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Construction of the elliptic Gaudin system based on Lie algebra

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Abstract Gaudin model is a very important integrable model in both quantum field theory and condensed matter physics. The integrability of Gaudin models is related to classical r -matrices of simple Lie algebras and semi-simple Lie algebra. Since most of the constructions of Gaudin models works concerned mainly on rational and trigonometric Gaudin algebras or just in a particular Lie algebra as an alternative to the matrix entry calculations often presented, in this paper we give our calculations in terms of a basis of the typical Lie algebra, A_n, B_n, C_n, D_n , and we calculate a classical r -matrix for the elliptic Gaudin system with spin.

Keywords Gaudin model, classical r -matrix, Lie algebra, elliptic function

PACS numbers 03.65.Fd, 04.20.Jb, 05.30.-d, 75.10.Jm

1 Introduction

The Gaudin system is a very important integrable model in both quantum field theory and condensed matter physics, which was originally introduced as an integrable spin chain system associated with N particles with $su(2)$ or $su(1, 1)$ in statistical quantum mechanics [1, 2]. This system is very different from the well-known Heisenberg models. The most significant property of the Gaudin model is its N commutative Hamiltonians expressed as:

$$\mathcal{H}_i = 2 \sum_{j \neq i}^N \frac{(S_i, S_j)}{z_i - z_j} \quad (1)$$

where $z_1 > z_2 > \dots > z_N$ are free parameters and S_i ($i = 1, \dots, N$) is the $su(2)$ or $su(1, 1)$ spin operator of the i -th particle.

The total spin of the system $S^2 = \sum_{i=1}^N (S^i)^2$ and $S_z = \sum_{i=1}^N S_z^i$ are the remaining conserved moving constants of motion. Here, the spin operators S_i belong to the Lie algebra $SL(2)$ generators $\{E_{\pm}, H\}$, which satisfy the commutation relations in Chevalley basis:

$$[H^i, E_{\pm}^j] = \pm \delta_{ij} 2E_{\pm}^i, \quad [E_+^i, E_-^j] = \delta_{ij} H^i \quad (2)$$

This model has been used as a testing ground for ideas such as the functional Bethe Ansatz [3] and the general procedure of separation of variables [8, 9], and also applied in quantum optics [10]. The Gaudin models based on the face-type elliptic quantum groups and the XYZ Gaudin models in Sl_2 algebra are studied in Ref. [4]. A method to construct a basis of singular and non-singular common eigenvectors for Gaudin Hamiltonians in a tensor product module of the Lie algebra $SL(2)$ was proposed in [7]. $Osp(1|2)$ Gaudin algebra was discussed in the trigonometric and rational cases in Ref. [6]. All of these are fixed on a special case with some simple symmetry. Since then, constructing a new kind of integrable Gaudin system has become quite interesting due to its possible applications in other physical fields. This motivates us to investigate the problem.

2 Integrability

We know that investigating an integrable system needs two ingredients: a classical r -matrix and a quantum L -operator [13–15]. The proof of the complete integrability of a system given in terms of a Lax pair involves several stages. The first, an immediate consequence of a Lax pair formulation, is

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Received February 13, 2007

the observation that the quantities $\text{Tr}_A L^n$ are conserved. Here, the trace is taken over the representation A of the Lie algebra \mathcal{G} to which the operator L is associated. Another stage is to show that these provide enough functionally independent conserved quantities. Finally, and this is perhaps the most tedious step, one must show that the quantities are in involution, i.e., $\{\text{Tr}_A L^n, \text{Tr}_A L^m\} = 0$. This step is model dependent. We can get the result by arguments based on asymptotics, inverse scattering or direct recursion. Given an L -operator, an alternative approach to proving this Poisson commutativity proceeds via the r -matrix [13, 15].

The L -operators associated with Lie algebra \mathcal{G} must satisfy the standard linear r -matrix algebra:

$$\{L_1(u), L_2(v)\} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)] \quad (3)$$

$$L_1(u) \equiv L(u) \otimes 1, \quad L_2(v) \equiv 1 \otimes L(v) \quad (4)$$

So r_{12} and r_{21} are matrices belonging to the tensor product $\mathcal{G} \otimes \mathcal{G}$. In general the r -matrix is a dynamical object and possesses no special symmetry, and r_{21} can be expressed as:

$$r_{21}(v, u) \equiv Pr_{12}(u, v)P \quad (5)$$

and P is the permutation operator such that: $Px \otimes yP = y \otimes x$.

Integrability in the sense of Liouville does not automatically follow from the existence of the Lax representation. One needs to check that the conserved quantities $\text{Tr}_A L^n$ have vanishing Poisson brackets [16]. When an r -matrix structure is known as Eq. (3), this result is trivial [17]. The proof runs as follows:

$$\begin{aligned} \{\text{Tr}_A L_1^n, \text{Tr}_A L_2^m\} &= \text{Tr}_{A \otimes A} \{L_1^{n \otimes}, L_2^m\} \\ &= nm \text{Tr}_{A \otimes A} \{L^{n-1} \otimes L^{m-1} ([r_{12}, L_1] - [r_{21}, L_2])\} \\ &= 0, \text{ by cyclicity of the trace} \end{aligned} \quad (6)$$

which also implies the existence of an r -matrix.

In our case r is antisymmetric. Thus, Eq. (3) becomes

$$\{L_1(u), L_2(v)\} = [r(u, v), L_1(u) + L_2(v)] \quad (7)$$

More generally, integrable systems can be associated to any semi-simple complex Lie algebra. The infinite dimensional algebra generated by the components of $L(u)$ is called a Gaudin algebra. If a Gaudin algebra describes the Gaudin system corresponding to the Lie algebra \mathcal{G} , then we term this kind of a Gaudin algebra a \mathcal{G} Gaudin algebra. To any simple Lie algebra, one can associate different types of Gaudin algebras, corresponding to different types of r -matrices [18]. With such an algebraic nature, the study of Gaudin algebras becomes an important issue.

The Gaudin models related to classical r -matrices of simple Lie algebras were studied intensively (see [8–10, 19–24], and references therein). In Ref. [12] the Gaudin model was generalized to any semi-simple Lie algebra. However, most of the previous works concerned mainly on rational and trigonometric Gaudin algebras or just in a particular Lie algebra [4–7]. As an alternative to the matrix entry calculations often presented, we give our calculations in terms of a

basis of the typical Lie algebra, A_n, B_n, C_n, D_n . We shall calculate a classical r -matrix for the elliptic Gaudin system with spin.

3 Construction

Before discussing how to construct the Gaudin model, let us review some basics about Lie algebras and give some notations, which is closely related to our work.

Let X_μ denote a Cartan-Weyl basis for semi-simple Lie algebra $\mathcal{G} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$. Here, \mathcal{H} and \mathcal{N}_\pm are the corresponding Cartan and Borel subalgebra of \mathcal{G} , respectively.

Denote the root system of \mathcal{G} by Δ . In the Cartan-Weyl bases, they satisfy:

$$[H_\alpha, E_\alpha] = \alpha_\alpha E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H \quad (8)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \Delta \quad (9)$$

Define the Jacobian elliptic function as:

$$\theta(u; q) = \prod_{n \geq 0} (1 - q^n u) \prod_{n \geq 1} (1 - q^n u^{-1}) \quad (10)$$

where q is related to the module and u as the argument of function $\theta(u) = \theta(u; q)$. Sometimes, we represent brief notations by $\dot{\theta}(u) \equiv u \frac{d\theta}{du}(u)$, $\theta'(u) \equiv \frac{d\theta}{du}(u)$. The Jacobian

elliptic functions have many properties, which can be found in standard text books. Here, we give a few key important identities:

$$\theta(u^{-1}) = -u^{-1} \theta(u), \quad \frac{\dot{\theta}}{\theta}(u/v) = 1 - \frac{\dot{\theta}}{\theta}(v/u) \quad (11)$$

$$\begin{aligned} &\frac{\theta(tuw^{-1})\theta(tt'w)}{\theta(t)\theta(u/w)\theta(t')\theta(w)} - \frac{\theta(t^{-1}uw^{-1})}{\theta(t^{-1})\theta(u/w)} \frac{\theta(tt'u)}{\theta(t')\theta(u)} \\ &= \frac{\theta(tu)}{\theta(t)\theta(u)} \frac{\theta(t'w)}{\theta(t')\theta(w)} \end{aligned} \quad (12)$$

$$\begin{aligned} &\frac{1}{\theta'(1)} t \frac{d}{dt} \left[\frac{\theta(tw)}{\theta(t)\theta(w)} \right] + \frac{1}{\theta'(1)} \frac{\dot{\theta}}{\theta}(u) \frac{\theta(tw)}{\theta(t)\theta(w)} \\ &= -\frac{\theta(t^{-1}u/w)}{\theta(t^{-1})\theta(u/w)} \frac{\theta(tu)}{\theta(t)\theta(u)} + \frac{1}{\theta'(1)} \frac{\dot{\theta}}{\theta}(u/w) \frac{\theta(tw)}{\theta(t)\theta(w)} \end{aligned} \quad (13)$$

$$\begin{aligned} &\frac{\theta(t^{-1}u)}{\theta(t^{-1})\theta(u)} \frac{\theta(tw)}{\theta(t)\theta(w)} + \frac{\theta(t^{-1}u/w)}{\theta(t^{-1})\theta(u/w)} \frac{1}{\theta'(1)} \left[\frac{\dot{\theta}}{\theta}(u) - \frac{\dot{\theta}}{\theta}(w) \right] \\ &= \frac{\theta(t^{-1}u/w)}{\theta(t^{-1})\theta(u/w)} \frac{1}{\theta'(1)} \left[\frac{\dot{\theta}}{\theta}(t) + \frac{\dot{\theta}}{\theta}(t^{-1}u/w) \right] \end{aligned} \quad (14)$$

In terms of the canonical variables p, z ($\{p_a, z_b\} = \delta_{ab}$) and $t_\alpha = e^{\alpha \cdot z}$,

$$\{p_a, t_\alpha\} = \alpha_a t_\alpha, \quad \{p_a, p_b\} = \{t_\alpha, t_\beta\} = 0, \quad a, b = 1, \dots, m \quad (15)$$

In the case of $L = \sum_a L_a X_a^i \otimes X_a$, the Poisson bracket has the form:

$$\begin{aligned} \{L_1(u), L_2(v)\} &= \sum_{a,b} \{L_a X_a^i, L_b X_b^j\} X_a \otimes X_b \\ &= \sum_{a,b} L_a L_b [X_a^i, X_b^j] X_a \otimes X_b \\ &\quad + \{L_a, L_b\} X_a^i X_b^j (X_a \otimes X_b) \end{aligned} \quad (16)$$

With these notations we can express the L operator as:

$$L(u) = \sum_{i=1}^N \left\{ \sum_{\alpha \in \Delta} W_\alpha^i(u) E_\alpha^i \otimes E_{-\alpha} + \sum_a (p_a + \zeta^i(u) H_a^i) \otimes H_a \right\} \quad (17)$$

where

$$W_\alpha^i(u) \equiv \theta'(1) \frac{\theta(t_\alpha u / z_i)}{\theta(t_\alpha) \theta(u / z_i)}, \quad \zeta(u) \equiv u \frac{\theta'_{11}(u, q)}{\theta_{11}(u, q)} = \frac{\dot{\theta}}{\theta}(u) - \frac{1}{2} \quad (18)$$

The following discussions are based on all the A_n, B_n, C_n, D_n Lie algebra. Now we can construct the r -matrix via Eq.(7) and Eq. (16), noticing that $\zeta^i(u)$ does not contain variables p, z ,

$$\begin{aligned} \{L_1(u), L_2(v)\} &= \sum_{i=1}^N \left\{ \sum_{\alpha, \beta \in \Delta} [E_\alpha^i, E_\beta^j] E_{-\alpha}^1 E_{-\beta}^2 W_\alpha^i(u) W_\beta^j(v) \right. \quad (I) \\ &\quad + \sum_{\alpha, a} \{W_\alpha^i(u), p_a\} E_\alpha^i E_{-\alpha}^1 H_a^2 \quad (II) \\ &\quad + \sum_{\alpha, a} [E_\alpha^i, H_a^j] E_{-\alpha}^1 H_a^2 W_\alpha^i(u) \zeta(v)^j \quad (III) \\ &\quad + \sum_{\beta, a} \{p_a, W_\alpha^j(v)\} E_\beta^j H_a^1 E_{-\beta}^2 \quad (IV) \\ &\quad \left. + \sum_{\beta, a} [H_a^i, E_\beta^j] H_a^1 E_{-\beta}^2 \zeta(u)^i W_\beta^j(v) \right\} \quad (V) \end{aligned}$$

Using Eqs. (8), (9), (15) and the properties of the theta function, we calculate each part of the above equation.

$$\text{Introducing a zero term (i) as } \left[\sum_\alpha E_\alpha^1 E_{-\alpha}^2, \sum_b p_b (H_b^1 + H_b^2) \right] \equiv 0 \text{ to part (I), and defining}$$

$$\rho_\alpha(u, v) = -t_\alpha \frac{d}{dt_\alpha} W_\alpha(u/v)$$

$$\tilde{r}(u, v) = -\sum_\alpha W_{-\alpha}(u/v) E_\alpha^1 E_{-\alpha}^2$$

we can get

$$\begin{aligned} \text{(I) + 'zero term(i)'} &= [\tilde{r}(u, v), L_1(u) + L_2(v)] \\ &\quad + \sum_i \sum_\alpha \left\{ W_\alpha(u/v) W_\alpha^i(v) E_\alpha^i E_{-\alpha}^1 (\alpha \cdot H)^2 \right. \\ &\quad + W_{-\alpha}(u/v) W_\alpha^i(u) E_\alpha^i (\alpha \cdot H)^1 E_{-\alpha}^2 \\ &\quad \left. + [\theta'(1)]^2 \rho_\alpha(u, v) (\alpha \cdot H^i) E_{-\alpha}^1 E_\alpha^2 \right\} \\ &= [\tilde{r}(u, v), L_1(u) + L_2(v)] \\ &\quad + \sum_i \sum_\alpha \left\{ \left[W_\alpha(u/v) E_{-\alpha}^1 E_\alpha^2, W_\alpha^i(v) E_\alpha^i E_{-\alpha}^2 \right] \right. \\ &\quad + \left[W_{-\alpha}(u/v) E_\alpha^1 E_{-\alpha}^2, W_\alpha^i(u) E_\alpha^i E_{-\alpha}^1 \right] \\ &\quad \left. + [\theta'(1)]^2 \rho_\alpha(u, v) (\alpha \cdot H^i) E_{-\alpha}^1 E_\alpha^2 \right\} \quad (19) \end{aligned}$$

$$\begin{aligned} \text{(II) + (III)} &= \sum_i \left\{ -\sum_\alpha \left[W_\alpha(u/v) E_{-\alpha}^1 E_\alpha^2, W_\alpha^i(v) E_\alpha^i E_{-\alpha}^2 \right] \right. \\ &\quad \left. - \sum_{\alpha, a} \left[\zeta(u/v) H_a^1 H_a^2, W_\alpha^i(u) E_\alpha^i E_{-\alpha}^1 \right] \right\} \quad (20) \end{aligned}$$

$$\begin{aligned} \text{(IV) + (V)} &= \sum_i \left\{ -\sum_\alpha \left[W_{-\alpha}(u/v) E_\alpha^1 E_{-\alpha}^2, W_\alpha^i(u) E_\alpha^i E_{-\alpha}^1 \right] \right. \\ &\quad \left. - \sum_{\alpha, a} \left[\zeta(u/v) H_a^1 H_a^2, W_\alpha^i(v) E_\alpha^i E_{-\alpha}^2 \right] \right\} \quad (21) \end{aligned}$$

With $\tilde{L}_1(u) = \sum_{\alpha, i} W_\alpha^i(u) E_\alpha^i E_{-\alpha}^1$, it's easy to see that

$$\begin{aligned} \{L_1(u), L_2(v)\} &= [\tilde{r}(u, v), L_1(u) + L_2(v)] \\ &\quad - \sum_a \left[\zeta(u/v) H_a^1 H_a^2, \tilde{L}_1(u) + \tilde{L}_2(v) \right] \\ &\quad + \sum_i \sum_\alpha [\theta'(1)]^2 \rho_\alpha(u, v) (\alpha \cdot H^i) E_{-\alpha}^1 E_\alpha^2 \end{aligned} \quad (22)$$

We can find another zero term (ii) as

$$\begin{aligned} &-\left\{ \sum_a \zeta(u/v) H_a^1 H_a^2, \sum_i \sum_b [p_b + \zeta^i(u) H_b^i] H_b^1 \right. \\ &\quad \left. + [p_b + \zeta^i(v) H_b^i] H_b^2 \right\} = 0 \quad (23) \end{aligned}$$

Adding this term to Eq.(22), we obtain

$$\begin{aligned} \{L_1(u), L_2(v)\} &= [r(u, v), L_1(u) + L_2(v)] \\ &\quad + \sum_\alpha [\theta'(1)]^2 \rho_\alpha(u, v) [E_{-\alpha}^1 E_\alpha^2, \mathbb{H} - \mathbb{H}^2] \end{aligned} \quad (24)$$

Here,

$$\mathbb{H} = \sum_i \sum_a \frac{1}{2} H_a^i \otimes H_a \quad (25)$$

and

$$\begin{aligned} r(u, v) &= \tilde{r}(u, v) - \sum_a \zeta(u/v) H_a^1 H_a^2 \\ &= -\sum_\alpha W_{-\alpha}(u/v) E_\alpha \otimes E_{-\alpha} - \sum_\alpha \zeta(u/v) H_\alpha \otimes H_\alpha \end{aligned} \quad (26)$$

We can see clearly that r -matrix is antisymmetric indeed. r -matrix encodes the Hamiltonian structure of the Lax equation, provides the involution of integrals of motion, and gives a natural framework for quantizing integrable systems in a quantum group theoretical setting.

Let us now turn to the conserved quantities of the system.

Denoting $T = \frac{1}{2} \text{Tr} L^2$, where Tr_k is the trace taken only over the second factor in $\mathcal{G} \otimes \mathcal{G}$.

$$\begin{aligned} \{T(u), T(v)\} &= \text{Tr}_{12} \{L_1(u) L_2(v) [r(u, v), L_1(u) + L_2(v)]\} \\ &+ \sum_i \sum_{\alpha, a} [\theta^i(1)]^2 \rho_\alpha(u, v) \alpha_a H_a^i \text{Tr}_{12} \left\{ L_1(u) \overset{1}{E}_{-\alpha} L_2(v) \overset{2}{E}_\alpha \right\} \\ &= 0 \end{aligned} \quad (27)$$

In this procession, we use the conclusion of Eq. (6) and the fundamental representation of the classical Lie algebra.

4 Conclusion

In contrast to the well-studied case of purely numerical r -matrices, no general theory of dynamical r -matrices exists at present, apart from some concrete examples and observations. Still, the collection of examples is rather sparse, and any new example of dynamical r -matrix could possibly contribute to a better understanding of the model's algebraic and geometric nature.

Acknowledgements This work was supported by the National Natural

Science Foundation of China (Grant No. 90403019).

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