)}(E_{n+1} - E_n)/\hbar$ [2, 12, 13].

(5) The particle possesses an entirely new intrinsic uncertainty. Evidently, the microscopic particle has neither definite position nor definite momentum for $H(f^2) \neq (H(f))^2$ $(f = x, p)$. In the classical limit, $H(f^2)$ and $(H(f))^2$ are identical, then the particle can be said to have both definite position and definite momentum. This implies that once $H(f^2) \neq (H(f))^2$, there is an uncertainty associated with the quantity f, independent of the measurement. We will call the new uncertainty the *intrinsic fuzziness* (INFU) δf which can be defined by

$$\delta f = \sqrt{\left|H(f^2) - [H(f)]^2\right|} \tag{16}$$

Another useful quantitative measure of the INFU is the relative INFU $\delta f/\sqrt{(f^2)_{nn}}$. However, if the position (momentum) of the single particle is measured, we can get a value centered about, and not greatly deviated from $H(x)$ $(H(p))$. The ensemble average of position (momentum) is $\overline{H(x)} = x_{nn}$ $(\overline{H(p)} = p_{nn})$. Moreover, the inherent consistency of Huang's expectation value requires that both $H(x^2) \prec 0$ in some time interval and $H(p^2) \prec 0$ in another time interval can appear. This fact is analogous to a single particle dwelling in a classically forbidden region because if classical mechanics is (improperly) used we would find $p^2 \prec 0$. The presence of INFU sets up limits beyond which the concept of classical physics cannot be employed. Note that the relation $H(p^2) \prec 0$ does not mean that the momentum is imaginary. On the contrary, $H(p)$ is always real and the measurement value of the momentum is therefore always real. The nonvanishing value of INFU δf including the relation $H(p^2) \prec 0$ offers a measure of deviation from classical mechanics.

(6) The usual uncertainty is similar to the standard deviation. In the classical limit, $H(f^2) - (H(f))^2 = 0$, there is no INFU. However, it is easy to see that

$$\Delta f = \sqrt{(f^2)_{nn} - (f_{nn})^2} = \sqrt{\overline{H(f^2)} - \overline{H(f)}^2} \tag{17}$$

is really the standard deviation in statistics, and has nothing to do with the measurement.

So far, the core of the new theory has been built up. On one hand, the single particle behaves like a conventional quantum particle in some respects. For instance, the particle has no definite trajectory in either position or momentum space. On the other hand, a physical quantity for the particle of definite energy $H(f)$ is in general time-dependent, similar to a classical particle. It should be emphasized that even though there appears a connection (10) between our theory and the orthodox quantum mechanics, our theory and the orthodox one differ fundamentally. For instance, either $H(x^2) \prec 0$ or $H(p^2) \prec 0$ is forbidden in quantum mechanics. In classical limit, the expectation value of a quantity in a quasiclassical wave packet goes over to the classical quantity in terms of the Fejér average of the Fourier series [12−14], whereas in our theory $H(f)$ gives the classical quantity in terms of the Fourier series itself (15).

3 An experimentally testable prediction

For a particle of mass μ in the Earth's gravitational field, the potential is

$$V(x) = \begin{cases} \mu g x, & x > 0 \\ \infty, & x \leqslant 0 \end{cases} \tag{18}$$

The stationary states $\psi_n(x)$, $n = 1, 2, 3, \cdots$ are

$$\psi_n(x) = \begin{cases} N_n A(-\gamma_n + x/l), & x > 0 \\ 0, & x \leqslant 0 \end{cases} \tag{19}$$

where $A(x)$ is the Airy function with zeros $-\gamma_n$, and $N_n = (2\mu^2 g/\hbar^2)^{1/6}\, \mathrm{d}A(x)/\mathrm{d}x|_{x\to-\gamma_n}$ are the normalization constants, $l = (\hbar^2/2\mu^2 g)^{1/3}$ is the characteristic length, and $n (\geqslant 1)$ are the quantum numbers.

The corresponding energy eigenvalues E_n are

$$E_n = (\mu g^2 \hbar^2/2)^{1/3}\gamma_n \tag{20}$$

When $f = x, x^2, p, p^2$, the matrix elements are [15, 16]

$$\begin{aligned} \langle\psi_m|x|\psi_n\rangle &= \begin{cases} \dfrac{2}{3} l\gamma_n, & m = n \\ -\dfrac{2}{(\gamma_m - \gamma_n)^2} l, & m \neq n \end{cases} \\ \langle\psi_m|x^2|\psi_n\rangle &= \begin{cases} \dfrac{8}{15}(l\gamma_n)^2, & m = n \\ -\dfrac{24}{(\gamma_m - \gamma_n)^4} l^2, & m \neq n \end{cases} \\ \langle\psi_m|p|\psi_n\rangle &= \frac{\mathrm{i}\mu}{\hbar}(E_m - E_n)\langle\psi_m|x|\psi_n\rangle \\ \langle\psi_m|p^2|\psi_n\rangle &= 2\mu E_n \delta_{m,n} - 2\mu^2 g\langle\psi_m|x|\psi_n\rangle \end{aligned} \tag{21}$$

When the particle is a neutron having the ground-state-energy E_1, the quasiclassical period T_{cl}, fractional revival period T_{fr}, and super-revival period T_{sr} are from Eq. (13) 3.3348 ms, 15.0066 ms, and 25.3236 ms, respectively, but the actual quasiclassical period and fractional revival period are 3.93 ms and 21.8 ms, respectively. One may argue why the actual revival period 21.8 ms is quantitatively different from either T_{fr} = 15.0066 ms or T_{sr} =25.3236 ms above. This is because those periods (13) apply better in the quasiclassical case; and the larger the quantum number n is, the closer the fractional revival period approaches to the revival period. The probability density for the ground state is plotted in Fig. 1, from which we know that a macroscopic spatial height, $15 \mu m$, is necessary to accommodate the state. Huang's expectation values $H(f)$ (11) for x, x^2, $v (= p/\mu)$ and v^2 are plotted in Fig. 2–Fig. 5, respectively.

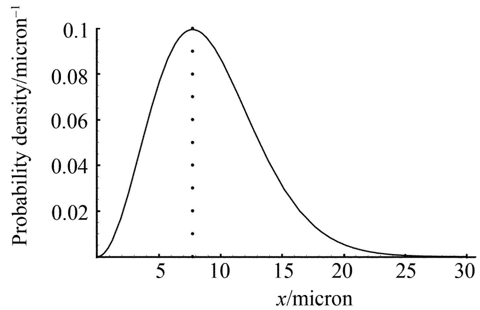

Fig. 1 The position probability density $|\psi_1(x)|^2$ for ground state of a neutron in the Earth's gravitational field. The maximum occurs at point x = 7.746 μm, and $1/e$ of the maximum occurs at x = 2.980 μm, 14.413 μm between which the probability is 0.852.

From these figures, the mean value of relative INFU for position $\delta x/\sqrt{(x^2)_{nn}}$ in the actual revival period is 0.388, whereas that for velocity $\delta v/\sqrt{(v^2)_{nn}}$ is 0.827. The physical quantities x (therefore velocity v for $H(v) = \mathrm{d}H(x)/\mathrm{d}t$), x^2, and v^2 share the same revival period 21.8 ms.

Note that the apparatus devised in 2002 to identify the quantization of neutron's energy in the Earth's gravitational field might be adapted to observe these quasiclassical periodic effects [6, 7]. In their experiment [6, 7], Nesvizhevsky and his colleague took a beam of ultracold neutrons and let them fly between a reflecting mirror below and a neutron absorber above, and both the mirror and the absorber are parallel with the Earth surface. By altering the height of the absorber, they could control the vertical component of a particle's velocity as it traversed a parabolic path through the trap. By quantum mechanics, as can be seen from Fig. 1, no neutron is transmitted through the absorber-mirror gap until the absorber is raised to a height of 15 μm which suffices to accommodate the dominative spatial part of the ground state of vertical motion of the neutron. With the ultracold neutron beam at about horizontal velocity 10 m/s, the distance between two sequential

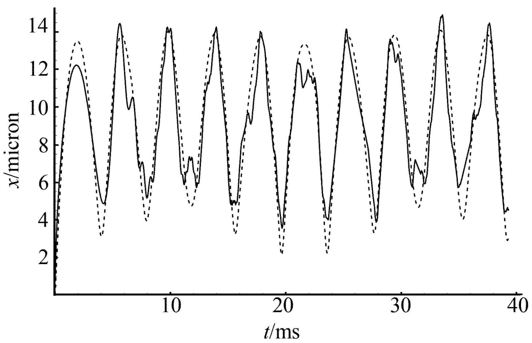

Fig. 2 The position x (in unit micron) versus time t (in unit ms) for the ground state neutron in Earth's gravitational field. The solid line shows $H(x)$, and the dashed line shows $\sqrt{H(x^2)}$, from which we see that $H(x)$ and $\sqrt{H(x^2)}$ differ a little. The intrinsic fuzziness of position is therefore small. The quasiclassical period is 3.93 ms. If the ultracold neutron has horizontal velocity 10 m/s , the distance between two sequential collisions with the mirror is 3.93 cm which is a macroscopic size.

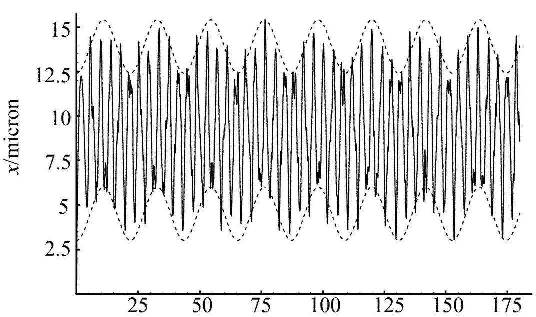

Fig. 3 The revival motion of the ground state neutron in Earth's gravitational field, viewed from position x. The solid line shows $H(x)$, and the dashed contour line is used to indicate the revival period: 21.8 ms. If the ultracold neutron has horizontal velocity 10 m/s, this revival period corresponds to the distance 21.8 cm which is also a macroscopic size.

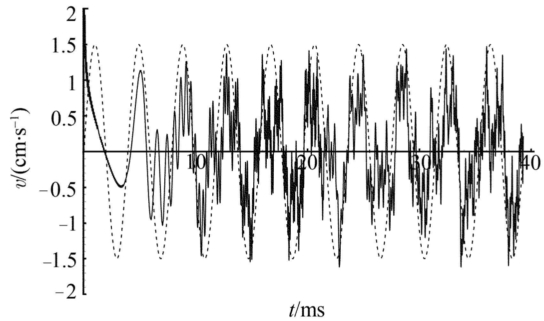

Fig. 4 The velocity v (in unit cm/s) versus t (in unit ms) for the ground state neutron in Earth's gravitational field. The solid line shows $H(v)$, and the dashed contour line is used to indicate the quasiclassical period: 3.93 ms. The velocity appears fluctuating and its intrinsic fuzziness is larger (see also Fig. 5).

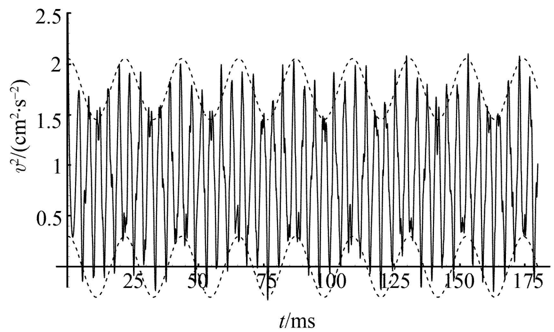

Fig. 5 The revival motion of the ground state neutron in Earth's gravitational field, viewed from the velocity v^2. The solid line shows $H(v^2)$, and the dashed contour line is used to indicate the revival period: 21.8 ms. When the particle approaches and dwells in the classically forbidden area, the square of velocity v^2 is negative.

collisions with the mirror is about 3.93 cm, which is a macroscopic size, and the distance corresponding to the revival period is about 21.8 cm, which is also a macroscopic size. If the neutron counter is suitably positioned, these macroscopic phenomena should be observable.

4 Conclusions

An alternative quantum theory for a single particle is developed in this paper from which a single particle state has a sharply defined energy which is one of the eigenvalues of the quantum Hamiltonian. The usual quantum mechanics for the particle in the stationary state holds only in statistical ensemble. In light of the theory, the particle has a time-dependent INFU δf which is independent of measurement, and the usual uncertainty is rather of a statistical origin. The presence of INFU sets up limits beyond which the concepts of classical physics cannot be employed. By and large, this theory supports the statistical interpretation of quantum mechanics. Even if this theory lacks a deeper insight into the wave or corpuscle nature of a quantum particle, it has an experimentally testable prediction wherein a particle of definite energy, ground-state-energy for instance, can exhibit the quasiclassical motion, and the collapse and revival phenomena. Since the ground-state-energy neutron in the Earth's gravitational field has quasiclassical period 3.93 ms, and revival period 21.8 ms, with the ultracold neutron beam having a horizontal velocity of about 10 m/s, these periodic phenomena will occur in macroscopic size and therefore in principle observable.

Finally, I would like to close this paper with three remarks. (1) Why ultracold atoms or molecules cannot be used to test our theory is due to the strong electric interaction between them and the bouncing plate. In contrast, the experiment [6, 7] successfully demonstrated that ultracold neutron beam has only a negligible interaction with the mirror. (2) Because of the presence of INFU, the present theory can only be categorized into the non-local hidden variables theory. It therefore does not contradict any known experiments, including those tested against the Bell's inequality, which rests upon the local realism. As accepted, no experiment rejects the possible existence of *non-local* hidden variables [17]. (3) It should be emphasized that our theory does not mean to offer the full foundation for quantum mechanics. More likely, it may finally turn out to be a supplementary part of quantum mechanics.

Notes added by the author The draft form of this paper was complete during my two months' visit at the Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, Italy, during February 15 to April 11, 2004. From then on, I have been constantly revising the consistency of the theory, and am confident that there is no problem on the theoretical side. However, even the theory intrigues me so I invested a huge amount of time on revising its basic framework in which no inconsistency was found, it does not mean that I favor or disfavor the theory. During April 6−7, 2006, I gave a talk on the workshop "Resonance Transitions Between Gravitationally Bound Quantum States of Neutrons" held at the Institute of Laue Langevin, Grenoble, France, where the work [6, 7] was done. During the workshop, I was indebted in discussions with many participants on both the theoretical and the experimental problem, and I was happily told that my proposed experiment is in principle feasible in Prof. Nesvizhevsky's laboratory, the Institute of Laue Langevin. The final version of the present paper was then formed.

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